The University of Chicago

ON INEQUALITIES OF CERTAIN TYPES IN GENERAL LINEAR INTEGRAL EQUATION THEORY

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On Inequalities of Certain Types in General Linear Integral Equation Theory.

BY MARY EVELYN WELLS.

PART I.

§ 1. *Introduction.*

In the theory of the classical linear integral equation

$$\xi(s)=\eta(s)-\lambda\int_a^b \varkappa(s,\,t)\eta(t)\,dt$$

we find the inequality of Schwarz *

$$\int_a^b[\xi(s)]^2ds\int_a^b[\eta(s)]^2ds-\left[\int_a^b\xi(s)\eta(s)\,ds\right]^2\geqq 0.$$

In the theory of the general linear integral equation

$$\xi(s)=\eta(s)-\lambda J\varkappa(s,\,t)\eta(t), \tag{1}$$

in which appear more general functions and operator, explained later, E. H. Moore has found, among other inequalities, the analogue of the Schwarz inequality given above, namely,

$$J\xi\bar{\xi}J\eta\bar{\eta}-J\xi\bar{\eta}J\eta\bar{\xi}\geqq 0, \tag{2}$$

where — denotes the conjugate imaginary. We proceed to the discussion of such inequalities.

In a memoir "On the Foundations of the Theory of Linear Integral Equations",† E. H. Moore has given the basis Σ_5, and system of postulates, of a theory of the general linear integral equation (1) which we shall write

$$\xi=\eta-\lambda J\varkappa\eta. \tag{1}$$

E. H. Moore has defined the properties of class $\mathfrak{M}\equiv[\mu]$ of functions μ on a general range $\mathfrak{P}\equiv[p]$ to the class $\mathfrak{A}\equiv[a]$ of all real or complex numbers, properties of class $(\mathfrak{M}\mathfrak{M})_*$ to which the kernel function $\varkappa$ belongs, and also the properties of the functional operation J, necessary for the theory of the gen-

* Heywood-Fréchet, "L'Equation de Fredholm."

† *Bulletin of the American Mathematical Society*, Ser. 2, Vol. XVIII.

eral linear integral equation. For the convenience of the reader some of the definitions are given here.

$\mathfrak{M}$ is linear (L), in notation $\mathfrak{M}^L$, in case

$$\mathfrak{M}=\mathfrak{M}_L\equiv[\text{all } \mu=a_1\mu_1+a_2\mu_2+\cdots+a_n\mu_n].$$

$\mathfrak{M}$ is real (R), in notation $\mathfrak{M}^R$, in case the class $\mathfrak{M}$ is the same as the class of conjugate elements,

$$\mathfrak{M}=\overline{\mathfrak{M}}.$$

J is an operator of binary quality eliminating both the arguments s and t when it operates on such a function as $\mu_1(s)\mu_2(t)$ or $\varkappa(s,t)$ giving a number of class $\mathfrak{A}$. That J operates on a function $\varkappa$ to give a number a, we indicate by the notation $J^{\text{on }\mathfrak{K}\text{ to }\mathfrak{A}}$. The notation $J_{(s,t)}\varkappa(s,t)$ will be replaced, throughout, by $J\varkappa$, and $J_{(s,t)}\mu_1(s)\mu_2(t)$ by $J\mu_1\mu_2$. That $J\mu_1\mu_2$ is in general different from $J\mu_2\mu_1$ is seen by the examples of J used in § 4.

The operator J is linear (L), in notation J^L, in case

$$a_1\varkappa_1+a_2\varkappa_2=\varkappa \quad \text{implies} \quad a_1J\varkappa_1+a_2J\varkappa_2=J\varkappa.$$

The operator J is hermitian (H), in notation J^H, in case

$$\overline{J\mu_1\mu_2}=J\bar{\mu}_2\bar{\mu}_1.$$

The operator J is positive (P), in notation J^P, in case for every function μ of $\mathfrak{M}$

$$J\bar{\mu}\mu\geqq 0.$$

Using the foundation

$$\Sigma=(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}^{LR};\ \mathfrak{K}\equiv(\mathfrak{M}\mathfrak{M})_*;\ J^{\text{on }\mathfrak{K}\text{ to }\mathfrak{A}\cdot LHP}),$$

it is the purpose of the first part of this paper to show the existence of certain inequalities of the type

$$\sum_{ijkl}^{(12)} a_{ijkl}J_{ij}J_{kl}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ J^{LHP});\tag{3}$$

where $ijkl$ is an arrangement of the digits 1234 which refer to the functions in the first, second, third and fourth positions in the product $\xi\bar{\xi}\eta\bar{\eta}$ of which the arguments are omitted for convenience. For instance the term $a_{1324}J_{13}J_{24}\xi\bar{\xi}\eta\bar{\eta}$ indicates $a_{1324}J\xi\eta J\bar{\xi}\bar{\eta}$. Of the twenty-four terms to correspond with the twenty-four arrangements of the subscripts 1234 in $J_{ij}J_{kl}$ only twelve are distinct since $J_{ij}J_{kl}=J_{kl}J_{ij}$. Thus, the inequality is one of twelve terms as indicated by the notation in (3). For definiteness the digit 1 will be kept in the first place or the last place in the arrangements of 1234, and the other digits will be in dictionary order. With this plan the coefficients of the twelve distinct terms are as indicated in table (7). In further explanation of (3) it should be said that the parenthesis indicates that the inequalities are valid for all functions ξ and η of the class $\mathfrak{M}$ and all operators which have the properties *LHP*. Since throughout this paper ξ and η will always be considered of the class $\mathfrak{M}$, and each J will have the properties *LHP*, these superscripts will often be omitted, and the superscripts used will usually indicate still more special properties of ξ, η, and J.

Further, it will be shown that the inequalities exhibited form a fundamental set for the two cases, (1°) where $J=\check{J}$, *i. e.*, $J_{(s,t)}\varkappa(s,t)=J_{(s,t)}\varkappa(t,s)$*, (2°) where ξ and η are real. By saying that certain inequalities form a fundamental set, it is meant that any inequality of type (3) which satisfies conditions shown to be necessary, can be expressed as a sum of positive or zero multiples of the inequalities forming the fundamental set.

The three following inequalities, containing a numerical parameter u, together with those obtainable from these three by transformations on ξ and η, form the fundamental set (1°) when $J=\check{J}$, (2°) when ξ and η are real.

$$\{(1+u\bar{u})J_{12}J_{34}+(u+\bar{u})J_{14}J_{32}\}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}). \qquad (4)$$

$$\{J_{12}J_{43}+\bar{u}J_{14}J_{23}+uJ_{32}J_{41}+u\bar{u}J_{34}J_{21}\}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}). \qquad (5)$$

$$\{J_{13}J_{42}+\bar{u}J_{13}J_{24}+uJ_{42}J_{31}+u\bar{u}J_{24}J_{31}\}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}). \qquad (6)$$

§ 2. *Proofs of the Inequalities.*

A special case of the first inequality (4) is the generalized Schwarz inequality (2) stated and proved by E. H. Moore. The proof of inequality (4) was exhibited by E. H. Moore in class lectures given in January, 1914. For the derivation of (4) and (5) E. H. Moore proved that when a set of classes of functions $\mathfrak{M}', \mathfrak{M}'', \ldots, \mathfrak{M}^{(n)}$, each of which is linear (L) and real (R), is used to form the class $\mathfrak{M}\equiv(\mathfrak{M}'\mathfrak{M}''\ldots\mathfrak{M}^{(n)})_L$,† the resulting class is linear (L) and real (R); and that the corresponding functional operation $J\equiv J'J''\ldots J^{(n)}$, in which each $J^{(i)}$ has the properties *LP*, itself has the properties *LP*. Accordingly, when $m=2$, we have

$$J'_{13}J''_{24}(\xi\eta+u\eta\xi)\,\overline{(\xi\eta+u\eta\xi)}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J'^{LHP};\ J''^{LHP}).$$

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* E. H. Moore, *Bulletin of the American Mathematical Society*, April, 1912.

† "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations" published in the "Proceedings of the Fifth International Congress of Mathematicians, Cambridge, August, 1912."

Since the two operators are not necessarily the same, two inequalities are obtained, (4) by using JJ and (5) by using $J\breve{J}$.

To prove (6) it is only necessary to write the inequality as a product of a number and its conjugate:

$$(J\xi\eta+uJ\eta\xi)\,\overline{(J\xi\eta+uJ\eta\xi)}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}).$$

Table (7), in which the inequalities are tabulated by means of their coefficients a_{ijkl} and labeled $A_1, A_2, \ldots, A_{12}$ for convenience in reference, shows the complete set of inequalities (4), (5), (6) and those obtainable from (4), (5) and (6) by transformations on ξ and η. The transformations used may be indicated by the usual transformation notation:* $(\xi\bar{\xi})$, $(\xi\bar{\xi})(\eta\bar{\eta})$, $(\eta\bar{\eta})$.

	a_{1234}	a_{1243}	a_{1324}	a_{1342}	a_{1423}	a_{1432}	a_{2341}	a_{2431}	a_{3241}	a_{3421}	a_{4231}	a_{4321}	
$A_{1,u}$	$1+u\bar{u}$	0	0	0	0	$u+\bar{u}$	0	0	0	0	0	0	
$A_{2,u}$	0	$1+u\bar{u}$	0	$u+\bar{u}$	0	0	0	0	0	0	0	0	(4′)
$A_{3,u}$	0	0	0	0	0	0	0	$u+\bar{u}$	0	$1+u\bar{u}$	0	0	
$A_{4,u}$	0	0	0	0	0	0	$u+\bar{u}$	0	0	0	0	$1+u\bar{u}$	
$A_{5,u}$	0	1	0	0	$\bar{u}$	0	0	0	u	$u\bar{u}$	0	0	(7)
$A_{6,u}$	1	0	$\bar{u}$	0	0	0	0	0	0	0	u	$u\bar{u}$	(5′)
$A_{7,u}$	0	$u\bar{u}$	0	0	$\bar{u}$	0	0	0	u	1	0	0	
$A_{8,u}$	$u\bar{u}$	0	$\bar{u}$	0	0	0	0	0	0	0	u	1	
$A_{9,u}$	0	0	$\bar{u}$	1	0	0	0	$u\bar{u}$	0	0	u	0	
$A_{10,u}$	0	0	$\bar{u}$	$u\bar{u}$	0	0	0	1	0	0	u	0	(6′)
$A_{11,u}$	0	0	0	0	$\bar{u}$	$u\bar{u}$	1	0	u	0	0	0	
$A_{12,u}$	0	0	0	0	$\bar{u}$	1	$u\bar{u}$	0	u	0	0	0	

§ 3. *The Twelve Inequalities (7) Form a Fundamental Set When $J=\breve{J}$.*

PROOF. Since the operator J is self-transpose $(J=\breve{J})$, $J\mu_1\mu_2=J\mu_2\mu_1$. As an example of such an operator may be mentioned the classical unary J of the classical instances II_n, III, IV,† which, in the respective instances, is

$$\sum_{p=1}^{n},\quad \sum_{p=1}^{\infty},\quad \int_a^b dp.$$

When the self-transpose J operates upon $\xi\bar{\xi}\eta\bar{\eta}$, we have

$$\left.\begin{aligned} J_{12}J_{34}&=J_{12}J_{43}=J_{34}J_{21}=J_{43}J_{21},\\ J_{13}J_{24}&=J_{13}J_{42}=J_{24}J_{31}=J_{42}J_{31},\\ J_{14}J_{23}&=J_{14}J_{32}=J_{23}J_{41}=J_{32}J_{41}.\end{aligned}\right\} \tag{8}$$

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* Cf. Cajori, "Theory of Equations."

† E. H. Moore, "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations."

Consequently, we have a grouping of coefficients a_{ijkl} which we designate

$$\left.\begin{aligned} b_1 &= a_{1234}+a_{1243}+a_{3421}+a_{4321}, \\ b_2 &= a_{1324}+a_{1342}+a_{2431}+a_{4231}, \\ b_3 &= a_{1423}+a_{1432}+a_{2341}+a_{3241}. \end{aligned}\right\} \tag{9}$$

In this notation the general inequality which we wish to build from the given inequalities (7) is

$$(b_1 J_{12} J_{34}+b_2 J_{13} J_{24}+b_3 J_{14} J_{23})\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi;\ \eta;\ J=\breve{J}). \tag{10}$$

In this field, $J=\breve{J}$, the given inequalities (7), with grouping of coefficients in accordance with (9), reduce to four distinct inequalities B_1, B_2, B_3, B_4, tabulated by means of the coefficients b_1, b_2, b_3, as follows:

	b_1	b_2	b_3
$B_{1,u}$	$1+u\bar{u}$	0	$u+\bar{u}$
$B_{2,u}$	$1+u\bar{u}$	$u+\bar{u}$	0
$B_{3,u}$	0	$1+u\bar{u}+u+\bar{u}$	0
$B_{4,u}$	0	0	$1+u\bar{u}+u+\bar{u}$

(11)

To build the general inequality (10), as the sum of positive or zero multiples of the fundamental inequalities (11), it is evident that the multipliers must be expressed in terms of the coefficients b_1, b_2, and b_3. First, then, we determine certain necessary conditions involving b_1, b_2 and b_3, which are certain expressions in b_1, b_2, b_3 found to be necessarily positive or zero numbers. By suitable choice of certain of these expressions we build (10) as a sum of positive or zero multiples of the inequalities (11).

For the determination of necessary conditions on b_1, b_2, b_3 we need only to use a binary operator, and use for ξ and η the vectors (x_1, x_2) and (y_1, y_2). In this binary algebraic case, the linearity of J demands that $J\xi\eta$ have the form

$$j_{11}x_1y_1+j_{12}x_1y_2+j_{21}x_2y_1+j_{22}x_2y_2.$$

Therefore, the binary operator may be written

$$J=\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix},$$

which effects the ordinary matricial combination with the product $\xi\eta$ which is

$$\begin{pmatrix} x_1y_1, & x_1y_2 \\ x_2y_1, & x_2y_2 \end{pmatrix}.$$

And, conversely, this form of J implies the linearity of J.

The fact that this operator is hermitian (H)* is expressed in the equality

$$\overline{j_{11}x_1y_1+j_{12}x_1y_2+j_{21}x_2y_1+j_{22}x_2y_2}=j_{11}\overline{x_1y_1}+j_{21}\overline{x_1y_2}+j_{12}\overline{x_2y_1}+j_{22}\overline{x_2y_2} \qquad (x_1,\, x_2,\, y_1,\, y_2),$$

for which it is necessary and sufficient that the matrix (j_{rs}) be hermitian, viz., $\bar{j}_{rs}=j_{sr}$, $(r=1,\,2;\; s=1,\,2)$.

The property P of J^* demands

$$j_{11}x_1\bar{x}_1+j_{12}x_1\bar{x}_2+j_{21}x_2\bar{x}_1+j_{22}x_2\bar{x}_2 \geqq 0 \qquad (x_1\,,\, x_2).$$

For the positiveness of this hermitian form the necessary and sufficient conditions are

$$j_{11}\geqq 0, \qquad j_{22}\geqq 0, \qquad j_{11}j_{22}-j_{12}j_{21}\geqq 0.$$

Hence the matrix (j_{rs}) is both hermitian and positive. In general, it is true that such an operator, binary or n-ary, is hermitian (H) if, and only if, the matrix (j_{rs}) is hermitian; and such an operator is positive (P) if, and only if, each principal minor is positive or zero.

The binary operator $\begin{pmatrix} j_{11} & 0 \\ 0 & j_{22}\end{pmatrix}$ affords, as a special instance of (10),

$$\left.\begin{array}{l} b_1(j_{11}x_1\bar{x}_1+j_{22}x_2\bar{x}_2)\,(j_{11}y_1\bar{y}_1+j_{22}y_2\bar{y}_2) \\ +b_2(j_{11}x_1y_1+j_{22}x_2y_2)\,(j_{11}\bar{x}_1\bar{y}_1+j_{22}\bar{x}_2\bar{y}_2) \\ +b_3(j_{11}x_1\bar{y}_1+j_{22}x_2\bar{y}_2)\,(j_{11}\bar{x}_1y_1+j_{22}\bar{x}_2y_2) \end{array}\right\}\geqq 0 \quad \begin{pmatrix} x_1\,, & y_1\,, & j_{11}\geqq 0 \\ x_2\,, & y_2\,, & j_{22}\geqq 0\end{pmatrix}. \tag{12}$$

The cases

$$\begin{aligned}(\xi;\ \eta;\ J) &= (x_1\,,\, x_2;\ y_1\,,\, y_2;\ j_{11}\,,\, j_{22}) \\ &= (1,\,0;\,1,\,0;\,1,\,0),\ (0,\,-1;\ 1,\,0;\ 1,\,1),\ (1,\,i;\,1,\,-i;\,1,\,1), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1,\,i;\,1,\,i;\,1,\,1),\end{aligned}$$

where $i=\sqrt{-1}$, show that we have as necessary conditions:

$$b_1+b_2+b_3\geqq 0, \qquad b_1\geqq 0, \qquad b_1+b_2\geqq 0, \qquad b_1+b_3\geqq 0. \tag{13}$$

With these conditions on the coefficients we are able at once to build the general inequality (10) in the field indicated by $J=\check{J}$ from the fundamental inequalities (11).

If $b_3\geqq 0$, we secure the desired inequality by using

$$\tfrac{1}{2}\,b_1B_{2,-1}+\,(b_1+b_2)\,B_{3,\,0}+b_3B_{4,\,0}\,,$$

which is the sum of positive or zero multiples of positive or zero forms, hence is positive or zero. If $b_3<0$, we use

$$-\tfrac{1}{2}\,b_3B_{1,\,-1}+\tfrac{1}{2}\,(b_1+b_3)\,B_{2,\,-1}+\,(b_1+b_2+b_3)\,B_{3,\,0}\,,$$

which is the sum of positive or zero multiples of positive or zero forms, hence is positive or zero.

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* Cf. Section 1.

§ 4. *The Twelve Inequalities* (7) *Form a Fundamental Set When* ξ *and* η *Are Real.*

PROOF. In case ξ and η are real we have:

$$\left.\begin{aligned} &J_{12}J_{34}=J_{12}J_{43}=J_{34}J_{21}=J_{43}J_{21},\\ &J_{13}J_{42}=J_{14}J_{32}=J_{23}J_{41}=J_{24}J_{31},\\ &J_{13}J_{24}=J_{14}J_{23},\\ &J_{32}J_{41}=J_{42}J_{31}.\end{aligned}\right\} \tag{14}$$

We shall designate the consequent grouping of coefficients a_{ijkl},

$$\left.\begin{aligned} &c_1=a_{1234}+a_{1243}+a_{3421}+a_{4321},\\ &c_2=a_{1342}+a_{1432}+a_{2341}+a_{2431},\\ &c_3=a_{1324}+a_{1423},\\ &c_4=a_{3241}+a_{4231}.\end{aligned}\right\} \tag{15}$$

The corresponding general inequality which is to be expressed as the sum of positive or zero multiples of the fundamental inequalities is

$$(c_1J_{12}J_{34}+c_2J_{13}J_{42}+c_3J_{13}J_{24}+c_4J_{32}J_{41})\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^R;\ \eta^R;\ J). \tag{16}$$

With ξ and η real, the given inequalities (7), with grouping of coefficients in accordance with (15), reduce to three distinct inequalities C_1, C_2, C_3, tabulated by means of their coefficients c_1, c_2, c_3, c_4 as follows:

	c_1	c_2	c_3	c_4
$C_{1,u}$	$1+u\bar{u}$	$u+\bar{u}$	0	0
$C_{2,u}$	$1+u\bar{u}$	0	$\bar{u}$	u
$C_{3,u}$	0	$1+u\bar{u}$	$\bar{u}$	u

(17)

The multipliers in terms of c_1, c_2, c_3, c_4 with which to build the general inequality (16), as a sum of positive or zero multiples of known inequalities (17) are to be chosen from necessary conditions involving the coefficients c_1, c_2, c_3, c_4. Again, it is possible to determine the necessary conditions by use of a binary operator; but in order to secure sufficient conditions, a more general binary operator must be used than served for the situation $J=\check{J}$. We use j_{12} pure imaginary (I) and have

$$J=\begin{pmatrix} j_{11} & j_{12}\\ -j_{12} & j_{22}\end{pmatrix}.$$

The inequality (16) becomes, for this special case,

$$\left.\begin{aligned} &c_1(j_{11}x_1^2+j_{22}x_2^2)(j_{11}y_1^2+j_{22}y_2^2) \\ &+c_2\{j_{11}x_1y_1+j_{12}(x_1y_2-x_2y_1)+j_{22}x_2y_2\} \\ &\qquad\qquad\{j_{11}x_1y_1+j_{12}(x_2y_1-x_1y_2)+j_{22}x_2y_2\} \\ &+c_3\{j_{11}x_1y_1+j_{12}(x_1y_2-x_2y_1)+j_{22}x_2y_2\}^2 \\ &+c_4\{j_{11}x_1y_1+j_{12}(x_2y_1-x_1y_2)+j_{22}x_2y_2\}^2 \end{aligned}\right\} \geqq 0 \begin{pmatrix} x_1^R & y_1^R \\ x_2^R & y_2^R \\ j_{11}\geqq 0 & \\ j_{22}\geqq 0 & j_{12}^I \\ j_{11}j_{22}+j_{12}^2\geqq 0 & \end{pmatrix}. \tag{18}$$

By making suitable choice of the variables and elements of the matrix J, we secure the necessary conditions. In this instance it is convenient to use the most general matrix whose determinant is zero. Such a matrix is

$$\begin{pmatrix} j_1\bar{j}_1 & j_1\bar{j}_2 \\ j_2\bar{j}_1 & j_2\bar{j}_2 \end{pmatrix} \text{ where } j_2=eij_1 \text{ and } e^R. \tag{19}$$

In this work we need frequently to remember that

$$me^2+2ne+p\geqq 0 \qquad (e^R) \tag{20}$$

implies

$$m^R,\ n^R,\ p^R, \quad m\geqq 0,\ p\geqq 0, \quad mp-n^2\geqq 0. \tag{21}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j_{11}, j_{12}, j_{22})$

$$=(1, 0;\ 1, 0;\ 1, 0, 1),\ (1, 0;\ 0, 1;\ 1, i, 1),\ (1, 1;\ 1, -1;\ 1, 0, 1)$$

give as necessary conditions

hence,

$$\left.\begin{aligned} &c_1+c_2+c_3+c_4\geqq 0, \quad c_1+c_2-c_3-c_4\geqq 0, \quad c_1\geqq 0, \\ &(c_1+c_2)^R\geqq 0, \quad (c_3+c_4)^R, \quad c_1^R,\ c_2^R. \end{aligned}\right\} \tag{22}$$

The case $(x_1, x_2;\ y_1, y_2;\ j_{11}, j_{12}, j_{22})=(1, 0;\ 1, 1;\ 1, -e'i, e'^2)$ gives the inequality

$$(c_1+c_2-c_3-c_4)e'^2-2i(c_3-c_4)e'+(c_1+c_2+c_3+c_4)\geqq 0 \qquad (e'^R)$$

which, by (21), ensures the conditions

$$(c_3-c_4)^I, \quad \therefore c_3=\bar{c}_4, \quad \text{and} \quad (c_1+c_2)^2-4c_3c_4\geqq 0. \tag{23}$$

With the conditions expressed in (22) and (23) it is possible to build the general inequality (16) in this field ($\xi^R;\ \eta^R$), from the fundamental inequalities (17).

If $c_2>0$, we may build first for the case $c_1=0$, using the consequent condition:

$$c_2^2-4c_3c_4\geqq 0.$$

The desired inequality is given by

$$\tfrac{1}{2}\{c_2+(c_2^2-4c_3c_4)^{\frac{1}{2}}\}C_{3,\,2c_4/[c_2+(c_2^2-4c_3c_4)^{\frac{1}{2}}]}. \tag{24}$$

If $c_1 \neq 0$ it follows from (22) that $c_1 > 0$, and we may obtain the desired inequality by adding $c_1 C_{2,0}$ to the inequality (24) already built.

If $c_2 = 0$ the necessary conditions (22) and (23) reduce to

$$c_1 \geqq 0, \quad c_1^2 - 4c_3c_4 \geqq 0.$$

Hence, the desired inequality may be expressed as

$$\tfrac{1}{2}\{c_1 + (c_1^2 - 4c_3c_4)^{\frac{1}{2}}\} C_{2,\, 2c_4/[c_1+(c_1^2-4c_3c_4)^{\frac{1}{2}}]}.$$

If $c_2 < 0$ and $c_1 + c_2 \neq 0$, we use

$$-\tfrac{1}{2}c_2 C_{1,-1} + dC_{2,\, c_4/d}, \quad \text{where} \quad d = \tfrac{1}{2}[c_1 + c_2 + \{(c_1+c_2)^2 - 4c_3c_4\}^{\frac{1}{2}}].$$

If $c_1 + c_2 = 0$, it follows that $c_3 = c_4 = 0$. Hence, we secure the desired inequality by using $\frac{1}{2} c_1 C_{1,-1}$.

Part II.

§ 5. *Definition of Polarizable Inequality.*

The inequality

$$\sum_{ijkl}^{(12)} a_{ijkl} J_{ij} J_{kl} \xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ J^{LHP})$$

is said to be polarizable if it is true that

$$\sum_{ijkl}^{(12)} a_{ijkl}(J'_{ij}J''_{kl} + J''_{ij}J'_{kl})\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ J'^{LHP};\ J''^{LHP}). \tag{25}$$

It will be shown in this part of the paper that the first eight inequalities of (7) are polarizable, while the last four are not polarizable, even for the case $J = \check{J}$, or for the case ξ and η real. Also, it will be shown that these eight polarizable inequalities (4′) and (5′) form a fundamental set of polarizable inequalities (1°) when $J = \check{J}$, and (2°) when ξ and η are real.

§ 6. *Proof of the Polarized Forms.*

In § 2 we have

$$J'_{13}J''_{24}(\xi\eta + u\eta\xi)\overline{(\xi\eta + u\eta\xi)} \geqq 0 \qquad (\xi;\ \eta;\ u^{\mathfrak{A}};\ J';\ J'').$$

We may equally well have

$$J''_{13}J'_{24}(\xi\eta + u\eta\xi)\overline{(\xi\eta + u\eta\xi)} \geqq 0 \qquad (\xi;\ \eta;\ u^{\mathfrak{A}};\ J';\ J'').$$

The sum of these two inequalities is

$$\{(1+u\bar{u})(J'_{12}J''_{34} + J''_{12}J'_{34}) + (u+\bar{u})(J'_{14}J''_{32} + J''_{14}J'_{32})\}\xi\bar{\xi}\eta\bar{\eta} \geqq 0$$
$$(\xi;\ \eta;\ u^{\mathfrak{A}};\ J';\ J''). \tag{26}$$

This is the polarized form of (4). Similarly, by using $J'\check{J}''$ and $J''\check{J}'$ we obtain the polarized form of (5). Transformations* on ξ and η afford the polarized

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*Cf. Section 2.

forms of the remaining six inequalities of (4′) and (5′). It will be seen from conditions (30) and (34) that the inequalities (6′) are not polarizable even when J is self-transpose or ξ and η are real.

§ 7. *The Eight Polarizable Inequalities* (4′) *and* (5′) *Form a Fundamental Set of Polarizable Inequalities When* $J=\breve{J}$.

PROOF. Since $J'=\breve{J}'$ and $J''=\breve{J}''$, and the operand is $\xi\bar{\xi}\eta\bar{\eta}$, it follows that

$$\left.\begin{aligned} J'_{12}J''_{34}&=J'_{12}J''_{43}=J'_{21}J''_{34}=J'_{21}J''_{43}, \\ J'_{34}J''_{12}&=J'_{34}J''_{21}=J'_{43}J''_{12}=J'_{43}J''_{21}, \\ J'_{13}J''_{24}&=J'_{13}J''_{42}=J'_{31}J''_{24}=J'_{31}J''_{42}, \\ J'_{24}J''_{13}&=J'_{24}J''_{31}=J'_{42}J''_{13}=J'_{42}J''_{31}, \\ J'_{14}J''_{23}&=J'_{14}J''_{32}=J'_{41}J''_{23}=J'_{41}J''_{32}, \\ J'_{23}J''_{14}&=J'_{23}J''_{41}=J'_{32}J''_{14}=J'_{32}J''_{41}. \end{aligned}\right\} \tag{27}$$

The corresponding grouping of coefficients a_{ijkl} is

$$a_{1234}+a_{1243}+a_{3421}+a_{4321}, \quad a_{1324}+a_{1342}+a_{2431}+a_{4231}, \quad a_{1423}+a_{1432}+a_{2341}+a_{3241},$$

which have already been defined (9) as b_1, b_2, b_3, respectively.

The general inequality which we wish to build from the eight given inequalities (4′) and (5′) is

$$(b_1J_{12}J_{34}+b_2J_{13}J_{24}+b_3J_{14}J_{23})\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \quad (\xi;\ \eta;\ J=\breve{J}). \tag{10}$$

However, since this inequality is polarizable, it must be true that

$$\{b_1(J'_{12}J''_{34}+J'_{34}J''_{12})+b_2(J'_{13}J''_{24}+J'_{24}J''_{13})+b_3(J'_{14}J''_{23}+J'_{23}J''_{14})\}\xi\bar{\xi}\eta\bar{\eta}\geqq 0$$
$$(\xi;\ \eta;\ J'=\breve{J}';\ J''=\breve{J}''). \tag{28}$$

Using the operators

$$J'=\begin{pmatrix} j'_{11} & 0 \\ 0 & j'_{22}\end{pmatrix} \text{ and } J''=\begin{pmatrix} j''_{11} & 0 \\ 0 & j''_{22}\end{pmatrix},$$

we have as a special instance of (28),

$$\left.\begin{aligned} &b_1\left\{\begin{aligned} &(j'_{11}x_1\bar{x}_1+j'_{22}x_2\bar{x}_2)(j''_{11}y_1\bar{y}_1+j''_{22}y_2\bar{y}_2) \\ +&(j'_{11}y_1\bar{y}_1+j'_{22}y_2\bar{y}_2)(j''_{11}x_1\bar{x}_1+j''_{22}x_2\bar{x}_2)\end{aligned}\right\} \\ +&b_2\left\{\begin{aligned} &(j'_{11}x_1y_1+j'_{22}x_2y_2)(j''_{11}\bar{x}_1\bar{y}_1+j''_{22}\bar{x}_2\bar{y}_2) \\ +&(j'_{11}\bar{x}_1\bar{y}_1+j'_{22}\bar{x}_2\bar{y}_2)(j''_{11}x_1y_1+j''_{22}x_2y_2)\end{aligned}\right\} \\ +&b_3\left\{\begin{aligned} &(j'_{11}x_1\bar{y}_1+j'_{22}x_2\bar{y}_2)(j''_{11}\bar{x}_1y_1+j''_{22}\bar{x}_2y_2) \\ +&(j'_{11}\bar{x}_1y_1+j'_{22}\bar{x}_2y_2)(j''_{11}x_1\bar{y}_1+j''_{22}x_2\bar{y}_2)\end{aligned}\right\}\end{aligned}\right\}\geqq 0 \quad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ j'_{11}\geqq 0 & j'_{22}\geqq 0 \\ j''_{11}\geqq 0 & j''_{22}\geqq 0\end{pmatrix}. \tag{29}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22}, j''_{11}, j''_{22})$

$=(1, 0;\ 1, 0;\ 1, 0;\ 1, 0)$, $(1, i;\ 1, -i;\ 0, 1;\ 1, 0)$,

$(1, i;\ 1, i;\ 0, 1;\ 1, 0)$, $(-1, 1;\ 1, 1;\ 0, 1;\ 1, 0)$, show that we have as necessary conditions

$$b_1+b_2+b_3 \geqq 0, \quad b_1+b_2-b_3 \geqq 0, \quad b_1-b_2+b_3 \geqq 0, \quad b_1-b_2-b_3 \geqq 0. \tag{30}$$

Since all polarizable inequalities for the field $J=\breve{J}$ must satisfy condition (30), and since neither $B_{3,u}$ nor $B_{4,u}$ of (11) satisfies the last condition of (30), it is now seen, as was suggested in § 5, that neither $B_{3,u}$ nor $B_{4,u}$ is polarizable. It follows that the inequalities (6′) from which $B_{3,u}$ and $B_{4,u}$ were obtained do not satisfy the definition of polarizable inequality, as that definition demands that the polarized form be positive or zero for all J^{LHP}, of which we have $J=\breve{J}^{LHP}$ as an instance.

The eight inequalities (4′) and (5′) which were proved polarizable, reduce to two when $J=\breve{J}$, designated as $B_{1,u}$ and $B_{2,u}$ under (11).

The desired inequality (10) is shown to be the sum of positive or zero multiples of the fundamental polarizable inequalities $B_{1,u}$ and $B_{2,u}$ as follows:

When $b_3 \geqq 0$ and $b_1 \neq b_3$, the desired inequality has the form

$$\tfrac{1}{2} b_3 B_{1,1}+\tfrac{1}{2} d_1 B_{2,\, b_2/d_1},$$

where

$$d_1=b_1-b_3+\{(b_1-b_3)^2-b_2^2\}^{\frac{1}{2}}.$$

When $b_3<0$ and $b_1+b_3 \neq 0$, the desired inequality has the form

$$-\tfrac{1}{2} b_3 B_{1,-1}+\tfrac{1}{2} d_2 B_{2,\, b_2/d_2},$$

where

$$d_2=b_1+b_3+\{(b_1+b_3)^2-b_2^2\}^{\frac{1}{2}}.$$

When $b_1=b_3$, we use $\frac{1}{2} b_1 B_{1,1}$, since $b_2=0$. When $b_1=-b_3$, we use $\frac{1}{2} b_1 B_{1,-1}$, since $b_2=0$.

§ 8. *The Eight Polarizable Inequalities* (4′) *and* (5′) *Form a Fundamental Set When* ξ *and* η *Are Real.*

Proof. In this instance (ξ^R, η^R)

$$\left.\begin{aligned}
J'_{12}J''_{34}&=J'_{12}J''_{43}=J'_{21}J''_{34}=J'_{21}J''_{43},\\
J'_{34}J''_{21}&=J'_{43}J''_{21}=J'_{34}J''_{12}=J'_{43}J''_{12},\\
J'_{13}J''_{24}&=J'_{14}J''_{23}=J'_{24}J''_{13}=J'_{23}J''_{14},\\
J'_{13}J''_{42}&=J'_{14}J''_{32}=J'_{23}J''_{41}=J'_{24}J''_{31},\\
J'_{42}J''_{13}&=J'_{32}J''_{14}=J'_{41}J''_{23}=J'_{31}J''_{24},\\
J'_{32}J''_{41}&=J'_{42}J''_{31}=J'_{41}J''_{32}=J'_{31}J''_{42}.
\end{aligned}\right\} \tag{31}$$

The corresponding grouping of coefficients is that indicated in (15). Hence the inequality (16) is the general inequality which is to be expressed as the sum of positive or zero multiples of the known polarizable inequalities, $C_{1,u}$ and $C_{2,u}$, to which (4′) and (5′) reduce when ξ and η are real. The positive or zero multipliers will be chosen from necessary conditions on c_1, c_2, c_3, c_4. The conditions (22) and (23), which must be satisfied by the coefficients of all inequalities (16) must necessarily be satisfied by the coefficients of the polarizable inequality (16). In addition to these conditions (22) and (23), necessary conditions are obtained by using the fact that the inequality (16) is polarizable; *i. e.*

$$\begin{aligned}\{c_1(J'_{12}J''_{34}+J'_{34}J''_{12})+c_2(J'_{13}J''_{42}+J'_{42}J''_{13})+c_3(J'_{13}J''_{24}+J'_{24}J''_{13})\\ +c_4(J'_{32}J''_{41}+J'_{41}J''_{32})\}\xi\bar\xi\eta\bar\eta\geqq 0 \quad (\xi^R;\ \eta^R;\ J';\ J'').\end{aligned} \tag{32}$$

The use of

$$J'=\begin{pmatrix} j'_{11} & j'_{12}\\ -j'_{12} & j'_{22}\end{pmatrix},\quad J''=\begin{pmatrix} j''_{11} & j''_{12}\\ -j''_{12} & j''_{22}\end{pmatrix},$$

where j'_{12} and j''_{12} are pure imaginary, gives as a special case of (32):

$$\left.\begin{aligned} &c_1\{(j'_{11}x_1^2+j'_{22}x_2^2)(j''_{11}y_1^2+j''_{22}y_2^2)\\ &\qquad+(j'_{11}y_1^2+j'_{22}y_2^2)(j''_{11}x_1^2+j''_{22}x_2^2)\}\\ &+c_2\left[\begin{aligned}&\{j'_{11}x_1y_1+j'_{12}(x_1y_2-x_2y_1)+j'_{22}x_2y_2\}\\ &\qquad\{j''_{11}x_1y_1+j''_{12}(x_2y_1-x_1y_2)+j''_{22}x_2y_2\}\\ &+\{j'_{11}x_1y_1+j'_{12}(x_2y_1-x_1y_2)+j'_{22}x_2y_2\}\\ &\qquad\{j''_{11}x_1y_1+j''_{12}(x_1y_2-x_2y_1)+j''_{22}x_2y_2\}\end{aligned}\right]\\ &+2c_3\{j'_{11}x_1y_1+j'_{12}(x_1y_2-x_2y_1)+j'_{22}x_2y_2\}\\ &\qquad\{j''_{11}x_1y_1+j''_{12}(x_1y_2-x_2y_1)+j''_{22}x_2y_2\}\\ &+2c_4\{j'_{11}x_1y_1+j'_{12}(x_2y_1-x_1y_2)+j'_{22}x_2y_2\}\\ &\qquad\{j''_{11}x_1y_1+j''_{12}(x_2y_1-x_1y_2)+j''_{22}x_2y_2\}\end{aligned}\right\}\geqq 0\quad\begin{pmatrix} x_1^R & x_2^R\\ y_1^R & y_2^R\\ j'_{11}\geqq 0 & j'_{22}\geqq 0\\ j''_{11}\geqq 0 & j''_{22}\geqq 0\\ j'^I_{12} & j''^I_{12}\\ j'_{11}j'_{22}+j'^2_{12}\geqq 0 & \\ j''_{11}j''_{22}+j''^2_{12}\geqq 0 & \end{pmatrix}. \tag{33}$$

The special values

$$\begin{aligned}&(x_1,\ x_2;\ y_1,\ y_2;\ j'_{11},\ j'_{12},\ j'_{22};\ j''_{11},\ j''_{12},\ j''_{22})\\ &\quad=(1,0;\ 0,1;\ 1,i,1;\ 1,-i,1),\ (1,1;\ 1,-1;\ 0,0,1;\ 1,0,0)\end{aligned}$$

give the necessary conditions

$$\text{whence}\qquad \left.\begin{aligned} c_1-c_2+c_3+c_4\geqq 0,\quad c_1-c_2-c_3-c_4\geqq 0,\\ c_1-c_2\geqq 0.\end{aligned}\right\} \tag{34}$$

Again, it is convenient to use, for J, a matrix whose determinant is zero. We choose

$$J'=\begin{pmatrix} j'_1\bar{j}'_1 & j'_1\bar{j}'_2\\ j'_2\bar{j}'_1 & j'_2\bar{j}'_2\end{pmatrix},\quad J''=\begin{pmatrix} j''_1\bar{j}''_1 & j''_1\bar{j}''_2\\ j''_2\bar{j}''_1 & j''_2\bar{j}''_2\end{pmatrix},$$

where $j_2'=e'ij_1'$ and $j_2''=e''ij_1''$ with e' and e'' real. Then (33) becomes a quadratic in e' and also in e''. As an instance we set

$(x_1, x_2; y_1, y_2; j_{11}', j_{12}', j_{22}'; j_{11}'', j_{12}'', j_{22}'')=(1, 1; 1, -1; 1, -e'i, e'^2; 1, -e''i, e''^2)$,

and (33) gives the quadratic expression $P(e', e'')$ in e' and e'' which must be positive or zero for every real e' and e'', and for which, therefore, the discriminant must be positive or zero.

$$\left.\begin{aligned} P(e', e'')\equiv 2c_1(1+e'^2)(1+e''^2) & \\ +c_2\{(1+2e'i-e'^2)(1-2e''i-e''^2) & \\ +(1-2e'i-e'^2)(1+2e''i-e''^2)\} & \\ +2c_3(1+2e'i-e'^2)(1+2e''i-e''^2) & \\ +2c_4(1-2e'i-e'^2)(1-2e''i-e''^2) & \end{aligned}\right\} \geqq 0 \quad (e'^R, e''^R).$$

The discriminant of $P(e', 0)$ gives the condition

$$c_1^2-(c_2+c_3+c_4)^2+(c_3-c_4)^2\geqq 0;$$

the discriminant of $P(e', 1)$ gives

$$c_1^2-(c_2-c_3-c_4)^2+(c_3-c_4)^2\geqq 0,$$

whence

$$c_1^2-c_2^2-4c_3c_4\geqq 0,$$

and, thus,

$$c_1^2+c_2^2-4c_3c_4\geqq 0. \tag{35}$$

The discriminant of $P(e' e'')$ as to e'' is a homogeneous quadratic expression in $(1-e'^2, 2e')$, and since for e' real, even between the values -1 and $+1$, $2e'/(1-e'^2)$ takes every real value, the discriminant of this homogeneous expression must be positive or zero. Hence, we have the condition

$$(c_1^2+c_2^2-4c_3c_4)^2-4c_1^2c_2^2\geqq 0,$$

and, therefore, by virtue of (35)

$$c_1^2+c_2^2-4c_3c_4\geqq \pm 2c_1c_2$$

or

$$(c_1\pm c_2)^2-4c_3c_4\geqq 0. \tag{36}$$

It can be seen from (34) that $C_{3,u}$ of (17) is not polarizable. By use of (22), (23), (34) and (36) we build all polarizable inequalities in this field $(\xi^R; \eta^R)$ as the sum of positive or zero multiples of the polarizable inequalities $C_{1,u}$ and $C_{2,u}$ of (17) in the following ways:

If $c_2\geqq 0$ and $c_1\neq c_2$, the desired inequality may be expressed as

$$\tfrac{1}{2}c_2C_{1,1}+d_1C_{2,c_4/d_1},$$

where

$$d_1=\tfrac{1}{2}[c_1-c_2+\{(c_1-c_2)^2-4c_3c_4\}^{\frac{1}{2}}].$$

If $c_2<0$ and $c_1+c_2\neq 0$, the desired inequality has the form

$$-\tfrac{1}{2}\,c_2C_{1,\,-1}+d_2C_{2,\,c_4/d_2},$$

where

$$d_2=\tfrac{1}{2}[c_1+c_2+\{(c_1+c_2)^2-4c_3c_4\}^{\frac{1}{2}}].$$

If $c_2\geqq 0$ and $c_1=c_2$, it follows from (36) that $c_3=c_4=0$, and we may express the inequality as

$$\tfrac{1}{2}\,c_1C_{1,1}.$$

If $c_2<0$ and $c_1+c_2=0$, it follows from (36) that $c_3=c_4=0$, and we may use as the desired inequality

$$\tfrac{1}{2}\,c_1C_{1,\,-1}.$$

PART III.

§ 9. *Bilinear Inequalities.*

Somewhat related to the preceding problem is that of determining all bilinear inequalities of the form

$$\sum_{ijkl}^{(24)} z_{ijkl}J'_{ij}J''_{kl}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;J';J'').$$

It is the purpose of this portion of the paper to exhibit sixteen such inequalities and prove that they form a fundamental set of bilinear inequalities for the cases $(1^\circ)\,J=\breve{J}$ and (2°) ξ and η real functions.

We have, in § 2, as a special instance of the theorem proved by E. H. Moore, the inequality

$$J'_{13}J''_{24}(\xi\eta+u\eta\xi)\,\overline{(\xi\eta+u\eta\xi)}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J''),$$

which is also written

$$(J'_{12}J''_{34}+\bar{u}J'_{14}J''_{32}+uJ'_{32}J''_{14}+u\bar{u}J'_{34}J''_{12})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J'').$$

By using J' instead of J', J'' instead of J'', and J' and J'' instead of J' and J'' we have also

$$(J'_{21}J''_{34}+\bar{u}J'_{41}J''_{32}+uJ'_{23}J''_{14}+u\bar{u}J'_{43}J''_{12})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J''),$$
$$(J'_{12}J''_{43}+\bar{u}J'_{14}J''_{23}+uJ'_{32}J''_{41}+u\bar{u}J'_{34}J''_{21})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J''),$$
$$(J'_{21}J''_{43}+\bar{u}J'_{41}J''_{23}+uJ'_{23}J''_{41}+u\bar{u}J'_{43}J''_{21})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J'').$$

Four more inequalities may be written by interchange of J' and J'', the eight being $Z_{1,u}$ to $Z_{8,u}$ inclusive, tabulated according to their coefficients z_{ijkl} in (37). By replacing ξ by $\bar{\xi}$ the inequalities $Z_{9,u},\ldots,Z_{16,u}$ of (37) are obtained from inequalities $Z_{1,u},\ldots,Z_{8,u}$.

	z_{1234}	z_{1243}	z_{1324}	z_{1342}	z_{1423}	z_{1432}	z_{2341}	z_{2431}	z_{3241}	z_{3421}	z_{4231}	z_{4321}	z_{3412}	z_{4312}	z_{2413}	z_{4213}	z_{2314}	z_{3214}	z_{4123}	z_{3124}	z_{4132}	z_{2134}	z_{3142}	z_{2143}
$Z_{1,u}$	1	0	0	0	0	$\bar{u}$	0	0	0	0	0	0	$u\bar{u}$	0	0	0	0	u	0	0	0	0	0	0
$Z_{2,u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	0	u	0	0	0	$\bar{u}$	1	0	0
$Z_{3,u}$	0	1	0	0	$\bar{u}$	0	0	0	u	$u\bar{u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{4,u}$	0	0	0	0	0	0	u	0	0	0	0	$u\bar{u}$	0	0	0	0	0	0	$\bar{u}$	0	0	0	0	1
$Z_{5,u}$	$u\bar{u}$	0	0	0	0	u	0	0	0	0	0	0	1	0	0	0	0	$\bar{u}$	0	0	0	0	0	0
$Z_{6,u}$	0	$u\bar{u}$	0	0	u	0	0	0	$\bar{u}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{7,u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	$\bar{u}$	0	0	0	u	$u\bar{u}$	0	0
$Z_{8,u}$	0	0	0	0	0	0	$\bar{u}$	0	0	0	0	1	0	0	0	0	0	0	u	0	0	0	0	$u\bar{u}$
$Z_{9,u}$	0	0	0	0	0	0	0	$\bar{u}$	0	$u\bar{u}$	0	0	0	0	0	0	0	0	0	u	0	1	0	0
$Z_{10,u}$	0	1	0	$\bar{u}$	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	0
$Z_{11,u}$	0	0	0	0	0	0	0	u	0	1	0	0	0	0	0	0	0	0	0	$\bar{u}$	0	$u\bar{u}$	0	0
$Z_{12,u}$	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	0	0	1	0	$\bar{u}$	0	0	0	0	0	0	0	0
$Z_{13,u}$	1	0	u	0	0	0	0	0	0	0	$\bar{u}$	$u\bar{u}$	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{14,u}$	0	0	0	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	$\bar{u}$	1
$Z_{15,u}$	$u\bar{u}$	0	$\bar{u}$	0	0	0	0	0	0	0	u	1	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{16,u}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	$\bar{u}$	0	0	0	0	0	0	0	u	$u\bar{u}$

(37)

§ 10. *The Sixteen Inequalities* (37) *Form a Fundamental Set of Bilinear Inequalities When* $J'=\breve{J}'$ *and* $J''=\breve{J}''$.

PROOF. In this field, $J'=\breve{J}'$ and $J''=\breve{J}''$, equations (27) are valid, and they indicate the new coefficients,

$$\left.\begin{aligned} w_1&=z_{1234}+z_{1243}+z_{2134}+z_{2143}, & w_2&=z_{1324}+z_{1342}+z_{3124}+z_{3142}, \\ w_3&=z_{1423}+z_{1432}+z_{4123}+z_{4132}, & w_4&=z_{2341}+z_{3241}+z_{2314}+z_{3214}, \\ w_5&=z_{2431}+z_{4231}+z_{2413}+z_{4213}, & w_6&=z_{3421}+z_{4321}+z_{3412}+z_{4312}, \end{aligned}\right\} \tag{38}$$

Accordingly the general inequality which we desire to build is

$$(w_1J'_{12}J''_{34}+w_2J'_{13}J''_{24}+w_3J'_{14}J''_{23}+w_4J'_{23}J''_{41}+w_5J'_{24}J''_{31}+w_6J'_{34}J''_{21})\xi\bar{\xi}\eta\bar{\eta}\geqq 0$$
$$(\xi;\,\eta;\,J'=\breve{J}';\,J''=\breve{J}''). \tag{39}$$

The known inequalities from which (39) is to be built in this field, $J'=\breve{J}'$ and $J''=\breve{J}''$, are the four inequalities to which (37) reduce on account of (38), and are tabulated in (40) according to their coefficients $w_1, \ldots, w_6$.

	w_1	w_2	w_3	w_4	w_5	w_6
$W_{1,u}$	1	0	$\bar{u}$	u	0	$u\bar{u}$
$W_{2,u}$	$u\bar{u}$	0	u	$\bar{u}$	0	1
$W_{3,u}$	1	u	0	0	$\bar{u}$	$u\bar{u}$
$W_{4,u}$	$u\bar{u}$	$\bar{u}$	0	0	u	1

(40)

As before, it is possible to secure the necessary conditions from the form corresponding to binary operators. Use of

$$J'=\begin{pmatrix} j'_{11} & 0 \\ 0 & j'_{22} \end{pmatrix}, \quad J''=\begin{pmatrix} j''_{11} & 0 \\ 0 & j''_{22} \end{pmatrix},$$

gives as an instance of (39):

$$\left.\begin{aligned} & w_1(j'_{11}x_1\bar{x}_1+j'_{22}x_2\bar{x}_2)(j''_{11}y_1\bar{y}_1+j''_{22}y_2\bar{y}_2) \\ & +w_2(j'_{11}x_1y_1+j'_{22}x_2y_2)(j''_{11}\bar{x}_1\bar{y}_1+j''_{22}\bar{x}_2\bar{y}_2) \\ & +w_3(j'_{11}x_1\bar{y}_1+j'_{22}x_2\bar{y}_2)(j''_{11}\bar{x}_1y_1+j''_{22}\bar{x}_2y_2) \\ & +w_4(j'_{11}\bar{x}_1y_1+j'_{22}\bar{x}_2y_2)(j''_{11}x_1\bar{y}_1+j''_{22}x_2\bar{y}_2) \\ & +w_5(j'_{11}\bar{x}_1\bar{y}_1+j'_{22}\bar{x}_2\bar{y}_2)(j''_{11}x_1y_1+j''_{22}x_2y_2) \\ & +w_6(j'_{11}y_1\bar{y}_1+j'_{22}y_2\bar{y}_2)(j''_{11}x_1\bar{x}_1+j''_{22}x_2\bar{x}_2) \end{aligned}\right\} \geqq 0 \quad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ j'_{11}\geqq 0 & j'_{22}\geqq 0 \\ j''_{11}\geqq 0 & j''_{22}\geqq 0 \end{pmatrix}. \tag{41}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22})$
$= (0, 1;\ 1, 0;\ 0, 1;\ 1, 0),\quad (1, 0;\ 0, 1;\ 0, 1;\ 1, 0)$

give as necessary conditions,

$$w_1\geqq 0, \quad w_6\geqq 0. \tag{42}$$

The case $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22}) = (1, k_1+ik_2;\ 1, k_1+ik_2;\ 0, 1;\ 1, 0)$, where k_1 and k_2 are real, gives as a special case of (41),

$$\begin{aligned}(w_1+w_3+w_4+w_6)(k_1^2+k_2^2)+(w_2+w_5)(k_1^2-k_2^2) \\ +2i(w_2-w_5)k_1k_2\geqq 0 \quad (k_1^R, k_2^R).\end{aligned} \tag{43}$$

In (43) the values $(k_1, k_2) = (1, 0), (0, 1)$ give the necessary conditions

$$w_1+w_2+w_3+w_4+w_5+w_6\geqq 0, \quad w_1-w_2+w_3+w_4-w_5+w_6\geqq 0. \tag{44}$$

Hence, from (42) and (44),

$$(w_2+w_5)^R, \quad (w_3+w_4)^R,$$

while from (43) it is now seen that w_2-w_5 is pure imaginary, and, therefore,

$$w_2=\bar{w}_5. \tag{45}$$

Similarly, the case

$$(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22}) = (1, k_1-ik_2;\ 1, k_1+ik_2;\ 0, 1;\ 1, 0)$$

yields the conditions

$$w_1+w_2-w_3-w_4+w_5+w_6\geqq 0, \quad (w_3-w_4)^I,$$

whence

$$w_3=\bar{w}_4. \tag{46}$$

The case $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22}) = (1, k_1+ik_2;\ 1, l_1+il_2;\ 0, 1;\ 1, 0)$, where k_1, k_2, l_1, l_2 are real, gives as an instance of (41),

$$\left.\begin{array}{l} w_6 l_1^2 + l_1\{(w_2+w_3+w_4+w_5)k_1 + i(w_2+w_3-w_4-w_5)k_2\} \\ + w_6 l_2^2 + l_2\{i(w_2-w_3+w_4-w_5)k_1 - (w_2-w_3-w_4+w_5)k_2\} \\ + w_1(k_1^2+k_2^2) \end{array}\right\} \geqq 0 \quad \begin{pmatrix} k_1^R & k_2^R \\ l_1^R & l_2^R \end{pmatrix}. \tag{47}$$

Inequality (47) shows that $w_6=0$ demands:

$$\left.\begin{array}{ll} w_2+w_3+w_4+w_5=0, & w_2+w_3-w_4-w_5=0, \\ w_2-w_3+w_4-w_5=0, & w_2-w_3-w_4+w_5=0. \end{array}\right\} \tag{48}$$

For if not, suppose any one of (48) is not zero, say

$$w_2+w_3+w_4+w_5 \neq 0.$$

Then by taking $(k_1, k_2, l_2) = (1, 0, 0)$ we have,

$$l_1(w_2+w_3+w_4+w_5) + w_1 \geqq 0, \quad (l_1^R),$$

which obviously is not true. Similarly, the other equations of (48) are proved when $w_6=0$. Also, equations (48) follow when $w_1=0$. Hence,

$$w_1=0 \text{ or } w_6=0 \text{ demands } w_2=w_3=w_4=w_5=0. \tag{49}$$

Thus for the remaining cases we may suppose $w_1 \neq 0$ and $w_6 \neq 0$ and omit w_1 or w_6 as a factor in any expression which is positive or zero, each of whose terms contains as factor one or more of the coefficients $w_2, \ldots, w_5$, or $w_1 w_6$.

When $l_1=0$, (47) becomes a quadratic in l_2, whose discriminant is a quadratic in k_1 and k_2, viz.:

$$\left.\begin{array}{l} \{4w_1w_6 + (w_2-w_3+w_4-w_5)^2\}k_1^2 + \{4w_1w_6 - (w_2-w_3-w_4+w_5)^2\}k_2^2 \\ \qquad + 2i(w_2-w_3+w_4-w_5)(w_2-w_3-w_4+w_5)k_1k_2 \end{array}\right\} \geqq 0 \quad (k_1^R k_2^R).$$

The discriminant of this quadratic in k_1 and k_2 gives the condition

$$16w_1w_6\{w_1w_6 + (w_2-w_3)(w_4-w_5)\} \geqq 0,$$

which by (49) gives as a condition always holding

$$w_1w_6 + (w_2-w_3)(w_4-w_5) \geqq 0. \tag{50}$$

Similarly, $l_2=0$ reduces (47) to a quadratic in l_1 whose discriminant is a quadratic in k_1 and k_2, which gives the necessary condition

$$w_1w_6 - (w_2+w_3)(w_4+w_5) \geqq 0. \tag{51}$$

Conditions (50) and (51) combine to give

$$\left.\begin{array}{ll} & w_1w_6 - w_3w_4 - w_2w_5 \geqq 0, \\ \text{whence} & w_1w_6 + w_3w_4 - w_2w_5 \geqq 0, \\ \text{and} & w_1w_6 - w_3w_4 + w_2w_5 \geqq 0. \end{array}\right\} \tag{52}$$

The discriminant for (47) as a quadratic in l_1 is a quadratic in l_2, viz.:

$$\left.\begin{aligned} 4w_6^2l_2^2+4w_6l_2\{i(w_2-w_3+w_4-w_5)k_1-(w_2-w_3-w_4+w_5)k_2\} \\ +\{4w_1w_6-(w_2+w_3+w_4+w_5)^2\}k_1^2 \\ +\{4w_1w_6+(w_2+w_3-w_4-w_5)^2\}k_2^2 \\ -2i(w_2+w_3+w_4+w_5)(w_2+w_3-w_4-w_5)k_1k_2 \end{aligned}\right\}\geqq 0 \quad \begin{pmatrix} k_1^R & k_2^R \\ & l_2^R \end{pmatrix},$$

whose discriminant, aside from a factor $16w_6^2$, reduces to

$$\left.\begin{aligned} \{w_1w_6-(w_2+w_4)(w_3+w_5)\}k_1^2+\{w_1w_6+(w_2-w_4)(w_3-w_5)\}k_2^2 \\ -2i(w_2w_3-w_4w_5)k_1k_2 \end{aligned}\right\}\geqq 0 \quad (k_1^R,\ k_2^R).$$

Hence,

$$\{w_1w_6-(w_2+w_4)(w_3+w_5)\}\{w_1w_6+(w_2-w_4)(w_3-w_5)\}+(w_2w_3-w_4w_5)^2\geqq 0,$$

or

$$(w_1w_6-w_3w_4-w_2w_5)^2-4w_2w_3w_4w_5\geqq 0. \tag{53}$$

Apart from exceptional cases, we have the desired general inequality (39) in the form

$$\frac{d_1}{2w_6}W_{1,\,2w_6w_4/d_1}+\frac{d_2}{2w_1}W_{4,\,2w_1w_5/d_2},$$

where

$$d_1=w_1w_6-w_2w_5+w_3w_4+\{(w_1w_6-w_2w_5-w_3w_4)^2-4w_2w_3w_4w_5\}^{\frac{1}{2}},$$
$$d_2=w_1w_6+w_2w_5-w_3w_4+\{(w_1w_6-w_2w_5-w_3w_4)^2-4w_2w_3w_4w_5\}^{\frac{1}{2}}.$$

The exceptional cases in which the above method will not be permissible are: (a) When $w_1w_6+w_2w_5-w_3w_4=0$; (b) when $w_1w_6-w_2w_5+w_3w_4=0$; (c) when $w_1w_6-w_2w_5-w_3w_4=0$; (d) when $w_1=0$; (e) when $w_6=0$.

In case $w_1w_6+w_2w_5-w_3w_4=0$ it follows that

$$w_2=w_5=0, \quad w_1w_6-w_3w_4=0,$$

and the desired inequality is

$$w_1W_{1,\,w_4/w_1}.$$

Similarly, $w_1w_6-w_2w_5+w_3w_4=0$ demands

$$w_3=w_4=0, \quad w_1w_6-w_2w_5=0;$$

and the desired inequality is

$$w_6W_{4,\,w_5/w_6}.$$

In case $w_1w_6-w_2w_5-w_3w_4=0$, it is seen by (53) that

$$w_2=w_5=0 \text{ or } w_3=w_4=0.$$

In case $w_2=w_5=0$, the desired inequality is

$$w_1W_{1,\,w_4/w_1}.$$

In case $w_3=w_4=0$, the desired inequality is

$$w_6W_{4,\,w_5/w_6}.$$

When $w_1=0$ or $w_6=0$, we have $w_2=w_3=w_4=w_5=0$, by (49), hence the desired inequality is

$$w_1 W_{1,0}+w_6 W_{4,0}.$$

§ 11. *The Sixteen Bilinear Inequalities* (37) *Form a Fundamental Set When ξ and η Are Real.*

PROOF. Equations (31) which are valid in this case $(\xi^R;\ \eta^R)$ indicate the coefficients,

$$\left.\begin{aligned} v_1&=z_{1234}+z_{1243}+z_{2134}+z_{2143}, & v_2&=z_{1324}+z_{1423}+z_{2413}+z_{2314}, \\ v_3&=z_{1342}+z_{1432}+z_{2341}+z_{2431}, & v_4&=z_{4213}+z_{3214}+z_{4123}+z_{3124}, \\ v_5&=z_{3241}+z_{4231}+z_{4132}+z_{3142}, & v_6&=z_{3421}+z_{4321}+z_{3412}+z_{4312}, \end{aligned}\right\} \tag{54}$$

Further, let

$$v=v_1+v_2+v_3+v_4+v_5+v_6.$$

The general inequality desired is

$$(v_1 J'_{12}J''_{34}+v_2 J'_{13}J''_{24}+v_3 J'_{13}J''_{42}+v_4 J'_{31}J''_{24}+v_5 J'_{31}J''_{42}+v_6 J'_{34}J''_{12})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^R;\ \eta^R;\ J';\ J''). \tag{55}$$

When ξ and η are real, the given inequalities (37) reduce, in accordance with (54), to four inequalities (56).

	v_1	v_2	v_3	v_4	v_5	v_6
$V_{1,u}$	1	0	$\bar{u}$	u	0	$u\bar{u}$
$V_{2,u}$	1	u	0	0	$\bar{u}$	$u\bar{u}$
$V_{3,u}$	$u\bar{u}$	0	$\bar{u}$	u	0	1
$V_{4,u}$	$u\bar{u}$	u	0	0	$\bar{u}$	1

(56)

Binary operators give necessary and sufficient conditions on $v_1,\ \ldots,\ v_6$.

$$J'=\begin{pmatrix} j'_{11} & j'_{12} \\ j'_{21} & j'_{22}\end{pmatrix},\quad J''=\begin{pmatrix} j''_{11} & j''_{12} \\ j''_{21} & j''_{22}\end{pmatrix}$$

give, as an instance of (55),

$$\left.\begin{aligned} & v_1\{j'_{11}x_1^2+(j'_{12}+j'_{21})x_1x_2+j'_{22}x_2^2\} \\ &\qquad\qquad \{j''_{11}y_1^2+(j''_{12}+j''_{21})y_1y_2+j''_{22}y_2^2\} \\ +\,& v_6\{j'_{11}y_1^2+(j'_{12}+j'_{21})y_1y_2+j'_{22}y_2^2\} \\ &\qquad\qquad \{j''_{11}x_1^2+(j''_{12}+j''_{21})x_1x_2+j''_{22}x_2^2\} \\ +\,& v_2\{j'_{11}x_1y_1+j'_{12}x_1y_2+j'_{21}x_2y_1+j'_{22}x_2y_2\} \\ &\qquad\qquad \{j''_{11}x_1y_1+j''_{12}x_1y_2+j''_{21}x_2y_1+j''_{22}x_2y_2\} \\ +\,& v_3\{j'_{11}x_1y_1+j'_{12}x_1y_2+j'_{21}x_2y_1+j'_{22}x_2y_2\} \\ &\qquad\qquad \{j''_{11}x_1y_1+j''_{12}x_2y_1+j''_{21}x_1y_2+j''_{22}x_2y_2\} \\ +\,& v_4\{j'_{11}x_1y_1+j'_{12}x_2y_1+j'_{21}x_1y_2+j'_{22}x_2y_2\} \\ &\qquad\qquad \{j''_{11}x_1y_1+j''_{12}x_1y_2+j''_{21}x_2y_1+j''_{22}x_2y_2\} \\ +\,& v_5\{j'_{11}x_1y_1+j'_{12}x_2y_1+j'_{21}x_1y_2+j'_{22}x_2y_2\} \\ &\qquad\qquad \{j''_{11}x_1y_1+j''_{12}x_2y_1+j''_{21}x_1y_2+j''_{22}x_2y_2\} \end{aligned}\right\}\geqq 0 \quad \begin{pmatrix} x_1^R & x_2^R \\ y_1^R & y_2^R \\ j'_{11}\geqq 0 & j'_{22}\geqq 0 \\ j''_{11}\geqq 0 & j''_{22}\geqq 0 \\ j'_{21}=\bar{j}'_{12} & \\ j''_{21}=\bar{j}''_{12} & \\ j'_{11}j'_{22}-j'_{12}j'_{21}\geqq 0 & \\ j''_{11}j''_{22}-j''_{12}j''_{21}\geqq 0 & \end{pmatrix}. \tag{57}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{12}, j'_{22};\ j''_{11}, j''_{12}, j''_{22}) = (1, 0;\ 0, 1;\ 1, 0, 0;\ 0, 0, 1)$, $(0, 1;\ 1, 0;\ 1, 0, 0;\ 0, 0, 1)$, $(1, 0;\ 1, 0;\ 1, 0, 0;\ 1, 0, 0)$, $(1, 0;\ 0, 1;\ 1, i, 1;\ 1, i, 1)$, $(1, 0;\ 0, 1;\ 1, i, 1;\ 1, -i, 1)$, $(1, 1;\ 1, -1;\ 0, 0, 1;\ 1, 0, 0)$, give as necessary conditions

$$\left.\begin{array}{ll} v_1 \geqq 0, \quad v_6 \geqq 0, & \\ v_1+v_2+v_3+v_4+v_5+v_6 \geqq 0, & v_1-v_2+v_3+v_4-v_5+v_6 \geqq 0, \\ v_1+v_2-v_3-v_4+v_5+v_6 \geqq 0, & v_1-v_2-v_3-v_4-v_5+v_6 \geqq 0, \end{array}\right\} \tag{58}$$

whence

$$(v_2+v_5)^R, \quad (v_3+v_4)^R.$$

The cases $(0, 1;\ 1, 0;\ 1, e', e'^2;\ 1, e'', e''^2)$, $(1, 0;\ 0, 1;\ 1, e', e'^2;\ 1, -e''i, e''^2)$, $(0, 1;\ 1, 0;\ 1, -e'i, e'^2;\ 1, e'', e''^2)$, $(1, 0;\ 0, 1;\ 1, -e'i, e'^2;\ 1, -e''i, e''^2)$ give the four quadratic forms

$$\left.\begin{array}{l} v_1e'^2 + (v_2+v_3+v_4+v_5)e'e'' + v_6e''^2 \geqq 0, \\ v_1e''^2 + (v_3+v_5-v_2-v_4)e'e''i + v_6e'^2 \geqq 0, \\ v_1e'^2 + (v_2+v_3-v_4-v_5)e'e''i + v_6e''^2 \geqq 0, \\ v_1e''^2 + (v_3+v_4-v_2-v_5)e'e'' + v_6e'^2 \geqq 0, \end{array}\right\} \tag{59}$$

whose discriminants give the necessary conditions,

$$\left.\begin{array}{ll} 4v_1v_6-(v_2+v_3+v_4+v_5)^2 \geqq 0, & 4v_1v_6+(v_3+v_5-v_2-v_4)^2 \geqq 0, \\ 4v_1v_6+(v_2+v_3-v_4-v_5)^2 \geqq 0, & 4v_1v_6-(v_3+v_4-v_2-v_5)^2 \geqq 0. \end{array}\right\} \tag{60}$$

From (59) it is seen also that v_2-v_5 and v_3-v_4 are pure imaginary, which with (58) gives the conditions,

$$v_2=\bar{v}_5, \quad v_3=\bar{v}_4 \tag{61}$$

Using the values

$$(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{12}, j'_{22};\ j''_{11}, j''_{12}, j''_{22}) = (1, 1;\ -1, 1;\ 0, 0, 1;\ 1, -e''i, e''^2),$$

the inequality (57) becomes

$$ve''^2-2e''i(v_2-v_3+v_4-v_5)+(v_1+v_6-v_2-v_3-v_4-v_5) \geqq 0 \quad (e''^R),$$

whence, since the discriminant of this quadratic in e'' must be positive or zero,

$$(v_1+v_6)^2-4(v_2+v_4)(v_3+v_5) \geqq 0. \tag{62}$$

The values $(1, 1;\ -1, 1;\ 1, -e'i, e'^2;\ 1, -e''i, e''^2)$ give:

$$\left.\begin{array}{l} e''^2\{ve'^2-2(v_2+v_3-v_4-v_5)e'i+(v_1+v_6-v_2-v_3-v_4-v_5)\} \\ \quad +2e''\{-2(v_2-v_3-v_4+v_5)e'+(v_2-v_3+v_4-v_5)i(1-e'^2)\} \\ \quad +\{(v_1+v_6-v_2-v_3-v_4-v_5)e'^2+2(v_2+v_3-v_4-v_5)e'i+v\} \end{array}\right\} \geqq 0 \quad (e'^R, e''^R),$$

whose discriminant may be written as a homogeneous quadratic expression in $(1-e'^2, 2e')$. Since for e' real, even between the values -1, and $+1$,

$2e'/(1-e'^2)$ takes every real value, the coefficient of $4e'^2$ and the discriminant of this homogeneous quadratic expression must be positive or zero. Hence,

$$\left.\begin{aligned} &(v_1+v_6)^2-4(v_2-v_4)(v_5-v_3)\geqq 0,\\ &\{(v_1+v_6)^2-4(v_2v_5+v_3v_4)\}^2-64v_2v_3v_4v_5\geqq 0.\end{aligned}\right\} \tag{63}$$

From (62) and (63) it follows that

$$(v_1+v_6)^2-4(v_2v_5+v_3v_4)\geqq 0,$$

whence

$$(v_1+v_6)^2-4(v_2v_5-v_3v_4)\geqq 0,$$

and

$$(v_1+v_6)^2\geqq 4v_3v_4,$$

and by (58)

$$v_1+v_6\geqq \pm 2(v_3v_4)^{\frac{1}{2}}. \tag{64}$$

The second condition of (63) can be written

$$\{(v_1+v_6)^2-4(v_2v_5-v_3v_4)\}^2-16v_3v_4(v_1+v_6)^2\geqq 0,$$

and hence by (64),

$$(v_1+v_6)^2-4(v_2v_5-v_3v_4)\geqq \pm 4(v_1+v_6)(v_3v_4)^{\frac{1}{2}},$$

and

$$\{v_1+v_6\pm 2(v_3v_4)^{\frac{1}{2}}\}^2\geqq 4v_2v_5,$$

giving, by (64),

$$v_1+v_6\pm 2(v_3v_4)^{\frac{1}{2}}\geqq \pm 2(v_2v_5)^{\frac{1}{2}}. \tag{65}$$

In building the general inequality (55) as the sum of positive or zero multiples of the fundamental inequalities (56), it is desirable first to build those for which $v_1=v_6=\frac{1}{2}(v_1+v_6)\neq 0$. For such, aside from exceptional cases, the general inequality has the form

$$\tfrac{1}{2}(v_3v_4)^{\frac{1}{2}}(V_{1,\,(v_4/v_3)^{\frac{1}{2}}}+V_{3,\,(v_4/v_3)^{\frac{1}{2}}})+\tfrac{1}{4}d(V_{2,\,2v_2/d}+V_{4,\,2v_2/d}),$$

where

$$d=v_1+v_6-2(v_3v_4)^{\frac{1}{2}}+[\{v_1+v_6-2(v_3v_4)^{\frac{1}{2}}\}^2-4v_2v_5]^{\frac{1}{2}}.$$

When $v_1+v_6-2(v_3v_4)^{\frac{1}{2}}=0$, it follows from (65) that $v_2=v_5=0$, hence the desired inequality may be expressed as

$$\tfrac{1}{4}(v_1+v_6)(V_{1,\,2v_4/(v_1+v_6)}+V_{3,\,2v_4/(v_1+v_6)}).$$

In case $v_1\neq v_6$, the desired inequality is secured by adding to that already built

$$(v_1-v_6)V_{1,0} \quad\text{or}\quad (v_6-v_1)V_{3,0},$$

according as v_1 is greater than or less than v_6.

If $v_1=0$, or $v_6=0$, then, by (60), $v_2=v_3=v_4=v_5=0$, hence the desired inequality has the form

$$v_1V_{1,0}+v_6V_{3,0}.$$

All portions of this paper which refer to ξ^R, η^R; namely, § 4, § 8, and § 11, are valid for ξ and η pure imaginary, as an inequality for ξ and η pure imaginary reduces at once to the same inequality for ξ and η real.

On the three corresponding larger problems, retaining general operator and general operand, the writer has made considerable progress. In each case the nature of the coefficients a_{ijkl} and z_{ijkl} has been determined, and many necessary conditions on the coefficients have been secured, by use of binary, ternary, quaternary and quinary operators. As to the nature of the coefficients it has been found that in the problems corresponding to those of Part I and Part II,

$$\begin{array}{cccc} a_{1234} \geqq 0, & a_{1243} \geqq 0, & a_{3421} \geqq 0, & a_{4321} \geqq 0, \\ a_{1342}{}^R, & a_{1432}{}^R, & a_{2341}{}^R, & a_{2431}{}^R, \\ & a_{1324} = \overline{a_{4231}}, & a_{1423} = \overline{a_{3241}}, & \end{array}$$

while in the problem on bilinear forms,

$$\begin{array}{cccc} z_{1234} \geqq 0, & z_{1243} \geqq 0, & z_{3421} \geqq 0, & z_{4321} \geqq 0, \\ z_{3412} \geqq 0, & z_{4312} \geqq 0, & z_{2134} \geqq 0, & z_{2143} \geqq 0, \\ z_{1324} = \overline{z_{4231}}, & z_{1342} = \overline{z_{4213}}, & z_{1423} = \overline{z_{3241}}, & z_{1432} = \overline{z_{3214}}, \\ z_{2341} = \overline{z_{4123}}, & z_{2431} = \overline{z_{3124}}, & z_{2413} = \overline{z_{3142}}, & z_{2314} = \overline{z_{4132}}. \end{array}$$

VITA.

Mary Evelyn Wells was born August 20, 1881, in LeRaysville, Pa. She graduated from the High School in Naugatuck, Conn., in 1900, and entered Mount Holyoke College in the same year, from which college she received the degree of Bachelor of Arts in 1904. In 1905 she entered the University of Chicago for two years of graduate work, where she received the degree of Master of Science in June, 1907. After teaching five years in Mount Holyoke College she returned to the University of Chicago and studied there during 1912 to 1914. In the University of Chicago she studied under Professors E. H. Moore, H. Maschke, L. E. Dickson, H. E. Slaught, A. C. Lunn, and E. J Wilczynski in Mathematics, and under Professors F. R. Moulton and W. D. MacMillan in Astronomy. To all of these professors she owes much, and wishes to take this opportunity to express her sincere gratitude; and most especially does she appreciate the help and inspiration given by Professor Moore both in class and in research.