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A GENERAL IMPLICIT FUNCTION THEOREM WITH AN APPLICATION TO PROBLEMS OF RELATIVE MINIMA

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BY

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A GENERAL IMPLICIT FUNCTION THEOREM WITH AN APPLICATION TO PROBLEMS OF RELATIVE MINIMA.

BY K. W. LAMSON.

Goursat has given a proof of the existence of a system of solutions of the equations

$$y_i = F_i(y_1, \cdots, y_n;\ z_1, \cdots, z_m) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where the functions F_i reduce to $y_i^{(0)}$ for $y = y^{(0)}$, $z = z^{(0)}$, and their difference from $y_i^{(0)}$ is of an order higher than the first in the variables y. He has further shown how, under certain conditions, the following system

$$G_i(y_1, \cdots, y_n;\ z_1, \cdots, z_m) = 0 \qquad (i = 1, 2, \cdots, n), \tag{2}$$

can be reduced to the form (1). A system of equations of type (2) arises in the theory of relative extrema of functions of a finite number of variables (referred to as theory I).

Equations (1) and (2) suggest the following problem of implicit functions in the theory of Functions of Lines. Let x, ξ be variables on the continuous range ab, and consider a functional operation $F[y(x), z(x);\ \xi]$ such that to a pair of functions $y(x)$, $z(x)$ and number ξ on ab corresponds a unique real number. Further suppose that $F[y(x), z(x);\ \xi]$ reduces to y_0 when $y = y_0$, $z = z_0$, and that its difference from y_0 is of an order higher than the first, with a suitable definition of order of difference. The subscript i, thought of as a variable with the discrete range $1, 2, \cdots n$, or $1, 2, \cdots m$, has been replaced by the variable ξ with the continuous range ab. The functions $y(x)$, $z(x)$ take the place of the sets of numbers y_i, z_i. To equations (1) and (2) correspond

$$y(\xi) = F[y(x), z(x);\ \xi], \tag{3}$$

$$G[y(x), z(x);\ \xi] = 0. \tag{4}$$

FRÉCHET uses the term "fonctionelle" for F or G, when ξ is fixed, and the term "functional" has come into use as the English equivalent. For equation (3), VOLTERRA* has suggested an existence proof analogous to that of Goursat for equation (1). An instance of equation (4) occurs in the Calculus of Variations in the case of problems in the plane (referred to as theory II).

The first purpose of this paper is to give an existence proof for equations

* *Leçons sur les Fonctions des Lignes*, p. 71.

which include as special cases equations (1), (2), (3) and (4). Equations (3) and (4), although suggested by (1) and (2) are not generalizations of them in the sense of including them as special cases. The general theory is to include also the systems of equations of type (4) appearing in the space problems of the Calculus of Variations (referred to as theory III). The existence theorems used in the theories I, II and III have similarities in hypothesis, proof and conclusion. In I a solution consists of a set of numbers y_i, a function of the variable i, with the range $i = 1, 2, \cdots n$; in II the solution is a function $y(x)$ of the continuous variable x, with the range $a \leqq x \leqq b$, and in III it is a function $y_i(x)$ of the composite variable (i, x) with the composite ranges $i = 1, 2, \cdots n, a \leqq x \leqq b$. The difference in the three theories lies in the difference in the range of the independent variable. Any general theory which includes the three as special cases will introduce a range which will specialize to the three just mentioned. For two reasons it has seemed best not to attempt to abstract common properties from these ranges, but to introduce the general range* of E. H. MOORE, not defined and on which no postulates bear explicitly. In the first place the dissimilarities make it hard to find useful common properties, and in the second place, the general theory is not to exclude problems involving double integrals or combinations of integrals and sums. The general range is a set $\mathfrak{P}$ of elements p, and the functions to be considered are such that to each p corresponds a real number $y(p)$.

Replace the i of equations (1) and (2) and the x of (3) and (4) by p. This leads to the equations

$$y(p) = F[y(q), z(q); p], \tag{5}$$

$$G[y(q), z(q); p] = 0, \tag{6}$$

where q has the range $\mathfrak{P}$ and where, by means of F and G, to each p and pair of functions y and z in a certain class $\mathfrak{M}$ of functions there corresponds a unique real number.

In § 1 below the basis and postulates for the solution of equations (5) and a special form of (6) are set down. In §§ 2, 3 are lemmas leading to the solution of (5) and to the reduction of (6) to the form (5). The last section of the paper contains an application to the problem of Lagrange in the Calculus of Variations.

§ 1. *The Basis.*

The independent variable of the theory has the general range $\mathfrak{P}$. An element of $\mathfrak{P}$ will be denoted by one of the letters p, q. The functions entering the theory belong to a class $\mathfrak{M}$, whose elements are real single-valued functions $y(p)$ or $z(p)$. In theory I the class $\mathfrak{M}$ is the set of n-

* Bolza, *Bulletin of the American Mathematical Society*, Vol. 16 (1910), p. 403; also *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 23 (1914), p. 251.

partite numbers or of points in n-space. In theories II and III $\mathfrak{M}$ is the class of functions or curves in the plane and in $(n + 1)$-space respectively, continuous with their first derivatives. To each element y of $\mathfrak{M}$ corresponds a positive or zero number, the "modulus" of y, which will be denoted by $\|y\|$. In theory I the modulus is interpreted as the largest of the numbers y_i, or as the distance of the point $(y_1, \cdots, y_n)$ from the origin. In theories II and III the modulus is interpreted as the number defining a neighborhood of the first order, namely the maximum absolute value of the functional value and of the derivative. In the general theory the modulus is not defined and is subject to postulates. These postulates and those on $\mathfrak{M}$ will be shown in § 4 to be satisfied in the case of the Lagrange problem.

Postulate 1. $\mathfrak{M}$ is linear, that is, contains all functions of the form $c_1y_1 + c_2y_2$, where c_1 and c_2 are real numbers, provided y_1 and y_2 are themselves in $\mathfrak{M}$.

Postulate 2. $\|y_1 + y_2\| \leqq \|y_1\| + \|y_2\|$.

Postulate 3. $\|cy\| = |c|\,\|y\|$, for every real number c.

Postulate 4. If $\|y\| = 0$, then $y(p) = 0$ for every p.

THEOREM 1. *If* $\{y_i\}$ *and* $\{y'_i\}$ *are sequences, and* y *and* y' *are functions, such that* $\lim_{i=\infty} \|y - y_i\| = \lim_{i=\infty} \|y_i - y'_i\| = 0$; *and if* $\lim_{i-\infty} \|y_i - y'_i\| < b$, *then* $\|y - y'\| \leqq b$.

This theorem follows at once from the preceding postulates.

Definition. The sequence $\{y_i\}$ is defined to be a Cauchy sequence if

$$\lim_{i=\infty,\, j=\infty} \|y_i - y_j\| = 0.$$

The sequence $\{y_i\}$ is said to have a limit y if $\lim_{i=\infty} \|y - y_i\| = 0$. The uniqueness of this limit is a result of postulates 2, 3, 4.

Postulate 5. For every Cauchy sequence in $\mathfrak{M}$ there exists a function in $\mathfrak{M}$ which is the limit of this sequence.

Definition. The symbol $(\bar{y})_a$ denotes the totality of functions y of $\mathfrak{M}$ such that $\|y - \bar{y}\| < a$.

Consider $F[y_1, \cdots, y_\kappa; p]$ real and single-valued for y_i in $(\bar{y}_i)_a$ $(i = 1, \cdots \kappa)$ and p in $\mathfrak{P}$, and such that when $y_1, \cdots, y_\kappa$ are fixed the resulting function of p is in $\mathfrak{M}$.

Definition. The functional F is continuous at a set of arguments $(y'_1, \cdots, y'_\kappa)$ if for every ϵ there exists a δ such that

$$\|F[y_1, \cdots, y_\kappa; p] - F[y'_1, \cdots, y'_\kappa; p]\| < \epsilon$$

whenever y_i is in $(y'_i)_\delta$.

§ 2. *Solution of the equation* $y(p) = F[y, z; p]$.

The proof of the existence of a solution, $y(p)$, of the equation

$$y(p) = F[y, z; p] \tag{5}$$

is similar to that given by Goursat,* who used the method of successive approximations to treat equation (1). Let y_0 and z_0 be two functions of the class $\mathfrak{M}$. The functional $F[y, z; p]$ is supposed to be real and single-valued for all elements $(y, z; p)$ for which y is in a neighborhood $(y_0)_a$, z in $(z_0)_a$, and p in $\mathfrak{P}$, and to have the property that when y and z are fixed in its range of definition the resulting function of p is also in $\mathfrak{M}$. It has further the properties

(1) $F[y_0, z_0; p] = y_0(p)$ for every p in $\mathfrak{P}$;

(2) it is continuous in y and z at each element y', z', in its range of definition;

(3) there exists a constant $0 < K < 1$ such that

$$\| F[y_1, z; p] - F[y_2, z; p] \| < K \| y_1 - y_2 \|$$

whenever (y_1, z) and (y_2, z) are in the range for which F is defined. This condition will be referred to as the Lipschitz condition.

Define a sequence of successive approximations by the equations

$$y_1 = F[y_0, z; p] \tag{7}$$

$$y_{i+1} = F[y_i, z; p] \qquad (i = 1, 2, 3, \cdots), \tag{8}$$

which is possible whenever every y_i is in $(y_0)_a$. It will first be shown that a neighborhood $(z_0)_{a_1}$ with $a_1 \leqq a$ can be chosen so that the elements of the sequence are well defined whenever z is in $(z_0)_{a_1}$.

LEMMA 1. *There exists a positive constant* $a_1 \leqq a$ *such that for* z *in* $(z_0)_{a_1}$, *and for every* i, y_i *is in* $(y_0)_a$.

To prove this, use the continuity of F in z, and choose $a_1 \leqq a$, so that

$$\| y_1 - y_0 \| = \| F[y_0, z; p] - F[y_0, z_0; p] \| < a(1 - K).$$

From the Lipschitz condition, if y is in $(y_0)_a$,

$$\| F[y, z; p] - F[y_0, z; p] \| < K \| y - y_0 \|.$$

From the addition of $\| F[y_0, z; p] - y_0 \|$ to both sides, and from Postulate 2, follows

$$\| F[y, z; p] - y_0 \| < K \| y - y_0 \| + a(1 - K). \tag{9}$$

In particular, putting $y = y_1$, this becomes

$$\| y_2 - y_0 \| < K \| y_1 - y_0 \| + a(1 - K) < a.$$

To complete the induction proof, assume $\| y_i - y_0 \| < a$, and put y_i in (9).

* Goursat, *Bulletin de la Société Mathématique de France*, Vol. 31 (1903), p. 184. Bliss, *Princeton Colloquium Lectures*, p. 8.

LEMMA 2. *The sequence $\{y_i\}$ is a Cauchy sequence and its limit, y, (Postulate 5) is in $(y_0)_a$.*

To prove this, the convergence of the series $\sum_i \| y_{i+1} - y_i \|$ is first shown, by using $\sum_i K^i a$ as a dominating series. From the definition of y_2 and y_1, and from the Lipschitz condition,

$$\| y_2 - y_1 \| = \| F[y_1, z; p] - F[y_0, z; p] \| < K \| y_1 - y_0 \| < Ka.$$

To complete the induction proof, assume

$$\| y_{i+1} - y_i \| < K^i a,$$

and apply the Lipschitz condition to $\| y_{i+2} - y_{i+1} \|$.

The convergence of $\Sigma \| y_{i+1} - y_i \|$, and Postulate 2 imply that the sequence $\{y_i\}$ is a Cauchy sequence. From Theorem 1 it follows that the limit y of $\{y_i\}$ is in $(y_0)_a$.

LEMMA 3. *The equation* (5) *is satisfied by the limit y of Lemma* 2.

For from the definition of y_i, and from Lemma 2,

$$\lim_{i=\infty} \| y - y_i \| = \lim_{i=\infty} \| y - F[y_i, z; p] \| = 0. \tag{10}$$

From the continuity of F,

$$\lim_{i=\infty} \| F[y_i, z; p] - F[y, z; p] \| = 0, \tag{10a}$$

and from the addition of (10) and (10*a*), and the application of Postulates 2 and 4,

$$y = F[y, z; p].$$

LEMMA 4. *The solution y of equation* (5) *described in the preceding lemmas is the only one in $(y_0)_a$ corresponding to a z in $(z_0)_{a_1}$.*

For the proof, assume two solutions, and apply the Lipschitz condition to their difference, using Postulate 4.

LEMMA 5. *As a functional of z, y is continuous in the neighborhood $(z_0)_{a_1}$.*

It is necessary to show that if $\| z - z' \|$ is small, then $\| y - y' \|$ is small, where y and y' are the solutions corresponding to z and z' respectively.

From Postulate 2,

$$\begin{aligned} \| y - y' \| &= \| F[y, z; p] - F[y', z'; p] \| \\ &\leqq \| F[y, z; p] - F[y', z; p] \| + \| F[y', z; p] - F[y', z'; p] \| \\ &\leqq K \| y - y' \| + \| F[y', z; p] - F[y', z'; p] \|. \end{aligned}$$

From the continuity in z, the last term can be made less than an ϵ as required, whence

$$\| y - y' \| < \frac{\epsilon}{1 - K}.$$

The results of this section may be summed up in the following

THEOREM 2. *When $F[y, z; p]$ has the solution $(y_0, z_0; p)$ and the properties described at the beginning of this section for elements $(y, z; p)$ with y in $(y_0)_a$, z in $(z_0)_{a_1}$ and p in $\mathfrak{P}$, there exists a constant $a_1 \leqq a$ such that the equation*

$$y = F[y, z; p]$$

has one and only one solution $y = Y[z; p]$ for each z in the neighborhood $(z_0)_{a_1}$. The functional $Y[z; p]$ so defined has the value $y = y_0$ for $z = z_0$ and is continuous at $z = z_0$.

§ 3. *The equation $G[y; p] = z(p)$.*

In order to transform equation (2) to the form (1), Goursat* assumes first that the derivatives $\partial G_i / \partial y_j$ exist and are continuous, and second that the functional determinant is different from zero for those values of y_i and z_i for which the G_i vanish. The equation (6) will be taken in the less general form,

(11) $$G[y; p] = z(p),$$

which is to be solved for y, given that

$$G[y_0; p] = z_0(p).$$

The equation (11) will be transformed to the form (5) treated in the preceding section, by a procedure following that of Goursat. Before prescribing the properties of the functional G it will be useful to describe those of a functional $A[y_1, y_2, \eta; p]$ which will be called a difference function for reasons which will presently appear. At each element $(y_1, y_2, \eta; p)$ with y_1 and y_2 in $(y_0)_a$, η in $\mathfrak{M}$, and p in $\mathfrak{P}$ the functional A has a single real value, and when the first three of its arguments are fixed defines a function of the class $\mathfrak{M}$. It has furthermore the following properties:

(1) it is linear in η, that is,

$$A[c_1\eta_1 + c_2\eta_2] = c_1 A[\eta_1] + c_2 A[\eta_2]$$

where η_1 and η_2 are functions of the class $\mathfrak{M}$ and c_1 and c_2 are constants. The three arguments other than η are suppressed for the moment in this equation;

(2) There exists a constant M such that

$$|| A[y_1, y_2, \eta; p] || \leqq M || \eta ||$$

whenever $(y_1, y_2, \eta; p)$ is in the set for which A is defined;†

(3) the functional A is uniformly continuous in (y_1, y_2) at (y_0, y_0) with

* Loc. cit., p. 191.

† Riesz, *Annales Scientifique de L'École Normale Supérieure*, 3me Série, Vol. 31 (1914), p. 10.

respect to the set of admissible arguments η for which $\|\eta\| = 1$, that is for every given ϵ there exists a δ such that

$$\| A[y_1, y_2, \eta; p] - A[y_0, y_0, \eta; p] \| < \epsilon$$

whenever y_1 and y_2 are in $(y_0)_\delta$, η is in $\mathfrak{M}$.

The functional $G[y; p]$ is supposed to be real single valued for all arguments (y, p) such that y is in $(y_0)_a$ and p in $\mathfrak{P}$, and to have the usual property that it is in the class $\mathfrak{M}$ when the argument y is fixed. It has furthermore a difference function A of the kind described above such that

$$G[y_1; p] - G[y_2; p] = A[y_1, y_2, y_1 - y_2; p]$$

whenever (y_1, p) and (y_2, p) are elements in the domain of definition of G. The functional $A[y_0, y_0, \eta; p]$ is called the differential of G at y_0. Since y_0 is a fixed element of the class $\mathfrak{M}$ the differential is a function of η and p alone.

The use which Goursat makes of his hypothesis concerning the non-vanishing of the functional determinant suggests the assumption that A has a "reciprocal" for $y_1 = y_2 = y_0$, namely that there exists a functional $\bar{A}[\eta; p]$ such that

$$\bar{A}[A[y_0, y_0, \eta; q]; p] = \eta(p)$$

$\bar{A}[\eta; p]$ has the properties (1) and (2) prescribed for A, where $\bar{M}$ denotes the number corresponding to the M of property (2). It has the further property that it vanishes identically in p only when $\eta(p) = 0$ for every p.

LEMMA 6. *The functional F defined by the equation*

$$F[y, z; p] = y - \bar{A}[G[y; p] - z; p]$$

has the properties of the functional F of § 2 near the element (y_0, z_0) where

$$z_0 = G[y_0; p].$$

As to the property (1) of § 2, it follows from the definition of F given in this lemma that

$$F[y_0, z_0; p] = y_0 - \bar{A}[0; p] = y_0.$$

The continuity, property 2, is proved by these inequalities,

$$\begin{aligned} \| F[y, z; p] - F[y', z'; p] & \\ &= \| y - y' + \bar{A}[G[y; p] - G[y'; p] - z + z'; p] \| \\ &\leqq \| y - y' \| + \bar{M} \| G[y; p] - G[y'; p] - z + z' \| \\ &\leqq (1 + M\bar{M}) \| y - y' \| + \| z - z' \|. \end{aligned}$$

To find the K of property 3, use the linearity of the functional $\bar{A}$.

$$\begin{aligned} \| F[y, z; p] - F[y', z; p] \| &= \| y - y' - \bar{A}[G[y; q] - G[y'; q]; p] \| \\ &= \| y - y' - \bar{A}[A[y, y', y - y'; q] - A[y_0, y_0, y - y'; q] \\ &\qquad + A[y_0, y_0, y - y'; q]; p] \|. \end{aligned}$$

From linearity again, from the fact that $\bar{A}$ is the reciprocal of A, and from Postulate 3, $\|-y\| = \|y\|$, this expression reduces to

$$\|\bar{A}[A[y, y', y-y'; q] - A[y_0, y_0, y-y'; q]; p].$$

Because $\bar{A}$ is bounded, this is less than

$$\bar{M}\left\| A\left[y, y', \frac{y-y'}{\|y-y'\|}; p\right] - A\left[y_0, y_0, \frac{y-y'}{\|y-y'\|}; p\right]\right\| \|y-y'\|.$$

The number a of Lemma 1 is then chosen to make the coefficient of $\|y-y'\|$ less than $K < 1$.

THEOREM 3. *The solution of the equation*

$$y = F[y, z; p] \tag{5}$$

where F is defined in Lemma 6, satisfies uniquely the equation

$$G[y; p] = z(p), \tag{11}$$

and is continuous as a functional of z.

For, from the definition of F, (5) reduces to

$$\bar{A}[G[y; q] - z; p] = 0$$

and since $\bar{A}[\eta; p]$ vanishes identically only when $\eta(p) \equiv 0$, it follows that

$$G[y; p] \equiv z(p).$$

Any other function y', a solution of (11), would make F reduce to y', and would satisfy (5). But the solution of (5) is unique (Lemma 4). The solutions of (5) and (11) have been shown to be the same, and the solution of (5) is continuous (Lemma 5). This proves the continuity asserted in the theorem.

§ 4. *An Application to the Calculus of Variations.*

The theorem of § 3 will now be applied to the differential equations of the problem of Lagrange in the Calculus of Variations. For this problem the functions y in the integral

$$\int_a^b f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

to be minimized are subject to two sets of conditions. They must satisfy, first, the $m < n$ differential equations,

$$\varphi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (\alpha = 1, \cdots, m), \tag{12}$$

and second, the end conditions,

$$y_i(a) - h_i = 0, \tag{13}$$

$$y_i(b) - k_i = 0 \qquad (i = 1, \cdots, n). \tag{14}$$

The equation (12) may be regarded as a single equation in the composite variable (α, x), whose range is a subset of the range of elements (i, x) where $i = 1, 2, \cdots n, a \leqq x \leqq b$.

Bliss* has given a treatment of a problem of which this is a special case by adjoining to (12) the $n - m$ new equations

$$(15) \qquad \varphi_r(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = Z_r(x) \quad (r = m + 1, \cdots, n).$$

In (15) the functions φ_r are arbitrary except that they are to be chosen so that the determinant $|\partial\varphi_i/\partial y_j'|$ is different from zero at every point of the minimizing arc to be studied. Equations (12) and (15) can then be written together in the single equation

$$(16) \qquad \varphi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = Z_i(x) \quad (i = 1, 2, \cdots, n),$$

with the understanding that $Z_i = 0$ identically in x, for $i \leqq m$.

Consider now a system of solutions $y_i^{(0)}(x)$, $Z_i^{(0)}(x)$ of class C' of the equations (16). In a neighborhood of the elements (x, y, y') of this solution the functions φ_i are supposed to have continuous first and second partial derivatives, and along the solution itself the functional determinant $|\partial\varphi_i/\partial y_j'|$ is different from zero. The partial derivatives $\partial\varphi_i/\partial y_j$ and $\partial\varphi_i/\partial y_j'$ will henceforth be denoted by φ_{ij} and ψ_{ij}, and their values at $x = a$, by $\varphi_{ij}(a)$ and $\psi_{ij}(a)$. It is proposed to show that the problem of determining a system of solutions of the equations (16) with initial conditions (13) is a special case of the theorem proved in § 3.

Equations (13) and (16) together are equivalent to the single system

$$G[y(q);\ p] = z(p),$$

where the independent variables are $p = (i, x)$, $q = (j, x_1)$ and G is the functional in the first member of the equation

$$(17) \quad \sum_j \psi_{ij}(a)(y_j(a) - h_j) + \int_a^x \varphi_i(x_1, y, y')dx_1 = z_i(x) \quad (i = 1, \cdots, n).$$

Equations (13) have been multiplied by a matrix of rank n. The z_i appearing in (17) are the integrals from a to x of the functions $Z_i(x)$ in (16), and so vanish for $x = a$. Equations (14) are discussed later.

The general theory of the preceding sections will be applied to the solution of (17) for y when z is given. With the $y^{(0)}$ which minimizes the integral is associated a $z^{(0)}$ by equations (17), and it is in a first order neighborhood of these functions that a solution is to be found. The range $\mathfrak{P}$ is specified to be the set of elements (i, x), $(i = 1, \cdots, n;\ a \leqq x \leqq b)$. The class $\mathfrak{M}$ is the class of functions $y_i(x)$ which for each i are continuous with their first derivatives on the interval ab. The modulus, $||y||$, is

* *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 307.

the maximum of the absolute values of y_i and y'_i $(i = 1, \cdots, n)$. The functional $G[y;\, p]$ is the left-hand member of equations (17).

It remains to exhibit the differential A, its reciprocal $\bar{A}$, and to prove that the postulates of § 1 and hypotheses of § 2 are satisfied. Postulates 1–4 are immediately seen to be satisfied. Postulate 5 can be proven from the fact that convergence of the moduli of a sequence of functions of $\mathfrak{M}$ implies the uniform convergence of the functions and of their first derivatives.

The differential A is given for the function (17) by Taylor's formula* in the form

$$\sum_j \psi_{ij}(a)\eta_j(a) + \sum_j \int_a^x \{C_{ij}(x_1)\eta_j(x_1) + C'_{ij}(x_1)\eta'_j(x_1)\}dx_1, \tag{18}$$

where

$$C_{ij}(x_1) = \int_0^1 \varphi_{ij}(x_1,\, y^{(1)} + u(y^{(2)} - y^{(1)}),\, y^{(1)\prime} + u(y^{(2)\prime} - y^{(1)\prime}))du,$$

$$C'_{ij}(x_1) = \int_0^1 \psi_{ij}(x_1,\, y^{(1)} + u(y^{(2)} - y^{(1)}),\, y^{(1)\prime} + u(y^{(2)\prime} - y^{(1)\prime}))du.$$

In C and C', $y^{(1)}$ and $y^{(2)}$ are the arguments of the functional A, and are in a first order neighborhood of the extremal $y^{(0)}$ such that the determinant $|\psi_{ij}| \neq 0$, and φ is defined. When $y^{(1)} = y^{(2)} = y^{(0)}$, A reduces to

$$\sum_j \psi_{ij}(a)\eta_j(a) + \sum_j \int_a^x \{\varphi_{ij}\eta_j(x_1) + \psi_{ij}\eta'_j(x_1)\}dx_1. \tag{19}$$

To exhibit the reciprocal $\bar{A}$ is to define an operation which will reduce (19) to $\eta_k(x)$. This operation will be taken in the form

$$\bar{A} = \sum_i \left[l_{ki}(x)\eta_i(a) + \int_a^x \lambda_{ki}(x,\, x_1)\eta_i(x_1)dx_1 + \nu_{ki}(x)\eta_i(x) \right]$$

with suitably chosen functions l, λ, ν, and it is to be proved that when the functions $\eta(q) = \eta_i(x_1)$ of the variable $q = (i,\, x_1)$ is replaced by A in this expression the result is $\eta(p)$ with $p = (k,\, x)$. To distinguish variables of integration from each other and from limits of integration, the notations x, x_1, x_2 are used. Summations are from 1 to n. To choose the functions l, λ, ν operate as follows. Put $x = a$ in (18) and multiply by undetermined factors $l_{ki}(x)$. Form (18) for x_1, $(a < x_1 < x)$, multiply by $\lambda_{ki}(x,\, x_1)$, and integrate from a to x. For $x_1 = x$, multiply by $\nu_{ki}(x)$. Add the terms so formed and sum as to i.

A method of choosing the functions l, λ and ν is to be given so that the expression,

$$\sum_{ij} \left[\left\{ l_{ki}(x) + \int_a^x \lambda_{ki}(x,\, x_1)dx_1 + \nu_{ki}(x) \right\} \psi_{ij}(a)n_i(a) \right.$$

* Jordan, *Cours d'Analyse*, 2d ed., Vol. 1, p. 247.

$$+ \int_a^x \lambda_{ki}(x, x_1) \int_a^{x_1} (\varphi_{ij}\eta_j + \psi_{ij}\eta_j')_{x_2} dx_2 dx_1$$
$$+ \int_a^x \nu_{ki}(x) \{\varphi_{ij}\eta_j + \psi_{ij}\eta_j'\}_{x_2} dx_2 \Big],$$

whose formation was described in the preceding paragraph, reduces to $\eta_k(x)$. By the change in order of integration in the second term, and the combination of the last two terms, this becomes

$$(20) \quad \sum_{ij} \Big[\Big\{ l_{ki}(x) + \int_a^x \lambda_{ki}(x, x_1)dx_1 + \nu_{ki}(x) \Big\} \psi_{ij}^{(a)}\eta_j(a)$$
$$+ \int_a^x (\varphi_{ij}\eta_j + \psi_{ij}\eta_j')_{x_2} \Big\{ \int_{x_2}^x \lambda_{ki}(x, x_1)dx_1 + \nu_{ki}(x) \Big\} dx_2 \Big].$$

A set of auxiliary functions $\mu_k(x, x_2)$ may be defined by means of the equations

$$(21) \qquad \int_{x_2}^x \lambda_{ki}(x, x_1)dx_1 + \nu_{ki}(x) = \mu_{ki}(x, x_2) \quad (\kappa, i = 1, 2, \cdots, n).$$

From (21) and the integration of the last term by parts, (20) is seen to become

$$(22) \quad \sum_{ij} \Big[\Big\{ l_{ki}(x) + \int_a^x \lambda_{ki}(x, x_1)dx_1 + \nu_{ki}(x) \Big\} \psi_{ij}(a)n_j(a)$$
$$+ \int_a^x \Big\{ \mu_{ki}(x, x_2)\psi_{ij}(x_2) - \int_a^{x_2} \mu_{ki}(x, x_1)\varphi_{ij}(x_1)dx_1 \Big\} \eta_j' dx_2$$
$$+ \eta_j(x) \int_a^x \mu_{ki}(x, x_1)\varphi_{ij}(x)dx_1 \Big].$$

Next it will be shown that the functions $\mu_{ki}(x, x_1)$ can be so chosen that the brace under the integral in the second term is independent of x_2 and therefore equal to a function $\kappa_{ki}(x)$ satisfying the following equation:

$$(23) \qquad \sum_i \mu_{ki}(x, x_2)\psi_{ij}(x_2) = \sum_i \int_a^{x_2} \mu_{ki}(x, x_1)\psi_{ij}(x_1)dx_1 + \kappa_{kj}(x),$$
$$(j = 1, \cdots, n).$$

The differentiation of (23) for x_2 as it stands would imply the existence of y''. To avoid this replace the μ's by linear combinations of them, $v_{kj}(x, x_2)$, determined by the following equations,

$$(24) \qquad v_{kj}(x, x_2) = \sum_i \mu_{ki}(x, x_2)\psi_{ij}(x_2).$$

The solution of these for the functions μ is possible since $|\psi_{ij}| \neq 0$, and it gives

$$(25) \qquad \mu_{ki}(x, x_2) = \sum_r c_{kr}(x_2) v_{ri}(x, x_2).$$

From (24) and (25), equation (23) becomes

$$v_{kj}(x, x_2) = \sum_{ir} \int_a^{x_2} c_{kr}(x_1) v_{ri}(x, x_1) \varphi_{ij}(x_1) dx_1 + \kappa_{kj}(x).$$

In this equation the right member is differentiable for x_2, and the equations for the determination of the functions v_{kj} may be written in the form

$$\frac{d}{dx_2} v_{kj}(x, x_2) = \sum_{ir} c_{kr}(x_2) v_{ri}(x, x_2) \varphi_{ij}(x_2). \tag{26}$$

These are linear differential equations which determine $v_{kj}(x, x_2)$ uniquely subject to the initial conditions,

$$v_{kj}(x, x) = \delta_{kj}, \tag{27}$$

where δ_{kj} is unity when $\kappa = j$ and zero otherwise. When the functions v_{kj} are known the μ's are given by (25), the κ's by (23) and the λ's and v's by (21). With the help of (23), (24) and (27) the expression (22) may be replaced by

$$\sum_{ij} \left[\left\{ l_{ki}(x) + \nu_{ki}(x) + \int_a^x \lambda_{ki}(x, x_1) dx_1 \right\} \psi_{ij}(a) \eta_j(a) \right] - \sum_j \kappa_{kj}(x) \eta_j(a) + \eta_k(x). \tag{28}$$

The functions l may now be determined by the equation

$$\sum_i l_{ki}(x) \psi_{ij}(a) = \kappa_{kj}(x) - \sum_i \nu_{ki}(x) \psi_{ij}(a) - \sum_i \psi_{ij}(a) \int_a^x \lambda_{ki}(x, x_1) dx_1 \tag{29}$$

so that everything in the expression (28) disappears except $\eta_k(x)$. This result is formulated in the following definition and theorem.

Definition. The differential $A[y_0, y_0, \eta; p]$ of the functional $G[y; q]$ in the equation (17) for the problem of Lagrange is the expression

$$\sum_j \psi_{ij}(a) \eta_j(a) + \sum_j \int_a^x (\varphi_{ij} \eta_j + \psi_{ij} \eta_j')_{x_1} dx_1. \tag{30}$$

The functional $\bar{A}[\eta; p]$ is given by the formula

$$\sum_i \left[l_{ki}(x) n_i(a) + \int_a^x \lambda_{ki}(x, x_1) \eta_i(x_1) dx_1 + \nu_{ki}(x) \eta_i(x) \right]. \tag{31}$$

In this definition the functions φ_{ij} and ψ_{ij} are formed for the extremal $y^{(0)}$, the functions λ and ν are determined by the equations (26), (27), (25) and (21), and the functions l by (29).

THEOREM 4. *The functional $\bar{A}$ is the reciprocal of A, that is if the η in* (31) *is replaced by the function* (30), *then* (31) *will reduce to* $\eta_k(x)$.

The differential A given by (30) is seen to satisfy the first and second assumptions of § 3. The reciprocal $\bar{A}$ is also seen to satisfy these assumptions. The third assumption as to A follows from the continuity properties of φ, and from the mean value theorem. It remains to show that the reciprocal vanishes identically only with the argument η.

LEMMA 7. *If the functions $\eta_i(x)$ are continuous with their first derivatives on the interval ab, and if the equation*

$$(31)\qquad \sum_i \left[l_{ki}(x)\eta_i(a) + \int_a^x \lambda_{ki}(x, x_1)\eta_i(x_1)dx_1 + \nu_{ki}(x)\eta_i(x) \right] = 0$$

holds identically in κ and x, it follows that $\eta_i(x) = 0$ identically in i and x.

To prove this, put $x = a$. From (29) with the help of equations (21) and (23) for $x = x_2 = a$, it follows that $l_{ki}(a) = 0$, and from (24) and (27) it is seen that $|\nu_{ki}(a)| \neq 0$. Therefore $\eta_k(a) = 0$ identically in κ, and it is correct to write

$$\eta_i(x_1) = \int_a^{x_1} \eta'_i(x_2)dx_2.$$

From (31) then

$$\sum_i \int_a^x \lambda_{ki}(x, x_1) \int_a^{x_1} \eta'_i(x_2)dx_2dx_1 + \nu_{ki}(x) \int_a^x \eta'_i(x_2)dx_2 = 0.$$

By change of order of integration, combination of terms and the use of (21), this becomes

$$(32)\qquad \sum_i \int_a^x \mu_{ki}(x, x_1)\eta'_i(x_1)dx_1 = 0.$$

From the theory of differential equations, the solutions of equations (26), and hence also the functions $\mu_k(x, x_1)$, are differentiable for x. Then differentiation of (32) with respect to x gives

$$(33)\qquad \sum_i \mu_{ki}(x, x)\eta'_i(x) = -\sum_i \int_a^x \frac{\partial}{\partial x} \mu_{ki}(x, x_1)\eta'_i(x_1)dx_1.$$

After multiplying by $\bar{\mu}_{rk}(x)$, the matrix reciprocal to $\mu_{ki}(x, x)$, summing with respect to κ and setting

$$-\sum_k \bar{\mu}_{rk}(x) \frac{\partial}{\partial x} \mu_{ki}(x, x_1) = \sigma_{ri}(x, x_1)$$

the equations (33) give

$$(34)\qquad \eta'_r(x) = \sum_i \int_a^x \sigma_{ri}(x, x_1)\eta'_i(x_1)dx_1.$$

The proof that no solution of (32) exists except $\eta'_r(x)$ identically zero is a slight modification of the corresponding proof for Volterra's integral equation.* If M and m are the maxima of $|\sigma_{ri}(x, x_1)|$ and $n'_i(x)$ respectively, for $r, i = 1, 2 \cdots n$ and values of x and x_1 on the interval ab, the

* Bôcher, *An Introduction to the Study of Integral Equations*, p. 15.

equations (34) give

$$m \leqq \int_a^x nMmdx = nMm(x - a),$$

and by repeated applications of this inequality it follows that

$$m \leqq n^\alpha M^\alpha m \frac{(x - a)^\alpha}{\alpha !}$$

for every positive integer α. As this last expression approaches zero with increasing α, it follows that

$$\eta_r'(x) = 0, \qquad a \leqq x \leqq b, \qquad r = 1, \cdots, n.$$

Since $\eta_r(a) = 0$, it is true that $\eta_r(x) \equiv 0$, as stated in the lemma.

The postulates and hypotheses of the general theory have been proved to be satisfied in the case of the Lagrange problem. The results of this section may be stated in the following theorem.

THEOREM. *Under the hypotheses made at the beginning of this section the system of equations*

$$\varphi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_u') = Z_i(x) \qquad (i = 1, \cdots, n)$$

with the initial conditions $y_i(a) = h_i$, $(i = 1, \cdots, n)$, *is equivalent to the single equation*

$$\sum_j \psi_{ij}(a)[y_j(a) - h_j] + \int_a^x \varphi_i dx = z_i(x).$$

This has the form

$$G[y(q);\ p] = z(p)$$

where p and q represent the pairs $p = (i, x)$, $q = (j, x)$. If $y^{(0)}(q)$, $z^{(0)}(p)$ is an initial solution of the last equation with properties as prescribed above, then there exist two neighborhoods $(y^{(0)})_a$ and $(z^{(0)})_{a_1}$ such that to every $z(p)$ in the latter there corresponds one and but one solution $y(q)$ in $(y^{(0)})_a$. The functional $y(q) = Y[z; q]$ so defined is continuous in $(z^{(0)})_{a_1}$ and reduces to $y = y^{(0)}$ for $z = z^{(0)}$.

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VITA

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