THE UNIVERSITY OF CHICAGO

Surfaces characterized by certain special properties of their directrix congruences

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

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REPRINTED FROM
ANNALI DI MATEMATICA
SERIE III - TOMO XXVI
1916.

VITA

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He acknowledges his indebtedness to all of his instructors and especially to Professors Moody of Bowdoin, Osgood, and Moore, to whose enthusiasm and encouragement he owes much of his interest in mathematics. This dissertation was written under the direction of Professor Wilczynski, whose instruction and helpful criticism will remain ever a source of inspiration to the writer.

Surfaces characterized by certain special properties of their directrix congruences.

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INTRODUCTION.

The projective differential geometry of non-developable surfaces, as developed by Wilczynski (*), is based on the theory of the invariants and covariants of a completely integrable non-involutory system of partial differential equations of the second order, which may be reduced to the canonical form (**)

$$\left.\begin{aligned} y_{uu} + 2\,b\,y_v + f\,y &= 0, \\ y_{vv} + 2\,a'\,y_u + g\,y &= 0, \end{aligned}\right\} \tag{1}$$

where a', b, f, g are analytic functions of the independent variables u and v, and satisfy the integrability conditions

$$\left.\begin{aligned} & a'_{uu} + g_u + 2\,a'_v\,b + 4\,a'\,b_v = 0, \\ & b_{vv} + f_v + 2\,a'\,b_u + 4\,a'_u\,b = 0, \\ & g_{uu} + 2\,b\,g_v + 4\,b_v\,g - f_{vv} - 2\,a'\,f_u - 4\,a'_u\,f = 0. \end{aligned}\right\} \tag{2}$$

The canonical form of such a system is left invariant by all transformations of the infinite group (***)

$$\bar{y} = k\sqrt{\varphi_u\,\psi_v}\,y, \quad \bar{u} = \varphi(u), \quad \bar{v} = \psi(v), \tag{3}$$

(*) E. J. Wilczynski, *Projective differential geometry of curved surfaces.* Five memoirs. Transactions of the American Mathematical Society, 1907-9. We shall hereafter refer to these as First Memoir, etc.

(**) First Memoir, p. 246.

(***) First Memoir, p. 256.

where $\varphi(u)$ and $\psi(v)$ are arbitrary functions of their respective arguments and where k is an arbitrary constant. An ***integral surface*** S_y of system (1) is obtained by interpreting four linearly independent solutions

$$y_1, \quad y_2, \quad y_3, \quad y_4,$$

as the homogeneous coördinates of a point P_y and, since the most general solution of system (1) is of the form

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4,$$

where c_1, c_2, c_3, c_4 are arbitrary constants, it follows that all of the integral surfaces of system (1) constitute a class of projectively equivalent surfaces. On each of these surfaces the parametric curves are asymptotic lines (*).

Through an arbitrary point P_y of an integral surface S_y there pass two asymptotic curves. The tangents at five neighboring points of either of these curves, in general, determine, as the five points approach coincidence at P_y, a definite linear complex which is said to osculate the curve at the point P_y. Since a straight line on a surface is always an asymptotic curve, at least one of the osculating linear complexes is indeterminate at a point of such a line; we shall, therefore, exclude integral surfaces which are ruled, that is, we shall assume that a' and b are not identically zero (**). The two complexes thus associated with a point P_y of a non-ruled surface have a linear congruence in common. This congruence is made up of the lines which intersect a certain pair of lines; one of these lies in the tangent plane at P_y and is called the *directrix of the first kind,* while the other pierces the surface at P_y and is called the *directrix of the second kind* (***). These directrices are hereafter designated as d and d' respectively. As P_y ranges over the surface, the lines d and d' generate two congruences, called *directrix congruences of the first and second kinds* (****) and designated as G and G' respectively. It turns out that the developables of the two congruences are determined by one and the same net of *directrix curves* on S_y.

In the present paper we obtain the conditions under which one or both of the focal sheets of the congruence G' are curves. Limiting our discussion

(*) First Memoir, p. 244.
(**) First Memoir, p. 260.
(***) Second Memoir, p. 95.
(****) Second Memoir, p. 114.

to the class of surfaces characterized by the vanishing of a certain invariant $\mathfrak{M}$, we then show that if the directrix curves form a conjugate system and are distinct from the asymptotic curves, the congruence G' may be linear; finally, we obtain the finite equation of a surface S to which all surfaces having these latter properties are projectively equivalent. The writer takes this opportunity to thank Professor WILCZYNSKI for his interest and guidance in the work leading to this paper.

1. INTRODUCTION OF A NEW LOCAL TETRAHEDRON OF REFERENCE AND SIMPLIFICATION OF THE CONDITIONS OF INTEGRABILITY.

The fundamental semi-covariants of system (1) are

$$y\,, \quad z = y_u\,, \quad \rho = y_v\,, \quad \sigma = y_{uv} \text{ (*)}. \tag{4}$$

In the second memoir the directrix d' is determined by the points P_y and P_τ, where

$$\tau = -a' b_v z - a'_u b \rho + 2 a' b \sigma.$$

The two focal points P'_1 and P'_2 on d', that is, the points in which d' touches the focal surface of the congruence G', are given by the expressions

$$\lambda_1 y + \mu_1 \tau\,, \quad \lambda_2 y + \mu_2 \tau\,, \quad (\mu_1\,, \mu_2 \neq 0), \tag{5}$$

where $\lambda_1 : \mu_1$ and $\lambda_2 : \mu_2$ are the two roots of the quadratic equation (**)

$$\left.\begin{aligned} &\left(\frac{\lambda}{\mu}\right)^2 - 4 a' b \left(\frac{C}{32 a'} + \frac{C'}{32 b} - 2 a' b + \alpha \beta\right) \frac{\lambda}{\mu} + \\ &+ 4 a'^2 b^2 \left(\frac{C}{32 a'} + \frac{C'}{32 b} - 2 a' b + \alpha \beta\right)^2 - \frac{1}{4 a'^2 b^2} (M^2 + a' b L N) = 0, \end{aligned}\right\} \tag{6}$$

where the notation is the same as that used in the second memoir, and

(*) The points P_y, P_z, P_ϱ, P_σ determine the local tetrahedron of reference used in the second memoir.

(**) Equation (143) of the second memoir has been put in this form by using equations (148) and (149). Equation (148) should read $8 M = a' b (a' C' - b C)$.

where in particular

$$\left.\begin{aligned} \alpha &= \frac{a'_u}{2\,a'}, & \beta &= \frac{b_v}{2\,b}, \\ C &= 16\,a'\,(\alpha_v - 2\,a'\,b), & C' &= 16\,b\,(\beta_u - 2\,a'\,b). \end{aligned}\right\} \tag{7}$$

If, now, we put

$$\gamma = \frac{C}{32\,a'} + \frac{C'}{32\,b} \qquad 2\,a'\,b + \alpha\,\beta = \frac{1}{2}\,(\alpha_v + \beta_u) - 4\,a'\,b + \alpha\,\beta, \tag{8}$$

the roots of (6) are

$$\frac{\lambda_1}{\mu_1} = 2\,a'\,b\,\gamma + \frac{1}{2\,a'\,b}\,\sqrt{M^2 + a'\,b\,L\,N}\,,$$

$$\frac{\lambda_2}{\mu_2} = 2\,a'\,b\,\gamma - \frac{1}{2\,a'\,b}\,\sqrt{M^2 + a'\,b\,L\,N}\,,$$

and the expressions (5) become

$$\left.\begin{aligned} &\frac{1}{2\,a'\,b}\,\sqrt{M^2 + a'\,b\,L\,N}\;y + 2\,a'\,b\,\gamma\,y + \tau, \\ -&\frac{1}{2\,a'\,b}\,\sqrt{M^2 + a'\,b\,L\,N}\;y + 2\,a'\,b\,\gamma\,y + \tau. \end{aligned}\right\} \tag{9}$$

From the form of (9) it follows that the focal points P'_1 and P'_2 on d' are harmonic conjugates with respect to the points P_y and P_t, where t is given by

$$2\,a'\,b\,\gamma\,y + \tau,$$

or, since we are using homogeneous coördinates, we may take

$$t = \gamma\,y - \beta\,z - \alpha\,\rho + \sigma. \tag{10}$$

The directrix d is determined by the points in which it intersects the asymptotic tangents, that is, by the points P_p and P_q where

$$p = -\,\alpha\,y + z, \qquad q = -\,\beta\,y + \rho. \tag{11}$$

Since the tetrahedron $P_y\,P_z\,P_\rho\,P_\sigma$ is non-degenerate (*), except for certain

(*) Second Memoir, p. 80.

singular points, the expressions (10) and (11) show that *the four points P_y, P_p, P_q, P_t determine a non-degenerate tetrahedron which may be used as a local tetrahedron of reference.* The unit point is so chosen that the point P_x whose coördinates in the local system are (x_1, x_2, x_3, x_4), shall be given by the expression

$$x = x_1 y + x_2 p + x_3 q + x_4 t.$$

Let P_y be displaced to $P_{y+z du+\rho dv}$; the corresponding directrix of the second kind joins this point to the point $P_{t+t_u du+t_v dv}$. By direct computation, we find

$$t_u = P y + Q p + R q + S t, \qquad t_v = P' y + Q' p + R' q + S' t, \tag{12}$$

where we have put

$$P = \gamma_u + 2 \alpha \gamma - \alpha \beta_u - \alpha_u \beta - 2 \alpha^2 \beta - b_v \beta + 2 b g - f_v + 2 a'_u b,$$

$$Q = \gamma - \beta_u - \alpha \beta + 4 a' b = - \frac{1}{2} (\beta_u - \alpha_v),$$

$$R = - \alpha^2 - \alpha_u - b_v - f,$$

$$S = - \alpha,$$

$$P' = \gamma_v + 2 \beta \gamma - \alpha_v \beta - \alpha \beta_v - 2 \alpha \beta^2 - a'_u \alpha + 2 a' f - g_u + 2 a' b_v,$$

$$Q' = - \beta^2 - \beta_v - a'_u - g,$$

$$R' = \gamma - \alpha_v - \alpha \beta + 4 a' b = \frac{1}{2} (\beta_u - \alpha_v),$$

$$S' = - \beta.$$

An arbitrary point of the displaced directrix will be given by

$$m' (y + z\, d u + \rho\, d v) + n' (t + t_u\, d u + t_v\, d v).$$

The coördinates of such a point are, therefore,

$$x_1 = m' (1 + \alpha\, d u + \beta\, d v) + n' (P\, d u + P'\, d v),$$

$$x_2 = m'\, d u + n' (Q\, d u + Q'\, d v),$$

$$x_3 = m'\, d v + n' (R\, d u + R'\, d v),$$

$$x_4 = \qquad n' (1 + S\, d u + S'\, d v).$$

This point will also be a point of d', if its coördinates satisfy the equations

$$x_2 = x_3 = 0,$$

that is, if

$$\left.\begin{aligned} m'\, d\, u + n'\,(Q\, d\, u + Q'\, d\, v) = 0, \\ m'\, d\, v + n'\,(R\, d\, u + R'\, d\, v) = 0, \end{aligned}\right\} \tag{13}$$

whence

$$\begin{vmatrix} d\, u & Q\, d\, u + Q'\, d\, v \\ d\, v & R\, d\, u + R'\, d\, v \end{vmatrix} = 0,$$

or

$$\mathfrak{L}\, d\, u^2 + 2\, \mathfrak{M}\, d\, u\, d\, v + \mathfrak{N}\, d\, v^2 = 0, \tag{14}$$

where we have put

$$\left.\begin{aligned} \mathfrak{L} &= R = -\alpha'^2 - \alpha'_u - b_v - f = \frac{1}{4\, a'^2\, b}\, L, \\ \mathfrak{M} &= R' = -Q = \frac{1}{2}\, (\beta_u - \alpha_v) = \frac{1}{4\, a'^2\, b^2}\, M, \\ \mathfrak{N} &= -Q' = \beta^2 + \beta_v + a'_u + g = -\frac{1}{4\, a'\, b^2}\, N. \end{aligned}\right\} \tag{15}$$

Equation (14) is the differential equation of the directrix curves on S_y (*). The net of directrix curves degenerates into a one-parameter family only when

$$\mathfrak{R}^2 = \mathfrak{M}^2 - \mathfrak{L}\, \mathfrak{N} \tag{16}$$

vanishes.

Considered as equations in $d\, u$ and $d\, v$, system (13) will be consistent if, and only if,

$$\begin{vmatrix} m' + Q\, n' & Q'\, n' \\ R\, n' & m' + R'\, n' \end{vmatrix} = 0,$$

or

$$m'^2 - \mathfrak{R}^2\, n'^2 = 0.$$

Hence, *the focal points P'_1 and P'_2 on d' are given by*

$$\Pi'_1 = \mathfrak{R}\, y + t, \qquad \Pi'_2 = -\mathfrak{R}\, y + t, \tag{17}$$

respectively, where $\mathfrak{R}$ is one of the square roots of $\mathfrak{M}^2 - \mathfrak{L}\,\mathfrak{N}$.

(*) Cf. Second Memoir, equation (137).

We next consider the directrix congruence G. The directrix d belonging to the point P_y is the edge $P_p P_q$ of the tetrahedron of reference. An arbitrary point of the displaced directrix, determined by the points $P_{p+p_u du+p_v dv}$ and $P_{q+q_u du+q_v dv}$, is given by

$$m\,(p+p_u\,d\,u+p_v\,d\,v)+n\,(q+q_u\,d\,u+q_v\,d\,v).$$

But, differentiating (11) and reducing, we find

$$\left.\begin{aligned} p_u &= \mathfrak{L}\,y-\alpha\,p-2\,b\,q, \quad p_v=-\left(\mathfrak{M}+\frac{C}{8\,a'}\right)y+\beta\,p+t,\\ q_u &= \left(\mathfrak{M}-\frac{C'}{8\,b}\right)y+\alpha\,q+t, \quad q_v=-\,\mathfrak{N}\,y-2\,a'\,p-\beta\,q, \end{aligned}\right\} \tag{18}$$

so that such a point has the coördinates

$$\begin{aligned} x_1 &= m\left[\mathfrak{L}\,d\,u-\left(\mathfrak{M}+\frac{C}{8\,a'}\right)d\,v\right]+n\left[\left(\mathfrak{M}-\frac{C'}{8\,b}\right)d\,u-\mathfrak{N}\,d\,v\right],\\ x_2 &= m\,(1-\alpha\,d\,u+\beta\,d\,v)-2\,a'\,n\,d\,v,\\ x_3 &= -\,2\,b\,m\,d\,u+n\,(1+\alpha\,d\,u-\beta\,d\,v),\\ x_4 &= \qquad m\,d\,v+n\,d\,u. \end{aligned}$$

This point will lie on the directrix d if, and only if,

$$\left.\begin{aligned} x_1 &= m\left[\mathfrak{L}\,d\,u-\left(\mathfrak{M}+\frac{C}{8\,a'}\right)d\,v\right]+n\left[\left(\mathfrak{M}-\frac{C'}{8\,b}\right)d\,u-\mathfrak{N}\,d\,v\right]=0,\\ x_4 &= m\,d\,v+n\,d\,u=0. \end{aligned}\right\} \tag{19}$$

As equations in m and n, these are consistent if, and only if, $d\,u$ and $d\,v$ satisfy equation (14); that is, *the curves defined by* (14) *determine the* developables *of both directrix congruences.* If system (19) is to be consistent when we regard $d\,u$ and $d\,v$ as variables, we must have

$$\begin{vmatrix} \mathfrak{L}\,m+\left(\mathfrak{M}-\dfrac{C'}{8\,b}\right)n, & -\left(\mathfrak{M}+\dfrac{C}{8\,a'}\right)m-\mathfrak{N}\,n \\ n & m \end{vmatrix}=0,$$

or

$$\mathfrak{L}\,m^2-2\,\mathfrak{M}\,m\,n+\mathfrak{N}\,n^2=0. \tag{20}$$

Therefore, *the focal points P_1 and P_2 on d are determined by the expressions*

$$\Pi_1=(\mathfrak{M}+\mathfrak{N})\,p+\mathfrak{L}\,q, \qquad \Pi_2=(\mathfrak{M}-\mathfrak{N})\,p+\mathfrak{L}\,q. \tag{21}$$

Further, *the coincidence of the two one-parameter families of directrix curves is the necessary and sufficient condition for the coincidence of the two focal sheets of either directrix congruence.*

The introduction of the invariants $\mathfrak{L}$ and $\mathfrak{N}$ enables us to simplify the integrability conditions (2). Substituting the values of g_u and f_v as obtained from (*)

$$f = -\alpha^2 - \alpha_u - b_v - \mathfrak{L}, \qquad g = -\beta^2 - \beta_v - a'_u + \mathfrak{N}, \tag{22}$$

in the first two of equations (2), we find

$$\left.\begin{aligned} -2\beta\beta_u - \beta_{uv} + \mathfrak{N}_u + 2a'_v b + 4a' b_v = 0, \\ -2\alpha\alpha_v - \alpha_{uv} - \mathfrak{L}_v + 2a' b_u + 4a'_u b = 0. \end{aligned}\right\} \tag{23}$$

But the differentiation of (7) gives

$$\left.\begin{aligned} C_u &= 16a'(\alpha_{uv} + 2\alpha\alpha_v - 2a'b_u - 4a'_u b), \\ C'_v &= 16b(\beta_{uv} + 2\beta\beta_u - 2a'_v b - 4a'b_v). \end{aligned}\right\} \tag{24}$$

Thus equations (23), that is, the first two integrability conditions become

$$16b\mathfrak{N}_u - C'_v = 0, \qquad 16a'\mathfrak{L}_v + C_u = 0. \tag{25}$$

Further, differentiating the first two equations of (2), we have

$$g_{uu} = -a'_{uuu} - 2a'_{uv}b - 2a'_v b_u - 4a'_u b_v - 4a' b_{uv},$$

$$f_{vv} = -b_{vvv} - 2a' b_{uv} - 2a'_v b_u - 4a'_u b_v - 4a'_{uv} b.$$

These together with the values of f_u and g_v as obtained from (22), substituted in the third equation of (2) give

$$\left.\begin{aligned} -a'_{uuu} + b_{vvv} + 4a'_u(\alpha^2 + \alpha_u + \mathfrak{L}) + 2a'(2\alpha\alpha_u + \alpha_{uu} + \mathfrak{L}_u) - \\ -4b_v(\beta^2 + \beta_v - \mathfrak{N}) - 2b(2\beta\beta_v + \beta_{vv} - \mathfrak{N}_v) = 0. \end{aligned}\right\} \tag{26}$$

But

$$\Big[(\alpha)_u\Big]_u = \left[\left(\frac{a'_u}{2a'}\right)_u\right]_u = \left[\frac{a'_{uu}}{2a'} - \frac{a'^2_u}{2a'^2}\right]_u = \frac{a'_{uuu}}{2a'} - \frac{3a'_{uu}a'_u}{2a'^2} + \frac{a'^3_u}{a'^3}.$$

and therefore,

$$-a'_{uuu} + 4a'_u(\alpha^2 + \alpha_u) + 2a'(2\alpha\alpha_u + \alpha_{uu}) = 0.$$

(*) Cf. Equations (15).

Similarly

$$b_{vvv} - 4\,b_v\,(\beta^2 + \beta_v) - 2\,b\,(2\,\beta\,\beta_v + \beta_{vv}) = 0.$$

Hence equation (26), that is, the third integrability condition becomes

$$2\,a'_u\,\mathfrak{L} + a'\,\mathfrak{L}_u + 2\,b_v\,\mathfrak{N} + b\,\mathfrak{N}_v = 0$$

or, multiplying by $a'\,b$, we find

$$b\,(a'^2\,\mathfrak{L})_u + a'\,(b^2\,\mathfrak{N})_v = 0. \tag{27}$$

Referring to equations (157) (*) of the second memoir and (15) of the present paper, we have

$$2^8\,a'^2\,\mathfrak{L} = \theta, \qquad 2^8\,b^2\,\mathfrak{N} = -\,\theta', \tag{28}$$

where θ and θ' are the invariants of weight four of the osculating ruled surfaces of S_y (**). Using (28) we obtain

$$b\,\theta_u - a'\,\theta'_v = 0. \tag{29}$$

So that *the integrability conditions* (2) *can be written in the simpler form given by* (25) *and* (29).

These results enable us to simplify the expressions for t_u and t_v given by (12). In fact, substituting from (8) and (22), we have

$$P = \frac{1}{2}\,(\beta_{uuu} - \alpha_{uuv}) + \alpha\,(\beta_u - \alpha_v) + 2\,(\alpha_{uv} + 2\,\alpha\,\alpha_v - 4\,a'_u\,b - 2\,a'\,b_u) +$$
$$+ 2\,b\,(\beta^2 + \beta_v + a'_u + g) + \mathfrak{L}_v\,,$$

which, by means of (15), (24), and (25), reduces to

$$P = \mathfrak{M}_u + 2\,\alpha\,\mathfrak{M} - \mathfrak{L}_v + 2\,b\,\mathfrak{N}.$$

Similarly

$$P' = -\,\mathfrak{M}_v - 2\,\beta\,\mathfrak{M} + \mathfrak{N}_u - 2\,a'\,\mathfrak{L}.$$

In order to simplify certain of our later results, we write

$$\mathfrak{P} = P + 2\,\alpha\,\mathfrak{M} - 2\,\beta\,\mathfrak{L}, \qquad \mathfrak{P}' = P' - 2\,\beta\,\mathfrak{M} + 2\,\alpha\,\mathfrak{N}. \tag{30}$$

(*) These equations should read $b\,\theta = 64\,L$ and $a'\,\theta' = 64\,N$.

(**) Second Memoir, pp. 81, 82.

The expressions (12) now become

$$\left.\begin{aligned} t_u &= (\mathfrak{P} - 2\,\alpha\,\mathfrak{M} + 2\,\beta\,\mathfrak{L})\,y - \mathfrak{M}\,p + \mathfrak{L}\,q - \alpha\,t\,, \\ t_v &= (\mathfrak{P}' + 2\,\beta\,\mathfrak{M} - 2\,\alpha\,\mathfrak{N})\,y - \mathfrak{N}\,p + \mathfrak{M}\,q - \beta\,t\,. \end{aligned}\right\} \tag{31}$$

2. Conditions for degeneracy of the focal surface of the directrix congruence of the second kind.

The locus S'_1 of the focal point P'_1 on the directrix d' will degenerate into a curve if, and only if, Π'_1 satisfies a partial differential equation of the form (*)

$$D\,\Pi'_1 + E\,(\Pi'_1)_u + F\,(\Pi'_1)_v = 0, \tag{32}$$

where D, E, and F are functions of u and v and where $(\Pi'_1)_u$ and $(\Pi'_1)_v$ denote $\dfrac{\partial\,\Pi'_1}{\partial\,u}$ and $\dfrac{\partial\,\Pi'_1}{\partial\,v}$ respectively. By direct computation from (17) and (31), we have

$$\left.\begin{aligned} \Pi'_1 &= \mathfrak{R}\,y + t, \\ (\Pi'_1)_u &= (\mathfrak{R}_u + \alpha\,\mathfrak{R} + \mathfrak{P} - 2\,\alpha\,\mathfrak{M} + 2\,\beta\,\mathfrak{L})\,y + (\mathfrak{R} - \mathfrak{M})\,p + \mathfrak{L}\,q - \alpha\,t, \\ (\Pi'_1)_v &= (\mathfrak{R}_v + \beta\,\mathfrak{R} + \mathfrak{P}' + 2\,\beta\,\mathfrak{M} - 2\,\alpha\,\mathfrak{N})\,y - \mathfrak{N}\,p + (\mathfrak{R} + \mathfrak{M})\,q - \beta\,t. \end{aligned}\right\} \tag{33}$$

The existence of a relation of the form (32) requires that the matrix

$$\begin{pmatrix} \mathfrak{R} & 0 & 0 & 1 \\ \mathfrak{R}_u + \alpha\,\mathfrak{R} + \mathfrak{P} - 2\,\alpha\,\mathfrak{M} + 2\,\beta\,\mathfrak{L} & \mathfrak{R} - \mathfrak{M} & \mathfrak{L} & -\alpha \\ \mathfrak{R}_v + \beta\,\mathfrak{R} + \mathfrak{P}' + 2\,\beta\,\mathfrak{M} - 2\,\alpha\,\mathfrak{N} & -\mathfrak{N} & \mathfrak{R} + \mathfrak{M} & -\beta \end{pmatrix}$$

be of rank not greater than two. Let D_i denote the third order determinant obtained by omitting the i^{th} column of this matrix. We find

$$\begin{aligned} D_1 &= \mathfrak{R}^2 - \mathfrak{M}^2 + \mathfrak{L}\,\mathfrak{N} = 0, \\ D_2 &= (\mathfrak{R}\,\mathfrak{P} + \mathfrak{M}\,\mathfrak{R}_u - \mathfrak{L}\,\mathfrak{R}_v) + (\mathfrak{N}\,\mathfrak{R}_u + \mathfrak{M}\,\mathfrak{P} - \mathfrak{L}\,\mathfrak{P}'). \\ D_3 &= -(\mathfrak{N}\,\mathfrak{P}' + \mathfrak{N}\,\mathfrak{R}_u - \mathfrak{M}\,\mathfrak{N}_v) - (\mathfrak{N}\,\mathfrak{R}_v + \mathfrak{N}\,\mathfrak{P} - \mathfrak{M}\,\mathfrak{P}'), \\ D_4 &= \mathfrak{R}\,(\mathfrak{R}^2 - \mathfrak{M}^2 + \mathfrak{L}\,\mathfrak{N}) = 0. \end{aligned}$$

(*) First Memoir, p. 237.

Hence, S'_1 *degenerates if, and only if,*

$$D_2 = D_3 = 0. \tag{34}$$

From (17) it follows that the corresponding expressions for the point P'_2 can be obtained from the above by replacing $\mathfrak{N}$ by $-\mathfrak{N}$. That is, ***the locus*** S'_2 ***of the point*** P'_2 ***degenerates if, and only if,***

$$\left.\begin{aligned} D'_2 &= -(\mathfrak{N}\mathfrak{P} + \mathfrak{M}\mathfrak{N}_u - \mathfrak{L}\mathfrak{N}_v) + (\mathfrak{N}\mathfrak{N}_u + \mathfrak{M}\mathfrak{P} - \mathfrak{L}\mathfrak{P}') = 0, \\ D'_3 &= (\mathfrak{N}\mathfrak{P}' + \mathfrak{N}\mathfrak{N}_u - \mathfrak{M}\mathfrak{N}_v) - (\mathfrak{N}\mathfrak{N}_v + \mathfrak{N}\mathfrak{P} - \mathfrak{M}\mathfrak{P}') = 0. \end{aligned}\right\} \tag{35}$$

If $\mathfrak{N} = 0$, then conditions (34) and (35) are identical and, provided that $\mathfrak{M} \neq 0$, they reduce to the single equation

$$\mathfrak{M}\mathfrak{P} - \mathfrak{L}\mathfrak{P}' = 0. \tag{36}$$

If S'_1 and S'_2 are distinct but degenerate, then (34) and (35) give

$$\mathfrak{N}\mathfrak{P} + \mathfrak{M}\mathfrak{N}_u - \mathfrak{L}\mathfrak{N}_v = 0, \qquad \mathfrak{N}\mathfrak{P}' + \mathfrak{N}\mathfrak{N}_u - \mathfrak{M}\mathfrak{N}_v = 0, \tag{37}$$

$$\mathfrak{N}\mathfrak{N}_u + \mathfrak{M}\mathfrak{P} - \mathfrak{L}\mathfrak{P}' = 0, \qquad \mathfrak{N}\mathfrak{N}_v + \mathfrak{N}\mathfrak{P} - \mathfrak{M}\mathfrak{P}' = 0. \tag{38}$$

But, multiplying equations (37) by $\mathfrak{M}$ and $-\mathfrak{L}$ respectively and adding, we find

$$\mathfrak{N}(\mathfrak{N}\mathfrak{N}_u + \mathfrak{M}\mathfrak{P} - \mathfrak{L}\mathfrak{P}') = 0;$$

also, multiplying by $\mathfrak{N}$ and $-\mathfrak{M}$ respectively and adding, we get

$$\mathfrak{N}(\mathfrak{N}\mathfrak{N}_v + \mathfrak{N}\mathfrak{P} - \mathfrak{M}\mathfrak{P}') = 0.$$

That is, assuming $\mathfrak{N} \neq 0$, equations (37) imply equations (38). Conversely, (37) follow from (38). The two systems are, therefore, equivalent and either may be taken as the conditions of the problem. This may be expressed as follows:

The focal sheets of the directrix congruence of the second kind are distinct and reduce to two curves if, and only if,

$$\left.\begin{aligned} &\mathfrak{N}\mathfrak{N}_u + \mathfrak{M}\mathfrak{P} - \mathfrak{L}\mathfrak{P}' = 0, \\ &\mathfrak{N}\mathfrak{N}_v + \mathfrak{N}\mathfrak{P} - \mathfrak{M}\mathfrak{P}' = 0, \end{aligned}\qquad \mathfrak{N} \neq 0.\right\} \tag{39}$$

If they coincide, the focal surface is the locus of the vertex P_t *of the local* *tetrahedron and this locus degenerates only if*

$$\left.\begin{aligned} \mathfrak{M}\,\mathfrak{P} - \mathfrak{L}\,\mathfrak{P}' &= 0, \\ \mathfrak{N}\,\mathfrak{P} - \mathfrak{M}\,\mathfrak{P}' &= 0, \end{aligned} \quad \mathfrak{N} = 0. \right\} \tag{40}$$

3. Conjugate directrix curves.

Thus far the values of the invariants $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ have been unrestricted, but we now limit our discussion to the case

$$\mathfrak{M} = 0,$$

that is, we shall consider only those surfaces S_y on which the directrix curves, if determinate, form a conjugate system. Among these surfaces we propose to discuss those cases for which the focal surface of the congruence G' degenerates and to study in detail the conditions under which this congruence is linear. There are three cases to be considered.

Case I. First let us suppose

$$\mathfrak{L} = \mathfrak{M} = \mathfrak{N} = 0.$$

This is the case of indeterminate directrix curves (*). Expressions (17) and (21) show that the focal points on d' coincide with P_t, while those on d are indeterminate. The conditions for the degeneracy of the focal surface of the congruence G' are satisfied identically. Wilczynski has shown that the point P_t remains fixed in space and that d lies in a fixed plane. *This directrix point and directrix plane are respectively the focal surfaces of the directrix congruences G' and G which they determine*; they are in united position only when the asymptotic curves belong to linear complexes (**), in which case they are corresponding elements in any one of the null systems determined by these osculating linear complexes.

(*) E. J. Wilczynski, *Ueber Flächen mit unbestimmten Direktrixkurven*. Mathematische Annalen, December 1914.

(**) C. T. Sullivan, *Properties of surfaces whose asymptotic curves belong to linear complexes*. Transactions of the American Mathematical Society, Vol. XV (1914).

Case II. If

$$\mathfrak{L} \neq 0, \qquad \mathfrak{M} = \mathfrak{N} = 0,$$

the two directrix curves through the surface point P_y coincide with the asymptotic curve $u = \text{const}$. The focal points on the directrix d' coincide at P_t; moreover, the locus of this point cannot degenerate into a curve, for conditions (40) now become

$$2\, a' \mathfrak{L}^2 = 0,$$

which cannot be satisfied since $\mathfrak{L} \neq 0$ and the surface S_y is not ruled. Expressions (21) show that the focal points on the directrix d coincide with the vertex P_q of the tetrahedron of reference. From (18) it follows that the expression q satisfies an equation of the form

$$D\, q + E\, q_u + F\, q_v = 0,$$

where D, E, and F are functions of u and v, only if the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -\dfrac{C'}{8b} & 0 & \varkappa & 1 \\ 0 & -2a' & -\beta & 0 \end{pmatrix}$$

is of rank not greater than two. But the determinant obtained by omitting the first column has the value $-2\, a'$; this cannot vanish identically and the matrix is, therefore, of rank three. That is, the locus of the vertex P_q cannot degenerate into a curve. These results may be summarized as follows:

If the two one-parameter families which form the net of directrix curves coincide with each other and with the asymptotic curves $u = \text{const.}$, then the two focal sheets of the congruence G' coincide with the surface S_t, while those of the congruence G coincide with the surface S_q, which is the locus of the intersection of the directrix d of the point P_y and the tangent to the asymptotic curve $u = \text{const.}$ Furthermore, neither of these focal surfaces can degenerate *into a curve.*

By symmetry we find a similar theorem for the case

$$\mathfrak{L} = \mathfrak{M} = 0, \qquad \mathfrak{N} \neq 0.$$

Case III. Finally suppose

$$\mathfrak{L} \neq 0, \qquad \mathfrak{M} = 0, \qquad \mathfrak{N} \neq 0.$$

The directrix curves constitute a non-degenerate conjugate net on the surface S_y and the two focal sheets of either directrix congruence are distinct. In order that the directrices d' may intersect two distinct curves, we must have, on account of (39),

$$\left.\begin{aligned} \mathfrak{N}\,\mathfrak{N}_u - \mathfrak{L}\,(\mathfrak{N}_u - 2\,a'\,\mathfrak{L} + 2\,\alpha\,\mathfrak{N}) &= 0, \\ \mathfrak{M}\,\mathfrak{N}_v - \mathfrak{N}\,(\mathfrak{L}_v - 2\,b\,\mathfrak{N} + 2\,\beta\,\mathfrak{L}) &= 0, \end{aligned}\right\} \tag{41}$$

where, now,

$$\mathfrak{M}^2 = -\,\mathfrak{L}\,\mathfrak{N}. \tag{42}$$

We proceed to require these focal curves to be straight lines, so that the directrix congruence G' becomes linear. Since $\mathfrak{L}$ and $\mathfrak{N}$ are different from zero, neither set of parametric curves can be directrix curves; in particular, then, the focal points P'_1 and P'_2 on d' must trace the above mentioned straight lines whenever the surface point P_y moves along a curve $v = \text{const.}$ So that, in addition to conditions (41), the expressions Π'_1 and Π'_2 must each satisfy an equation of the form

$$D\,\Pi + E\,\Pi_u + F\,\Pi_{uu} = 0, \tag{43}$$

where D, E, and F are functions of u and v. Putting $\mathfrak{M} = 0$, we find from (33)

$$\begin{aligned} \Pi'_1 &= \mathfrak{N}\,y + t, \\ (\Pi'_1)_u &= (\mathfrak{N}_u + \alpha\mathfrak{N} - \mathfrak{L}_v + 2b\mathfrak{N})\,y + \mathfrak{N}\,p + \mathfrak{L}\,q - \alpha\,t, \\ (\Pi'_1)_{uu} &= \left[(\mathfrak{N}_u + \alpha\mathfrak{N} - \mathfrak{L}_v + 2b\mathfrak{N})_u + \alpha\mathfrak{N}_u + (\mathfrak{L} + \alpha^2)\mathfrak{N} - \frac{\mathfrak{L}\,C'}{8b}\right] y + (2\mathfrak{N}_u - \mathfrak{L}_v + 2b\mathfrak{N})\,p + (\mathfrak{L}_u - 2b\mathfrak{N})\,q + (\mathfrak{L} + \alpha^2 - \alpha_u)\,t. \end{aligned}$$

Consequently Π'_1 satisfies a relation of the form (43) only if the matrix

$$\begin{pmatrix} \mathfrak{N} & 0 & 0 & 1 \\ \mathfrak{N}_u + \alpha\,\mathfrak{N} - \mathfrak{L}_v + 2\,b\,\mathfrak{N} & \mathfrak{N} & \mathfrak{L} & -\alpha \\ (\mathfrak{N}_u + \alpha\,\mathfrak{N} - \mathfrak{L}_v + 2\,b\,\mathfrak{N})_u + \alpha\,\mathfrak{N}_u + (\mathfrak{L} + \alpha^2)\,\mathfrak{N} - \dfrac{\mathfrak{L}\,C'}{8\,b} & 2\,\mathfrak{N}_u - \mathfrak{L}_v + 2\,b\,\mathfrak{N} & \mathfrak{L}_u - 2\,b\,\mathfrak{N} & \mathfrak{L} + \alpha^2 - \alpha_u \end{pmatrix} \quad (4$$

is of rank less than three. In particular, the determinant

$$D_1 = \begin{vmatrix} \mathfrak{N} & \mathfrak{L} \\ 2\,\mathfrak{N}_u - \mathfrak{L}_v + 2\,b\,\mathfrak{N} & \mathfrak{L}_u - 2\,b\,\mathfrak{N} \end{vmatrix} = \mathfrak{N}\,\mathfrak{L}_u - 2\,\mathfrak{N}_u\,\mathfrak{L} + \mathfrak{L}\,\mathfrak{L}_v$$

must vanish. Similarly, if Π'_2 is to satisfy a relation of the form (43), the matrix obtained from (44) by replacing $\mathfrak{N}$ by $-\,\mathfrak{N}$ must be of rank less than three and, in particular, the determinant

$$D'_1 = \begin{vmatrix} -\mathfrak{N} & \mathfrak{L} \\ -2\,\mathfrak{N}_u - \mathfrak{L}_v + 2\,b\,\mathfrak{N} & \mathfrak{L}_u + 2\,b\,\mathfrak{N} \end{vmatrix} = -(\mathfrak{N}\,\mathfrak{L}_u - 2\,\mathfrak{N}_u\,\mathfrak{L}) + \mathfrak{L}\,\mathfrak{L}_v$$

must vanish. The vanishing of D_1 and D'_1 gives

$$\mathfrak{L}_v = 0, \qquad \mathfrak{N}\,\mathfrak{L}_u - 2\,\mathfrak{N}_u\,\mathfrak{L} = 0, \tag{45}$$

the second of which may be written in the form

$$\mathfrak{N}^2\,\mathfrak{L}_u - (\mathfrak{N}^2)_u\,\mathfrak{L} = 0.$$

Whence, by means of (42), it follows that

$$\mathfrak{N}_u = 0.$$

Conditions (45) imply, therefore, that $\mathfrak{L}$ and $\mathfrak{N}$ are functions of u alone and v alone, respectively; that is,

$$\mathfrak{L} = \mathfrak{L}(u), \qquad \mathfrak{N} = \mathfrak{N}(v).$$

Moreover, $\mathfrak{L}$ and $\mathfrak{N}$ are invariants such that if the system (1) is transformed by means of (3), we have

$$\overline{\mathfrak{L}} = \frac{1}{\varphi_u^2}\,\mathfrak{L}, \qquad \overline{\mathfrak{N}} = \frac{1}{\psi_v^2}\,\mathfrak{N}.$$

By choosing $\varphi(u)$ and $\psi(v)$ so that

$$\varphi_u^2 = \mathfrak{L}(u), \qquad \psi_v^2 = -\mathfrak{N}(v),$$

we can make

$$\overline{\mathfrak{L}} = -\overline{\mathfrak{N}} = 1.$$

Let us assume that this transformation has already been made; the only transformations of form (3) which will not disturb the normal system obtained in this way, are those for which

$$\varphi_u^2 = \psi_v^2 = 1. \tag{46}$$

Putting

$$\mathfrak{L} = -\mathfrak{N} = 1 \tag{47}$$

in (41), we find

$$a' + \alpha = 0, \qquad b + \beta = 0. \tag{48}$$

Similarly the third integrability condition (27) reduces to

$$a'_u - b_v = 0. \tag{49}$$

This, combined with (48), gives

$$0 = a'_u - b_v = 2\,(a'\,\alpha - b\,\beta) = -\,2\,(a'^2 - b^2).$$

Hence, either

$$a' - b = 0 \tag{50}$$

or

$$a' + b = 0. \tag{51}$$

These cases are not essentially distinct. For, let us suppose that

$$a' + b = 0.$$

Since a' and b are relative invariants such that the effect of the transformation (3) upon them is given by the equations (*)

$$\overline{a'} = \frac{\varphi_u}{\psi_v^2}\,a', \qquad \overline{b} = \frac{\psi_v}{\varphi_u^2}\,b,$$

we shall have

$$\overline{a'} = a', \qquad \overline{b} = -\,b,$$

if we choose, consistent with the restrictions (46),

$$\varphi_u = -\,\psi_v = 1.$$

We shall then have

$$\overline{a'} - \overline{b} = 0.$$

Therefore a transformation of form (3) and satisfying the restriction (46), reduces (51) to (50), and we need consider only the case

$$a' = b = -\,\alpha = -\,\beta, \qquad \mathfrak{L} = -\,\mathfrak{N} = 1, \tag{52}$$

the condition $\mathfrak{M} = 0$ being implied by the equality $a' = b$.

As a result of (52), we find

$$\alpha_v = \beta_u = 2\,b^2 \tag{53}$$

and, by means of (7),

$$C = C' = 0, \tag{54}$$

(*) First Memoir, p. 249.

so that the first two integrability conditions are satisfied identically. Further, the matrix (44) and the corresponding matrix for the expression Π'_2 become

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -3b & 1 & 1 & b \\ 1+7b^2 & -2b & -2b & 1-b^2 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ -b & -1 & 1 & b \\ -1+b^2 & -2b & 2b & 1-b^2 \end{pmatrix}$$

respectively; each is of rank not greater than two. That is, as a consequence of (52), the integrability conditions are satisfied and both Π'_1 and Π'_2 satisfy equations of the forms

$$D\Pi + E\Pi_u + F\Pi_v = 0,$$
$$D'\Pi + E'\Pi_u + F'\Pi_{uu} = 0,$$

where $D, \ldots, F'$ are functions of u and v. Hence, *the directrix congruence G' is linear if conditions* (52) *hold.*

The conditions (52) enable us to determine the coefficients of system (1) explicitly as functions of u and v. In fact, we find

$$b_u = b_v = -2b^2, \tag{55}$$

whence

$$b = \frac{1}{2(u+v)+k},$$

where k is an arbitrary constant. Without disturbing the simplifications already secured, we may make the transformation

$$\bar{u} = u + \frac{k}{2}, \qquad \bar{v} = v$$

of the independent variables, so that

$$\bar{b} = \frac{1}{2(\bar{u}+\bar{v})}.$$

Assuming this transformation already made, we may write

$$a' = b = \frac{1}{2\,(u+v)}\,.$$

The conditions

$$\mathfrak{L} = -\,\mathfrak{N} = 1$$

determine the coefficients f and g. These results are summarized in the theorem.

Every surface with conjugate directrix curves whose directrix congruence of the second kind is linear, is determined, except for a projective transformation, as an integral surface of the completely integrable system of partial differential equations

$$\left.\begin{aligned} y_{uu} + 2\,b\,y_v + f\,y = 0, \\ y_{vv} + 2\,a'\,y_u + g\,y = 0, \end{aligned}\right\} \tag{56}$$

where

$$a' = b = \frac{1}{2\,(u+v)}\,, \qquad f = g = -\,1 - b^2, \tag{57}$$

the curves $u =$ const. and $v =$ const. being asymptotic lines. Since the coefficients of (56) involve no arbitrary elements, all such surfaces are projectively equivalent.

4. The asymptotic curves on the integral surfaces of system (56). The directrix congruence G. System (56) referred to the directrix curves.

We proceed to study the class of surfaces S_u defined by the theorem just stated. From (56) we find as the differential equation of the asymptotic curve $u =$ const.

$$y_{vvvv} + 4\,b\,y_{vvv} - 2\,(1 + 3\,b^2)\,y_{vv} - 4\,b\,(1 - 3\,b^2)\,y_v + (1 + 2\,b^2 - 15\,b^4)\,y = 0.$$

The dependent variable being transformed by means of

$$y = \frac{1}{\sqrt{u+v}}\,\bar{y}\,.$$

this equation becomes
$$\bar{y}_{vvvv} - 2\,\bar{y}_{vv} + \bar{y} = 0. \tag{58}$$

Its invariants are constants, in particular,

$$\theta_3 = \theta_{3\cdot 1} = 0, \qquad \theta_4 = \frac{16}{25}, \qquad \theta_{4\cdot 1} = \frac{16^3}{3^2\,5^3}.$$

We may recapitulate as follows.

All of the asymptotic curves $u = const.$ are projectively equivalent to each other. Each of them is anharmonic and belongs to a linear complex (*). *and, therefore, lies on a quadric* (**). *Besides, each is identically self-dual; in fact, any point of the curve and the osculating plane at that point are* correspond-ing *elements in the null system determined by the linear complex to which the curve belongs.*

Similarly, the asymptotic curves $v = const.$ are characterized by the differential equation
$$\bar{y}_{uuuu} - 2\,\bar{y}_{uu} + \bar{y} = 0.$$

Its invariants have the same constant values as those of equation (58) and, consequently, the curves of the two families of asymptotic curves are projectively equivalent to each other.

According to (21) the focal points P_1 and P_2 on the directrix d are given by the expressions
$$\text{II}_1 = p + q, \qquad \text{II}_2 = -\,p + q, \tag{59}$$

which show that *the focal points on d are harmonic conjugates with respect to the intersections of d and the corresponding asymptotic tangents.* Further, the differential equation of the directrix curves (14) reduces to

$$d\,u^2 - d\,v^2 = 0,$$

(*) C. T. Sullivan, l. c., p. 175. His conditions (15) may be written

$$\Omega' = -\frac{C}{8\,a'} = 0 \quad \text{and} \quad \Omega'' = -\frac{C'}{8\,b} = 0.$$

These are satisfied as a result of (54) of the present paper.

(**) E. J. Wilczynski, *Projective differential geometry of curves and ruled surfaces.* B. G. Teubner, Leipzig, 1906. p. 282. We shall hereafter refer to this work as Proj. Diff. Geom.

so that *the directrix curves are given by* $u + v = const.$ *and* $u - v = const.$, and the second equations of systems (13) and (19) become

$$n'\,d\,u + m'\,d\,v = 0, \qquad n\,d\,u + m\,d\,v = 0.$$

More specifically the notation is such that, when the surface point P_y moves along the directrix curve $u + v = \text{const.}$, the points P_1 and P'_1 trace the edges of regression of the corresponding developables in the directrix congruences G and G' respectively.

In order to study the loci of the points P_1 and P_2, we first write down the expressions

$$\left.\begin{aligned}
y_u &= -b\,y + p, & y_v &= -b\,y + q,\\
p_u &= y + b\,p - 2\,b\,q, & p_v &= -b\,p + t,\\
q_u &= -b\,q + t, & q_v &= y - 2\,b\,p + b\,q,\\
t_u &= -2\,b\,y + q + b\,t, & t_v &= -2\,b\,y + p + b\,t.
\end{aligned}\right\} \tag{60}$$

obtained from (11), (18), and (31) by means of (52). So that we find

$$\left.\begin{aligned}
\Pi_1 &= p + q,\\
(\Pi_1)_u &= y + b\,p - 3\,b\,q + t,\\
(\Pi_1)_v &= y - 3\,b\,p + b\,q + t,\\
(\Pi_1)_{uu} &= -2\,b\,y + (1 - b^2)\,p + (1 + 7\,b^2)\,q - 2\,b\,t,\\
(\Pi_1)_{uv} &= -6\,b\,y + (1 + 3\,b^2)\,p + (1 + 3\,b^2)\,q + 2\,b\,t,\\
(\Pi_1)_{vv} &= -2\,b\,y + (1 + 7\,b^2)\,p + (1 - b^2)\,q - 2\,b\,t,
\end{aligned}\right\} \tag{61}$$

and

$$\left.\begin{aligned}
\Pi_2 &= -p + q,\\
(\Pi_2)_u &= -y - b\,p + b\,q + t,\\
(\Pi_2)_v &= y - b\,p + b\,q - t,\\
(\Pi_2)_{uu} &= -2\,b\,y - (1 - b^2)\,p + (1 - b^2)\,q + 2\,b\,t,\\
(\Pi_2)_{uv} &= (1 + b^2)\,p - (1 + b^2)\,q,\\
(\Pi_2)_{vv} &= 2\,b\,y - (1 - b^2)\,p + (1 - b^2)\,q - 2\,b\,t.
\end{aligned}\right\} \tag{62}$$

From (61) we find

$$\left.\begin{aligned}
(\Pi_1)_{uu} + 2\,b\,(\Pi_1)_u - (1 + b^2)\,\Pi_1 = 0,\\
(\Pi_1)_{vv} + 2\,b\,(\Pi_1)_v - (1 + b^2)\,\Pi_1 = 0,
\end{aligned}\right\} \tag{63}$$

which show that the parametric curves on the locus of P_1 are straight lines. Moreover, the determinant

$$\begin{vmatrix} -6b & 1+3b^2 & 1+3b^2 & 2b \\ 1 & b & -3b & 1 \\ 1 & -3b & b & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = 64\,b^2$$

does not vanish. Also the matrix formed from the last three rows contains the determinant

$$\begin{vmatrix} b & -3b & 1 \\ -3b & b & 1 \\ 1 & 1 & 0 \end{vmatrix} = -8\,b,$$

which is not equal to zero and, therefore, the matrix is of rank three. We may conclude that Π_1 satisfies no relation of the form

$$D\,(\Pi_1)_{uv} + E\,(\Pi_1)_u + F\,(\Pi_1)_v + H\,\Pi_1 = 0,$$

where D, E, F, H are functions of u and v, the case $D=0$ being included. Consequently the locus of P_1 is a non-degenerate quadric surface.

Transforming (63) by means of

$$\Pi_1 = r^{-\frac{1}{2}}\,\overline{\Pi}_1\,, \qquad r = u+v, \qquad s = u-v,$$

we find

$$(\overline{\Pi}_1)_{rr} + (\overline{\Pi}_1)_{ss} - \overline{\Pi}_1 = 0,$$

$$(\overline{\Pi}_1)_{rs} = 0,$$

whence the differential equations of the curves $r=$ const. and $s=$ const. on the locus of P_1 are

$$(\overline{\Pi}_1)_{sss} - (\overline{\Pi}_1)_s = 0$$

and

$$(\overline{\Pi}_1)_{rrr} - (\overline{\Pi}_1)_r = 0$$

respectively. For each of these equations the invariant of weight three is zero. It therefore results that *the curves, along which the developables of the* *congruence G touch the quadric focal sheet, constitute a conjugate net of conics* (*).

There is one and only one relation of the first order between the expressions occurring in (62), namely

$$(\Pi_2)_u + (\Pi_2)_v - 2\, b\, \Pi_2 = 0,$$

and this shows that the locus of P_2 is a curve. The relation

$$(\Pi_2)_{uu} - 2\, b\, (\Pi_2)_u - (1 - 3\, b^2)\, \Pi_2 = 0$$

requires this curve to be a straight line. The focal points P_1 and P_2 being distinct, this line cannot be a ruling on the locus of P_1. Hence the theorem.

The directrices d of an integral surface S_y *of system* (56) *are tangent to a quadric and intersect a line not on that quadric. The two sets of rulings, that is, the asymptotic lines on the quadric correspond to the two* one-parameter *families of asymptotic curves on* S_y.

Let us now refer the surface S_y to its directrix curves by putting

$$r = \frac{1}{u+v}, \qquad s = u - v, \tag{64}$$

and, in order to simplify our results, transform the dependent variable by means of

$$y = r^{-\frac{1}{2}}\, \overline{y}. \tag{65}$$

We find

$$\left.\begin{aligned}
y_u &= r^{-\frac{1}{2}}\left(-r^2\, \overline{y}_r + \overline{y}_s + \frac{1}{2}\, r\, \overline{y}\right), \\
y_v &= r^{-\frac{1}{2}}\left(-r^2\, \overline{y}_r - \overline{y}_s + \frac{1}{2}\, r\, \overline{y}\right), \\
y_{uu} &= r^{-\frac{1}{2}}\left(r^4\, \overline{y}_{rr} - 2\, r^2\, \overline{y}_{rs} + \overline{y}_{ss} + r^3\, \overline{y}_r + r\, \overline{y}_s - \frac{1}{4}\, r^2\, \overline{y}\right), \\
y_{vv} &= r^{-\frac{1}{2}}\left(r^4\, \overline{y}_{rr} + 2\, r^2\, \overline{y}_{rs} + \overline{y}_{ss} + r^3\, \overline{y}_r - r\, \overline{y}_s - \frac{1}{4}\, r^2\, \overline{y}\right).
\end{aligned}\right\} \tag{66}$$

(*) Cf. DARBOUX, *Leçons sur la théorie des surfaces*, t. II, p. 12.

So that system (56) now becomes

$$\left.\begin{aligned} r^4\,\bar{y}_{rr} + \bar{y}_{ss} - \bar{y} &= 0, \\ \bar{y}_{rs} &= 0. \end{aligned}\right\} \tag{67}$$

That is, ***if the variables are suitably chosen, the differential equations of the surface S_y will have the simple form (67), the directrix curves being parametric.***

From (67) it follows that the differential equations of the directrix curves $r =$ const. and $s =$ const. are

$$\bar{y}_{sss} - \bar{y}_s = 0,$$

and

$$r^4\,\bar{y}_{rrr} + 4\,r^3\,\bar{y}_{rr} - y_r = 0$$

respectively. The invariant of weight three for the first equation is identically zero and, therefore, *the curves $r = const.$, that is, the directrix curves $u + v = const.$ are conics* (*). As we have already seen (**), the corresponding focal points P_1 and P'_1 trace a quadric and a straight line respectively.

5. Detailed discussion of a representative surface of the class.

We find that system (67) has the following four linearly independent solutions

$$\bar{y}_1 = r\,e^{\frac{1}{r}}, \qquad \bar{y}_2 = r\,e^{-\frac{1}{r}}, \qquad \bar{y}_3 = e^s, \qquad \bar{y}_4 = e^{-s}. \tag{68}$$

Interpreted as the homogeneous coördinates of a point, these determine a *surface S* to which all integral surfaces S_y of (67) are projectively equivalent. We introduce non-homogeneous coördinates by putting

$$x = \frac{\bar{y}_1}{\bar{y}_4} = r\,e^{\frac{1}{r}+s}, \qquad y = \frac{\bar{y}_2}{\bar{y}_4} = r\,e^{-\frac{1}{r}+s}, \qquad z = \frac{\bar{y}_3}{\bar{y}_4} = e^{2s}, \tag{69}$$

(*) *Proj. Diff. Geom.*, p. 61.

(**) Cf. p. 20.

and we may regard x, y, z as the rectangular cartesian coördinates of a point of the surface.

Eliminating r and s from (69), we find

$$z = \frac{1}{4}\, x\, y \left(\log_e \frac{x}{y}\right)^2 \tag{70}$$

as *the cartesian equation of* S, which shows that the planes $x + y = 0$ and $x - y = 0$ are planes of symmetry. If we transform to cylindrical coördinates by means of

$$x = w \cos \varphi, \qquad y = w \sin \varphi, \qquad z = z,$$

equation (70) becomes

$$z = \frac{1}{8}\, w^2 \sin 2\varphi\, (\log_e \cot \varphi)^2. \tag{71}$$

Taking account of the symmetry and interpreting negative values of w, we need consider only values of φ such that

$$-45^0 \leqq \varphi \leqq 45^0.$$

We can plot only real values of the variables and, since

$$\left.\begin{aligned} \log_e a &= \log_e a + 2k\pi i, && \text{if } a \text{ is positive and real,} \\ &= \log_e |a| + (2k+1)\pi i, && \text{if } a \text{ is negative and real,} \end{aligned}\right\} \tag{72}$$

where k is an arbitrary integer, it follows that z as a function of w and φ has more than one real value only when $\cot \varphi = 1$ or -1, that is, only when $\varphi = 45^0$ or -45^0. If $\varphi = 45^0$, we find from (71)

$$w^2 = -\frac{2}{k^2 \pi^2}\, z.$$

k being an arbitrary integer; in other words, the plane $\varphi = 45^0$ intersects the surface S in the denumerable set of parabolas having their vertices at the origin and their foci at the points $w = 0$, $z = -\dfrac{1}{2k^2\pi^2}$: for $k = 0$ this set includes the straight line $\varphi = 45^0$, $z = 0$. If $\varphi = -45^0$, we find

$$w^2 = \frac{8}{(2k+1)^2 \pi^2}\, z,$$

where k is an arbitrary integer; that is, the plane $\varphi = -45^0$ intersects the surface in a denumerable set of parabolas having their vertices at the origin and their foci at the finite points $w = 0$, $z = \frac{2}{(2k+1)^2 \pi^2}$. From (71) and (72) we see that there are no real points of S for values of φ in the interval $-45^0 < \varphi < 0^0$.

Consider now an arbitrary meridian plane for which $0^0 < \varphi < 45^0$. Referred to the rectangular axes in this plane, the intersection with S has the equation

$$w^2 = 4\,m\,z,$$

where

$$m = \frac{2 \csc 2\,\varphi}{(\log_e \cot \varphi)^2}.$$

The real part of this intersection consists of a single parabola with its vertex at the origin and whose focus is the point $w = 0$, $z = m$. Thinking of m as a function of φ, we find

$$\frac{d\,m}{d\,\varphi} = -\frac{4 \csc 2\,\varphi \cot 2\,\varphi}{(\log_e \cot \varphi)^3} (\log_e \cot \varphi - 2 \sec 2\,\varphi).$$

In the interval $0^0 < \varphi < 45^0$, this derivative vanishes only when

$$\log_e \cot \varphi - 2 \sec 2\,\varphi = 0,$$

that is, only when, $\varphi = 7^0 \cdot 2+$. This value of φ makes $\frac{d^2\,m}{d\,\varphi^2}$ positive and, therefore, determines the meridian plane section of minimum latus rectum. Since $\frac{d\,m}{d\,\varphi}$ is negative when $0^0 < \varphi < 7^0 \cdot 2+$, we see that m increases as φ decreases and becomes infinite when φ approaches zero, for

$$\underset{\varphi=0}{L}\, m = 2 \underset{\varphi=0}{L} \frac{\csc 2\,\varphi}{(\log_e \cot \varphi)^2} = \underset{\varphi=0}{L} \frac{\cot 2\,\varphi}{\log_e \cot \varphi} = \underset{\varphi=0}{L} \csc 2\,\varphi = \infty.$$

Hence, as the meridian plane approaches the $x\,z$-plane, the parabolic section straightens out and coincides with the x-axis when $\varphi = 0^0$. Also, from (70), we find

$$\frac{\partial z}{\partial y} = \frac{1}{4}\, x \left(\log_e \frac{x}{y}\right) \left(\log_e \frac{x}{y} - 2\right), \tag{73}$$

which becomes infinite when y approaches zero, provided $x \neq 0$; so that the surface becomes tangent to the xz-plane along the x-axis, except possibly at the origin. When $7^\circ \cdot 2+ < \varphi < 45^\circ$, then $\frac{dm}{d\varphi}$ is positive and, therefore, m increases as φ increases and becomes infinite when φ approaches 45°, that is, the parabolic section straightens out as the meridian plane approaches the plane $x - y = 0$ and coincides with the line $x - y = 0$, $z = 0$ when $\varphi = 45^\circ$. Putting $y = x$ in (73), we see that $\left.\frac{\partial z}{\partial y}\right|_{y=x=0} = 0$ whence it follows by means of the symmetry that the surface S is tangent to the xy-plane along the line $x - y = 0$, $z = 0$, except possibly at the origin.

Consider the plane section $z = \text{const.}$ (positive). From (71) we have

$$x^2 = w^2 \cos^2 \varphi = \frac{4 \cot \varphi}{(\log_e \cot \varphi)^2} z.$$

Then

$$\frac{d x^2}{d \varphi} = - \frac{4 z \csc^2 \varphi}{(\log_e \cot \varphi)^3} (\log_e \cot \varphi - 2).$$

In the interval $0^\circ < \varphi < 45^\circ$, this derivative vanishes only if

$$\log_e \cot \varphi - 2 = 0,$$

that is, only if

$$\varphi = 7^\circ \cdot 7+ .$$

We find that this value of φ makes $\frac{d^2 x^2}{d \varphi^2}$ positive and, therefore, gives the minimum value of x for every positive value of z. In other words, for an arbitrary section of the surface S, parallel to the xy-plane, the point with minimum abscissa in the interval $0^\circ < \varphi < 45^\circ$ lies in the meridian plane whose equation is $x - e^2 y = 0$.

By symmetry we obtain the complete surface S. We now proceed to consider certain of the associated loci. From (69) we find

$$\frac{x}{y} = e^{\frac{2}{r}}, \qquad z = e^{2s},$$

whence it follows that *the intersections of the surface S with the planes through and perpendicular to the z-axis are respectively the directrix curves $r = const.$ and $s = const.$: the directrix d' belonging to the surface point P_y intersects the* *z-axis perpendicularly and passes through the point P_y. Thus the z-axis and the line at infinity in the planes $z = const.$ are seen to constitute the focal loci of the congruence G'.*

The equations of transformation (64) and (65) enable us to compute the four values of the semi-covariants z, ρ, σ corresponding to the four values of $\bar{y}$ given by (68). Substituting the expressions thus obtained in the covariants p, q, t (*), we find

$$\left.\begin{aligned}
p_1 &= \sqrt{r}\, e^{\frac{1}{r}}, \quad p_2 = -\sqrt{r}\, e^{-\frac{1}{r}}, \quad p_3 = \frac{(r+1)}{\sqrt{r}}\, e^{s}, \quad p_4 = \frac{(r-1)}{\sqrt{r}}\, e^{-s}, \\
q_1 &= \sqrt{r}\, e^{\frac{1}{r}}, \quad q_2 = -\sqrt{r}\, e^{-\frac{1}{r}}, \quad q_3 = \frac{(r-1)}{\sqrt{r}}\, e^{s}, \quad q_4 = \frac{(r+1)}{\sqrt{r}}\, e^{-s}, \\
t_1 &= \sqrt{r}\, e^{\frac{1}{r}}, \quad t_2 = \sqrt{r}\, e^{-\frac{1}{r}}, \quad t_3 = -\frac{1}{\sqrt{r}}\, e^{s}, \quad t_4 = -\frac{1}{\sqrt{r}}\, e^{-s}.
\end{aligned}\right\} \tag{74}$$

These determine the other three vertices of the local tetrahedron belonging to P_y. We can now write the coördinates of a point referred to our fixed cartesian system; in particular, we have

$$\left.\begin{aligned}
&P'_1\left(-r\,e^{\frac{1}{r}+s},\ -r\,e^{-\frac{1}{r}+s},\ e^{2s}\right), \quad && P'_2\,(0,\ 0,\ e^{2s}), \\
&P_1\left(e^{\frac{1}{r}+s},\ -e^{-\frac{1}{r}+s},\ e^{2s}\right), && P_2\,(0,\ 0,\ -e^{2s}),
\end{aligned}\right\} \tag{75}$$

where the directrix curves are parametric. The homogeneous coördinates of the focal point P'_1 are $\left(e^{\frac{1}{r}},\ e^{-\frac{1}{r}},\ 0,\ 0\right)$. From (69) and (75) it follows that *the directrix d belonging to the point P_y, as determined by the points P_1 and P_2, is the intersection of the tangent plane to S and the plane determined by the z-axis and the point which is the image of P_y in the xz-plane. The focal sheets of the congruence G are the hyperbolic paraboloid S_1*

$$z + xy = 0 \tag{76}$$

and the z-axis. More specifically, let P_y trace the parabolic directrix curve

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(*) The reduced forms of p and q are given by (60); as a result of (52) the expression (10) becomes

$$t = -b^2 y + bz + b\rho + \sigma,$$

$r = \text{const.}$, which is determined by the plane

$$x - e^{\frac{2}{r}}\, y = 0,$$

then the directrix d intersects the z-axis in the point whose z-coördinate is equal to the negative of that of P_y, and envelopes the parabola in which the paraboloid S_1 is cut by the plane

$$x + e^{\frac{2}{r}}\, y = 0,$$

while the directrix d' remaining parallel to the xy-plane, sweeps over the upper half of the given meridian plane. If P_y traces the directrix curve $s = \text{const.}$, that is, the curve in which S is cut by the plane $z = e^{2s}$, then d generates the cone determined by the vertex $(0, 0, -e^{2s})$ and the hyperbola in which S_1 is cut by the plane of the directrix curve, while d' rotates about the z-axis.

The point P_t, being the harmonic conjugate of P_y with respect to the two focal points on the directrix d', has as its locus a surface which, in the terminology of Koenigs (*), may be called *the point conjugate of the surface S_y with respect to the directrix congruence G'*. Moreover, if the developables of a congruence determine a conjugate net on a surface S_y such that when S_y is referred to this net, the equation of the form

$$y_{rs} = a\, y_r + b\, y_s + c\, y$$

has equal Laplace-Darboux invariants, Koenigs has shown that these developables trace a conjugate net on the point conjugate of S_y also. From (69) and (74) it is clear that *the surface S is its own point conjugate*, corresponding points P_y and P_t being symmetrically situated with respect to the z-axis. The parametric curves of the locus S_t, coinciding with the directrix curves of S_y, form a conjugate net as is required by the theorem, since both Laplace-Darboux invariants of the second equation of system (67) vanish.

If in (69) we introduce the variables of the asymptotic curves by means of (64) and then eliminate r, we find that *the asymptotic curves $u = \text{const.}$*

(*) G. Koenigs, *Sur les systèmes conjugués à invariants égaux*. Comptes Rendus, v. 113 (1891).

on the surface S are determined by the family of hyperbolic paraboloids

$$x\,z - e^{4u}\,y = 0 \text{ (*)}. \tag{77}$$

Similarly the asymptotic curves $v = \text{const.}$ are the curves in which the family of hyperbolic paraboloids

$$y\,z - e^{-4v}\,x = 0$$

intersects the surface S.

As a result of (54) *both families of osculating ruled surfaces R_1 and R_2 have distinct straight line directrices* (**). The two directrices of each of these ruled surfaces are the two branches of its flecnode curve (***). From (56) the osculating ruled surface of the first kind R_1 ($u = \text{const.}$) is characterized by the equations

$$\left.\begin{aligned} y_{vv} + p_{11}\,y_v + p_{12}\,z_v + q_{11}\,y + q_{12}\,z = 0, \\ z_{vv} + p_{21}\,y_v + p_{22}\,z_v + q_{21}\,y + q_{22}\,z = 0, \end{aligned}\right\} \tag{78}$$

where

$$\begin{aligned} &p_{11} = 0, \qquad p_{12} = 0, \quad q_{11} = -1 - b^2, \quad q_{12} = 2\,b, \\ &p_{21} = -4\,b^2, \quad p_{22} = 0, \quad q_{21} = 2\,b + 6\,b^3, \quad q_{22} = -1 - 5\,b^2. \end{aligned}$$

The flecnodes P_η and P_ζ on the generator $P_y P_z$ are given by the factors of the quadratic

$$z^2 + 2\,b\,y\,z + (b^2 - 1)\,y^2,$$

so that, referring to (60), we may put

$$\eta = y + \rho, \qquad \zeta = -y + \rho.$$

That is, *the flecnodes are harmonic conjugates with respect to the points P_y and P_ρ* (****). In non-homogeneous coördinates the flecnodes are the points

$$P_\eta\left(2\,e^{2u},\ 0,\ \frac{1+b}{b}\,e^{2(u-v)}\right),$$

$$P_\zeta\left(0,\ \frac{2\,b}{1-b}\,e^{-2v},\quad -\frac{b}{1-b}\,e^{2(u-v)}\right).$$

(*) Cf. p. 19.

(**) C. T. Sullivan, l. c., p. 178. The reader will distinguish between the straight line directrices of a ruled surface and the directrices of the first and second kinds associated with an arbitrary point of a non-ruled surface.

(***) *Proj. Diff. Geom.*, p. 150.

(****) This is always true for the osculating ruled surfaces R_1.

Hence, as v varies, P_η and P_ζ trace the lines given by

$$x = 2\, e^{2u}, \qquad y = 0$$

and

$$x = 0, \qquad e^{2u}\, y + 2\, z = 0$$

respectively; that is, ***the directrices of R_1 are a line in the $x\,z$-plane parallel to the z-axis and a line in the $y\,z$-plane through the origin.***

Similarly the flecnodes P_θ and P_ξ on the generator $P_y\, P_\rho$ of the osculating ruled surface of the second kind ($v =$ const.) are harmonic conjugates with respect to P_y and P_q, in fact, we may write

$$\theta = y + q, \qquad \xi = -\, y + q.$$

In non-homogeneous coördinates the flecnodes are the points

$$P_\theta\left(\frac{2\, b}{1 + b}\, e^{2u},\ 0,\ \frac{b}{1 + b}\, e^{2(u - v)}\right),$$

$$P_\xi\left(0,\ -\, 2\, e^{-2v},\ -\, \frac{1 - b}{b}\, e^{2(u - v)}\right).$$

Hence, the equations of the directrices of R_2 are

$$e^{-2v}\, x - 2\, z = 0, \qquad y = 0$$

and

$$x = 0, \qquad y = -\, 2\, e^{-2v};$$

these represent respectively a line in the $x\,z$-plane through the origin and a line in the $y\,z$-plane parallel to the z-axis. SULLIVAN has shown that the directrices of the two families of ruled surfaces R_1 and R_2 are complementary reguli on a certain quadric surface, which he calls the ***directrix quadric*** (*). For our surface S this directrix quadric degenerates into the $x\,z$- and $y\,z$-planes.

(*) C. T. SULLIVAN, l. c., p. 185. In a later paper the same author has considered surfaces having a degenerate directrix quadric. Cf. abstract, « Bulletin of the American Mathematical Society », vol. 21 (1915), p. 431.

6. Duality.

The adjoint of system (1) is

$$\left.\begin{aligned} Y_{uu} + 2\bar{b}\, Y_v + \bar{f}\, Y = 0, \\ Y_{vv} + 2\bar{a}'\, Y_u + \bar{g}\, Y = 0, \end{aligned}\right\} \tag{79}$$

where

$$\bar{a}' = -a', \quad \bar{b} = -b, \quad \bar{f} = f + 2b_v, \quad \bar{g} = g + 2a'_u \text{ (*)}.$$

An integral surface S_Y, obtained by interpreting four linearly independent solutions of (79) as the homogeneous coördinates of a point P_Y, is dualistic to every integral surface S_y of (1). Relations among the invariants of (79) define projective properties of the surface S_Y: by duality, then, certain projective properties of the surface S_y are determined and, since the invariants of (79) are expressible in terms of the invariants of (1), in particular,

$$\bar{a}' = -a', \quad \bar{b} = -b, \quad \mathfrak{C} = \mathfrak{C}, \quad \bar{\mathfrak{M}} = \mathfrak{M}, \quad \mathfrak{N} = \mathfrak{N}, \tag{80}$$

we are able to write down the relations among the invariants of (1) which characterize these projective properties of the surface S_y. The dualistic transformation, by which the point P_Y of S_Y corresponds to the tangent plane of S_y at the point P_y, makes the directrices of the first and second kinds correspond to the directrices of the second and first kinds respectively; moreover, the directrix directions correspond. These considerations enable us to extend the results of the second and third paragraphs of this paper.

For the surface S_Y the two focal sheets of the directrix congruence of the second kind are distinct and degenerate if, and only if (**),

$$\begin{aligned} \bar{\mathfrak{N}}\,\bar{\mathfrak{N}}_u + \bar{\mathfrak{M}}\,\bar{\mathfrak{P}} - \mathfrak{C}\,\bar{\mathfrak{P}}' = 0, \\ \bar{\mathfrak{N}} \neq 0, \\ \bar{\mathfrak{N}}\,\bar{\mathfrak{N}}_v + \bar{\mathfrak{N}}\,\bar{\mathfrak{P}} - \bar{\mathfrak{M}}\,\bar{\mathfrak{P}}' = 0. \end{aligned}$$

(*) ***First Memoir***, p. 259. Note the corrected values of f and $\bar{g}$.

(**) Cf. equations (39).

But by means of (80) these become

$$\left.\begin{aligned} &\mathfrak{N}\,\mathfrak{N}_u + \mathfrak{M}\,(\mathfrak{P} - 4\,b\,\mathfrak{N}) - \mathfrak{L}\;(\mathfrak{P}' + 4\,a'\,\mathfrak{L}) = 0, \\ &\mathfrak{N}\,\mathfrak{N}_v + \mathfrak{N}\;(\mathfrak{P} - 4\,b\,\mathfrak{N}) - \mathfrak{M}\,(\mathfrak{P}' + 4\,a'\,\mathfrak{L}) = 0. \end{aligned}\quad \mathfrak{N} \neq 0, \right\} \tag{81}$$

Hence we may state the theorem for the surface S_{u}. ***The focal surface of the directrix congruence of the first kind consists of two distinct developable surfaces only if conditions*** (81) ***are satisfied.*** If the two focal sheets coincide and if the resulting surface is to be developable, we must have either

$$\mathfrak{M}\,(\mathfrak{P} - 4\,b\,\mathfrak{N}) - \mathfrak{L}\,(\mathfrak{P}' + 4\,a'\,\mathfrak{L}) = 0, \qquad \mathfrak{N} = 0, \qquad \mathfrak{M} = 0, \tag{82}$$

or

$$\mathfrak{L} = \mathfrak{M} = \mathfrak{N} = 0.$$

In the latter case the focal surface is a plane (*).

Conditions (39) and (81) cannot hold simultaneously, for, in that case, we must have

$$\left.\begin{aligned} \mathfrak{M}\,\mathfrak{N}\,b + \mathfrak{L}^2\,a' = 0, \\ \mathfrak{N}^2\,b + \mathfrak{L}\,\mathfrak{M}\,a' = 0, \end{aligned}\right\} \tag{83}$$

But, since $\mathfrak{N} \neq 0$, these equations can be satisfied by non-vanishing values of a' and b only if both $\mathfrak{L}$ and $\mathfrak{N}$ vanish: for, if one is zero the other must be also. Finally, if $\mathfrak{L} = \mathfrak{N} = 0$, conditions (39) imply that $\mathfrak{M} = 0$, which is impossible. Hence, ***if the focal surface of the congruence G' consists of two distinct curves, the two distinct focal sheets of the congruence G cannot be developable.***

From (52) and (80) it follows that, if a surface, on which the directrix curves form a non-degenerate conjugate net, has a linear directrix congruence G, it is possible to choose the variables so that

$$a' = b = \alpha = \beta, \qquad \mathfrak{L} = -\,\mathfrak{N} = 1. \tag{84}$$

These conditions enable us to determine the seminvariants a', b, f, g, and we find the theorem. ***Every surface with conjugate directrix curves, whose directrix congruence of the first kind is linear, is projectively equivalent to an arbitrary integral surface of the system whose canonical form*** (1) ***has the***

(*) Cf. p. 12.

coefficients

$$a' = b = \frac{-1}{2(u+v)}, \qquad f = g = -1 - 5b^2. \tag{85}$$

If now we put

$$y = r^{\frac{1}{2}} \bar{y}$$

and refer the surface to its directrix curves by means of the transformation (64), the system given by (85) becomes

$$\left. \begin{aligned} & r^4 \bar{y}_{rr} + \bar{y}_{ss} + 4 r^3 \bar{y}_r - \bar{y} = 0, \\ & \bar{y}_{rs} = 0. \end{aligned} \right\} \tag{86}$$

We find that these equations have the following four linearly independent solutions

$$\bar{y}_1 = -\left(\frac{1}{r} + 1\right) e^{-\frac{1}{r}}, \quad \bar{y}_2 = \left(\frac{1}{r} - 1\right) e^{\frac{1}{r}}, \quad \bar{y}_3 = e^{-s}, \quad \bar{y}_4 = e^{s}. \tag{87}$$

Hence, *every surface characterized by the conditions* (84), *is projectively equivalent to the surface defined by the parametric equations* (87). It is easy to verify that the system determined by (85) is the adjoint of system (56) and that the solutions (87) are the coördinates of the tangent plane to the surface S defined by the parametric equations (68).

Since a' and b are different from zero, it is impossible to satisfy conditions (52) and (84) simultaneously. In other words, *a surface whose directrix curves form a conjugate net, cannot have more than one linear directrix congruence.*

Estratto dagli *Annali di Matematica*
Tomo XXVI della Serie III, pag. 1 e seguenti.

Tipo-Lit. Rebeschini di Turati e C.
Milano, 1916.