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A Certain Type of Isoperimetric Problem,

IN PARTICULAR,

the Solid of Maximum Attraction.

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IX.—*A Certain Type of Isoperimetric Problem, in particular, the Solid of Maximum Attraction*

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INTRODUCTION.

In the fourth chapter of his "*Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*," Euler has given an ingenious method of transforming isoperimetric problems of a certain class to the non-isoperimetric type. In the first chapter of this paper we discuss in a general way the conditions implied in the transformations for Euler's examples, and the circumstances in which it is effective in removing the isoperimetric condition in other isoperimetric problems.[1] In the second chapter we consider the transformation in detail for the solid of revolution of maximum attraction, after giving a brief critique of the partial solutions that have already been given of this problem. We obtain the form that the solid must take in order to furnish the maximum attraction, and show that this actually does produce a maximum. The latter result depends upon certain relations between the senses of description at the points of intersection of a straight line with a simple closed curve of a special class. In the third chapter we establish these results of Analysis Situs.

CHAPTER I.

EULER'S METHOD OF REMOVING THE ISOPERIMETRIC CONDITION.

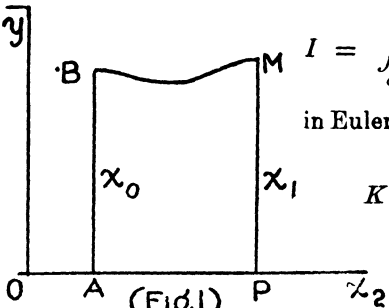
§ 1. *The Curve of Least Arc and Given Area.*

(a). The first of the problems which Euler has solved by the method above referred[2] to, he enunciates: "*Supra axe, AP, construere lineam, BM, ita comparatam ut abscissa area, ABMP, datae magnitudinis, arcus curvae, BM, illi respondens, sit omnium minimus.*" Euler does not explicitly state whether the end-point, M, is free to move in any way or not. Axes of x and y being chosen as in Fig. 1, the area, u, swept out

[1] See also Kneser Math. Annalen, 59, and Erdmann, Crelle's Journal.

[2] See pp. 143-4.

by an ordinate, is taken as the independent variable. If a denote the total area, the integral, I, which is to be maximised, takes the form :



(Fig.1)

$$I = \int_0^{x_1} \sqrt{dx^2 + dy^2} = \int_0^{a} \sqrt{dy^2 + \frac{du^2}{y^2}}$$

in Euler's notation. The isoperimetric condition

$$K = \int_{x_0}^{x_1} y \,.\, dx = \int_0^{a} du = a,$$

becomes an identical equation. The problem is thus reduced to that of finding a minimum for $\int_0^a \sqrt{dy^2 + \frac{du^2}{y^2}}$ without any isoperimetric condition.

(b). We propose to discuss briefly the transformation under more explicit conditions, and see under what circumstances it can be applied. We assume that our "admissible curves"[1] are the totality of curves representable in the form : $\mathfrak{L} : y = f(x)$, for $x_0 \leq x \leq x_1$, and satisfying the following conditions :

(α) $f(x)$ is of class C' [2] on the interval, $(x_0 \,.\, x_1)$;

(β) the end-point, $B\ (x_0, \ y_0)$, is fixed, and the end-point, $M\ (x_1, y_1)$.
 i) is fixed, or
 ii) is free to move on the curve, $\varphi\ (x, y) = 0$, or
 iii) is free to move in any way ;

(γ) the curve lies above O x ; i.e., $y > 0$ ....(1)

(δ) $K = \int_{x_0}^{x_1} y \,.\, dx = a$, for $y = f(x)$ .... (2)

We have to find among these curves, $\mathfrak{L}$, one which renders

$$I = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy^2}{dx^2}\right)} \,.\, dx \quad \text{..............................(3)}$$

a maximum.

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[1] See Bolza, Variations, p. 9.

[2] We say that $f(x)$ is of class C', $\{C'', C'' \ldots C^n\}$ in $(x_0\, x_1)$ if $f(x)$ is continuous, and $f'(x)\{f''(x) \ \ldots \ f_n(x)\}$ exist and are continuous ; it is of class D' $\{D'' \ldots\}$ if $(x_0\, x_1)$ can be divided into a finite number of intervals in which $f(x)$ is of class C', $\{C'', C'' \ldots\}$; cf. Bolza, Variations, p. 7.

(c). We now introduce, after Euler, the variable, u, where

$$u = \int_{x_0}^{x} y\, dx \quad\ldots\ldots\ldots\ldots (4)$$

taken along $\mathcal{L}$. From (4) and (1),

$$\frac{du}{dy} > 0,$$

whence we can solve (4) with respect to x,[1] and the inverse function,

$$x = x\,(u) \quad\ldots\ldots\ldots\ldots (5)$$

will be of class C″ in the interval $(0, a)$, and it will increase from x_0 to x_1, as u increases from 0 to a. Further, if we define, $y\,(u)$ by

$$y\,(u) = f\left\{x\,(u)\right\}$$

then $y\,(u)$ will be of class C′, on the interval, $(0, a)$, and between $x\,(u)$ and $y\,(u)$ holds the relations :

$$y\,.\,x' = 1 \quad\ldots\ldots\ldots\ldots (6)$$

By the introduction of u, the curve, $\mathcal{L}$, in the (x, y)—plane is transformed into a curve, $\mathcal{L}'$, in the (x, y, u)— space :

$$\mathcal{L}' : x = x\,(u), \quad y = y\,(u),$$

where .

(α') $x\,(u)$ is of class C″ and $y\,(u)$ is of class C′ in $(0\; a)$;

(β') $x\,(0) = x_0$, $y\,(0) = y_0$, and $x\,(a) = x_1$, $y\,(a) = y_1$, where (x_1, y_1, a) is

i) fixed, or

ii) $\varphi\,(x, y) = 0$, or

iii) x_1 and y_1 are arbitrary ;

(γ') $y\,(u) > 0$ in $0\; a$) ;

(δ') $y\,x' = 1$.

The isoperimetric condition is expressed by (δ'). For, if we introduce u into K by means of (δ'), we have

$$K = \int_{x_0}^{x_1} y\, dx = \int_0^a y\,.\,x'\, du = \int_0^a du = a$$

Transforming I, we obtain :

$$I' = \int_0^a \sqrt{x'^2 + y'^2}\,.\,du \quad\ldots\ldots\ldots\ldots (7)$$

[1] Osgood Funktionentheorie, p. 193.

an integral taken along a curve in the (u, x, y) space and defined by the differential equation, (6).

(d). Now it happens that in the present problem, x can be eliminated from (7) by (6), giving

$$I'' = \int_0^a \frac{\sqrt{1 + y^2 y'^2}}{y} \cdot du \quad \text{......................................} (8)$$

From (6) and (β'), we have:

$$x_1 - x_0 = \int_{x_0}^{x_1} dx = \int_0^a x' \cdot du = \int_0^a \frac{du}{y} \quad \text{..........................} (9)$$

whence $\mathcal{L}$ transforms into $\mathcal{L}''$,

$\mathcal{L}'' : y = y(u)$,

satisfying the conditions:

$(\alpha''$ $y(u)$ is of class C' in $(0, a)$;

(β'') $y(0) = y_0$, and $y(a) = y_1$, where:

i) $x_1 = x_0 + \int_0^a \frac{du}{y}$, and y_1 are fixed, or

ii) $\varphi \left\{ x_0 + \int_0^a \frac{du}{y}, y \right\} = 0$, or

iii) $x_0 + \int_0^a \frac{du}{y}$ and y_1 are arbitrary;

(γ'') $y(u) > 0$ in $(0\ a)$.

(e). *Cases in which the problems of (b) and (d) are equivalent.*

For each $\mathcal{L}''$ and the corresponding integral, I, there is an $\mathcal{L}''$ and a corresponding I'', such that $I = I''$. Now in order that the problems shall be equivalent, the converse must be true; *i.e.*, for each $\mathcal{L}''$ and integral I'', there is an $\mathcal{L}$ and the corresponding integral I, such that $I = I''$. To see under what circumstances this is the case, we apply the above transformations inversely; *i.e.*, we determine the function, $x(u)$, by the equation.

$$x(u) = x_0 + \int_0^{u_1} \frac{du}{y} \quad \text{......................................} (10)$$

We thus transform the integral, I'', given by (8) into I' given by (7), and the transformed path of integration, $\mathcal{L}''$, satisfies the conditions, $(\alpha'') \ldots (\delta'')$, except those to which the end-point, $M(x_1\ y_1)$ is subjected. In case $(\beta''$: i), in order that $\mathcal{L}''$ shall pass through $(x_1\ y_1\ a)$,

we must have, $\qquad x_1 = x_0 + \int_0^a \frac{du}{y}$

along $\mathcal{L}''$. In other words, we have a new isoperimetric condition on $\mathcal{L}$, and nothing is gained by the process. In case (β'' : iii), the curve, $\mathcal{L}''$, joining $(x_o, y_o, 0)$ and $(x_o + \int_0^a \frac{du}{y}, y_1, a)$ satisfies (β : iii) and is an $\mathcal{L}$, since (x_1, y_1) is perfectly arbitrary in position. Solving (10) for u as a function of x, we have; $u(x) = y$. Hence I' transforms into I, and $\mathcal{L}''$ to an $\mathcal{L}$ joining $(x_o\, y_o)$ to $(x_o + \int_0^a \frac{du}{y}, y_1)$. *In case ($\beta$: iii), the problems of finding a maximum of I for the totality of curves, $\mathcal{L}$, and of I'' for the totality, $\mathcal{L}''$, are equivalent.*

(f). In case (β'' : ii), the condition that the end-point shall lie upon $\varphi(x, y) = 0$, is

$$\varphi \left\{ x_o + \int_0^a \frac{du}{y}, y \right\} = 0.$$

This is a still more complicated isoperimetric condition if $\varphi(x, y)$ contains x. If not, i.e., *if y is constant*, then (β' : ii) is satisfied and the transformed $\mathcal{L}''$ is an $\mathcal{L}$. Transforming into the (x, y) plane by (10), *the problems of finding a maximum for the totality, $\mathcal{L}$, and of I'' for the totality, $\mathcal{L}''$, are equivalent if M is free to move on a line parallel to* O x.

(g). Excluding from the totality, $\mathcal{L}$, the cases in which we have seen that equivalence necessitates a new isoperimetric condition on $\mathcal{L}''$, we have the problems of finding a maximum for I along a totality, $\mathcal{L}$, $y = f(x)$, where: (α_o) $f(x)$ is of class C′ on $(x_o\, x_1)$;

(β_o) the end-point, B $(x_o\, y_o)$, is fixed and the end-point, M $(x_1\, y_1)$·

i) is free to move on a line parallel to Ox, or

ii) is free to move in any way;

(γ_o) $y > 0$;

$$(\delta_o)\ K = \int_{x_o}^{x} y\, dx = a, \text{ for } y = f(x);$$

and of finding a maximum for I'' along a totality, $\mathcal{L}''$, $y = y(u)$, where (α_o'') $y(u)$ is of class C′ in $(0, a)$;

(β_o'') the end-point, B'' $(0, y_o)$ is fixed, and the end-point, M'' (a, y).

i) is fixed, or

ii) is free to move on a line parallel to 0 x, or

iii) is free to move along $u = a$;

(γ_o'') $y > 0$;

are equivalent problems.

(h). *Determination of Constants.*

Euler finds for the minimizing curve in the (x, y)—plane a circle through B, centre in AP, (fig. 1). In case (β_o : i), the other end-point lies in $y = y_1$, and is determined by the isoperimetric condition. In case (β_0 : ii) we have from the condition of transversality in the (u, y)—plane[1]

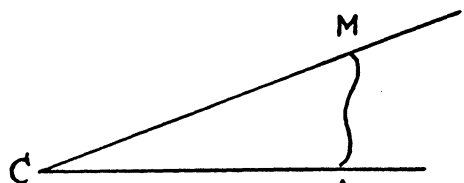
$$F_{y'}\Big/_a = \frac{yy'}{\sqrt{1 + y^2 y'^2}}\Big/_a = 0.$$

Since $y > 0$, we have $y' = 0$. Hence in the (x, y) plane $\frac{dy}{dx} = y' \frac{du}{dx} = 0$. The tangent to the circle at B is then parallel to $0\,x$. This, with the isoperimetric condition, fixes the circle.

(i). *End-point, B, not fixed.*

If the end-point, B. is free to move on the curves, $\varphi\ (x, y) = 0$ or to move arbitrarily, the isoperimetric and non-isoperimetric problems will be equivalent in the two cases above given. For, for each particular B, there is a 1 : 1 correspondence of the type described in (e), and therefore the same is true of the totality of points, B. It might appear at first sight that if $(x_0\ y_0)$ were free to move on $\psi\ (x, y) = 0$, the con dition (β : ii) need not be modified to (β_0 : i). But from (10), since $x_1 - x_0 = \int_0^a \frac{du}{y}$ along $\pounds''$, the difference is arbitrary. Hence at least one of the end-points, B or M, must be free to move parallel to $0\,x$.

§2. *Sector of Shortest Arc.*



(Fig.2)

Under similarly indefinite conditions, Euler solves the problem: "*Eductis ex puncto, C, radiis CM, CA; intra eos describere curvam, AM, quae pro dato spatio, ACM, habeat arcum, CM, brevissimum.*"[2] Here again the point, A, may be fixed, free to move on any curve, or to move arbitrarily. Transforming as in §1, we find it necessary to suppose, in order that the problems may be equivalent, that at least one end-point, M, is free to move on a circle about C as centre, or to move arbitrarily. The solution is again a circle through C and A, the third condition being determined as in §1, (g).

§ 3. *General Non-parameter Case.*

(a). Turning to the general non-parameter case, we are required to minimize: $I = \int_{x_0}^{x_1} F\left\{x, y, \frac{dy}{dx}\right\}.\ dx$ ....(11)

[1] Bolza, Variations, p. 36.

[2] Euler, l.c., pp. 144-5.

for a set of curves, $\mathcal{L}$, $y = f(x)$, joining $(x_0\ y_0)$ to $(x_1\ y_1)$, satisfying certain continuity conditions, and the isoperimetric condition,

$$K = \int_{x_0}^{x} G\left\{x, y, \frac{dy}{dx}\right\}. dx = 1 \quad \text{.........................}(12)$$

We suppose that at least one end-point, (x_1, y_1) is free to move in a straight line, $y = y_1$, parallel to $O\,x$. The other end-point, (x_0, y_0), may be fixed, free to move on a curve, or unrestricted in position. But a minimum in the last two cases will also be a minimum for the subset of curves passing through the end-points of the minimizing curve. Hence we find the extreme in all cases if we find it for (x_0, y_0) fixed. We suppose that $G\ (x, y, \lambda) > 0$ for every finite x, y.[1]

(b). *Transformation from the (x, y) to the (u, y) plane.*

As in § 1, we introduce the new variable, u by

$$u = \int_{x_0}^{x} G\left\{x, y, \frac{dy}{dx}\right\}. dx \quad \text{.........................}(13)$$

Since $G > 0$, we may solve this for x as a function of u, subject to certain continuity conditions. Substituting in $y = f(x)$, we obtain y as a function of $u = y\ (u)$, satisfying certain continuity conditions and

$$x'.\ G\left\{x, y, \frac{y'}{x'}\right\}. du = 1 \quad \text{.........................}(14)$$

where $x' = \dfrac{dx}{du}$, etc. The integral, I, transforms into

$$I' = \int_{0}^{l} x'\ .\ F\left(x, y, \frac{y'}{x'}\right)\ .\ du \quad \text{.....................}(15)$$

along a path, $\mathcal{L}''$, in the (u, x, y)—space, satisfying (15) and certain continuity conditions. If we can eliminate x from (15) by means of (14)

$$I' = I'' = \int_{0}^{l} F\ (y,\ y')\ .\ du \quad \text{..........}\qquad \text{...........}(16)$$

along a path, $\mathcal{L}'$, in the (u, y)—plane. Conversely, if we can solve (15) for x' as a function of x, y, y', and therefore of x and u, and if the continuity conditions are such as to permit, $x' = f\ (x, u)$[2] to be integrated, we can obtain $x\ \ u)$ satisfying (14) and such that $x_0 = x\ (0)$. Since $G > 0$, we have $x'\ (u) > 0$. We can therefore solve $x = x\ (u)$ for u as a function of x. Substituting in I'', we obtain

$$I'' = I = \int_{x_0}^{\xi} F\left(x, y, \frac{dy}{dx}\right) \quad \text{.........................}(17)$$

[1] If $G\,(x, y) < 0$, we change the sign of G and l.
See Picard, Traité d'analyse, II, chap. 11.

for a path in the (x, y)—plane from (x_0, y_0) to (ξ, y_1) where $\xi = x$ (a). The problems of finding a maximum for I among the curves, $\mathfrak{L}$, and I'' among the curves, $\mathfrak{L}''$, under suitable continuity conditions, are therefore equivalent.

(c). It is evident that the method will be effective also if H contains $u = \int_{x_0}^{x} G\left(x, y, \frac{dy}{dx}\right) dx$; i.e., if the integrand of (17) is of the form, $H(u, y, y')$, and F is of the Form $F\left\{\int_{x_0}^{x} G \,.\, dx, x, y, \frac{dy}{dx}\right\}$.

§4. *Example.*

As an example, illustrating the last remark and at the same time showing the necessity of definitely formulating the end-point conditions, let us take a third problem of Euler's: "*Inter omnes curvas isoperimetras, definire eam in qua sit $\int s^n \,.\, dx$ maximum ver minimum, denotanti s arcum curvae abscissae x respondentem.*" We suppose one end-point fixed and take it for origin, the other is free to move on $x = x_1$. We assume that our totality of admissible curves is the totality, $\mathfrak{L}$,

$$\mathfrak{L} : y = f(x),$$

where: (α) $f(x)$ is of class C'';

$$(\beta)\; K = \int_0^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx = 1 \quad \text{.......................... (19)}$$

We have to minimize, $I = \int_0^x s^n \,.\, dx$ .... (19)

Transforming as in § 1, to s as independent variable by the equation

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx, \quad \text{......... .... (20)}$$

we obtain, $I = I'' = \int_0^l s^n \,.\, x' \,.\, ds$ .... (21)

to be minimized for a totality, $\mathfrak{L}''$, in the (s, x)—plane:

$$\mathfrak{L}'' : x = x(s), \text{ for } 0 \leq s \leq 1,$$

where: (α'') $x(s)$ is of class C'';

(β'') $0 < x'' \leq 1$, ... (22)

and $\mathfrak{L}''$ joins the end-points, $(0, 0)$ and (l, x_1). Making the transformation of I'' back into the (x, y) plane by:

$$y' = \sqrt{1 - x'^2}, \quad \text{.......................... (23)}$$

it is not difficult to show that the problem of minimizing I for the totality, $\mathcal{L}$, and I'' for the totality, $\mathcal{L}''$, are equivalent. Applying the usual method to the latter problem, we obtain the solution,[1]

$$s = \text{const.}$$

This is not an admissible solution, since in $\mathcal{L}$, $\frac{ds}{dx} \geqq 1$. There is therefore no solution under the initial conditions specified. With unspecified conditions, Euler obtains a solution.[2]

§ 5. *General Parameter Case.*

(a). From the preceding, it might seem that the introduction of u as independent variable was equally essential with the possibility of eliminating x or y in the transformed integral; and therefore that the condition, $G > 0$ or $G < 0$, is indispensable to the success of the method. But when we permit ourselves to use parameter representation, this restriction may be removed. Denoting now by x', y', derivatives with respect to the curve parameter, t, we suppose given totality of admissible curves to be:

$$\mathcal{L} : x = \varphi(t), y = \psi(t),$$

joining 0 ($t = t_0$), and 1 ($t = t_1$), satisfying certain continuity conditions and

$$K = \int_{t_0}^{t_1} G(x, y, x', y') \,.\, dt = 1 \quad\text{(24)}$$

We assume again that one end-point, 0 (x_0, y), is fixed and that the other, 1, is free to move on $y = y_1$. We introduce the new variable, u by

$$u(t) = \int_{t_0}^{t} G(x, y, x', y') \,.\, dt \quad\text{(25)}$$

or

$$u'(t) = G(x, y, x', y') \quad\text{(26)}$$

The method is effective if x, (or y) can be eliminated from (26) and the integral,

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt \quad\text{(27)}$$

which is to be minimized. We obtain,

$$I = I'' = \int_{t_0}^{t_1} H(u, y, y')\, dt \quad\text{(28)}$$

for a curve, $\mathcal{L}''$, joining $(0, y_0)$ and (l, y_1), and satisfying certain continuity conditions. Conversely, if the equation, (26), can be solved for x' as a function of x, y, y', and therefore of x and t,

$$x' = f(x, t),$$

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[1] Bolza, Variations, p. 20.

[2] Euler, l.c., p. 208.

and the continuity conditions permit of the integration of this equation, we may obtain an $x(t)$ satisfying (26), and such that $x(t_0) = x_0$ Substituting for u' in I″ from (25), we obtain the integral, I, along a path, $\mathfrak{L}$, from (x_0, y_0) to $\left\{ x(t_1), y \right\}$. Under suitable continuity conditions the isoperimetric problems in the (x, y) plane, and the non-isoperimetric problem in the (u, y) plane are equivalent.

§6. *General Remarks.*

(a). In the first example of Euler's which we have given, (see §§ 1, 2) the set of admissible curves in the (u, y) plane contains the whole set of curves of class C′ in a certain region containing the end-points. Variations of the type used in the proof of the fundamental lemma on which Euler's differential equation depends can therefore be constructed.[1] In the third problem, however, the transformed curves are subject to the slope condition $0 < | x'(s) | \leq 1$. The question therefore arises whether it is possible to construct a variation of the required type without violating this condition. Indeed, if we use parameter representation and admit a corner in this problem we find in the (u, y)—plane that an admissible curve in the (s, x)—plane satisfies the conditions,

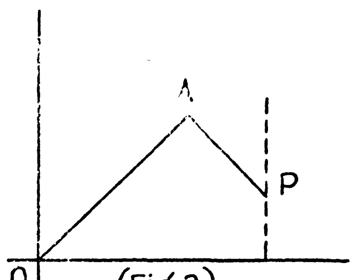
$$\mathfrak{L}'' : x = x(s), \; s \text{ on } (0, l).$$

where:

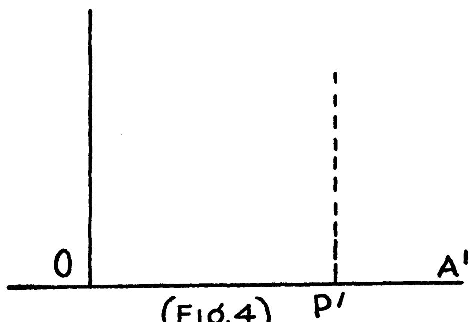
(α) $x(s)$ is of class D′;

(ρ) $0 < | x'(s) | \leq 1$;

(λ) $x(0) = 0, \; x(l) = x_1$.



(Fig.3)



(Fig.4)

The broken line, OAP, constructed as in fig. 3 so that $| x' | = 1$ along it is therefore an $\mathfrak{L}''$. But in order to obtain such a solution, the method employed is to vary OA, keeping AP fixed and vice versa.[2] It is evidently impossible to construct such a variation without violating the condition

$$| x' | \leq 1.$$

[1] See Bolza, Variations, §5.

[2] See Bolza, l. c., §9.

(b). Referring again to Euler's first two examples, (§§ 1, 2), in order that *the area between the arcs and lines shall have an arithmetic meaning*, we must suppose the curve made up of these has no double points. In the first case, for example, this is ensured by the fact that BM is of type, $y = f(x)$, and $y > 0$, conditions assumed for other reasons. On the other hand, if we use parameter representation, permitting the curve to turn back on itself, the inequalities equivalent to the non-existence of double points may be expected to take a somewhat complicated form in the transformed problem. The first question to be settled, therefore, is whether or no there exists in the transformed set of curves a variation of the type required in the fundamental lemma. Again, if this question can be answered in the affirmative, the transformed double point condition may be expected to play an important role in proving that the extremal so discovered actually furnishes a minimum.

Similar remarks apply to any problem in parameter representation in which there is a distinction between inside and outside points. Our purpose is to discuss these questions in detail for the *solid of revolution of maximum attraction*.

CHAPTER II.

THE SOLID OF MAXIMUM ATTRACTION.

§ 1. *Historical.*

As stated in the last paragraph, our object in the present chapter is to apply this method of Euler's in detail to determine the form of the solid of maximum attraction. More explicitly, *given a quantity of homogeneous matter, bounded by a surface of revolution and attracting, according to Newton's law, to find the form of the generating curve in order that the attraction upon a particle on the axis of revolution and in contact with the surface shall be a maximum.*

(b). The problem is first mentioned by *Gauss*,[1] (1830), in a paper on capillary attraction, where it is stated without proof that the maximum attraction is to that of a sphere of equal mass as $3 : \sqrt[3]{25}$.

(c) The first discussion of the problem seems to have been given by *Airy*.[2] He takes the attracted particle as origin for a system of rectangular co-ordinates, the x—axis coinciding with the axis of revolu-

[1] Gauss, Ges. Werke, V, s. 31.

[2] Airy, Math. Tracts: p. 309; Airy's solution is reproduced by Jellett (1850, Variations, p. 307), Todhunter (1871: Researches, p. 120), and Carll (1881: Variations, p. 141). Todhunter computes also the second variation for $I + \lambda K$, which is negative.

tion. Assuming the admissible curves in the form, $y = f(x)$, he obtains for the attraction the definite integral, (laying aside constant factors):

$$I = \int_{x_0}^{x_1} \left\{ 1 - \frac{x}{\sqrt{x^2 + y^2}} \right\} dx \quad \text{.........................(1)}$$

and for the mass

$$K = \int_{x_0}^{x_1} y^2 \, . \, dx, \quad \text{.........................(2)}$$

The application of the ordinary rules of isoperimetric problems leads immediately to the equation of the meridian curve of the maximizing solid in the form,

$$2 \lambda \left(x^2 + y^2\right)^{\frac{3}{2}} + x = 0 \quad \text{.........................(3)}$$

the constant, λ, being determined by the mass.

(d). *Moigno-Lindelöf*,[1] (1861), using polar co-ordinates, and assuming the admissible curves in the form, $r = f(\theta)$, obtains for the two integrals the values,

$$I = \int_0^{\theta_0} r \, . \sin \theta \cos \theta \, d\theta, \quad K = \int_0^{\theta_0} r^3 \, . \sin \theta \, . \, d\theta, \quad \text{...........(4)}$$

and, again by the ordinary method of isoperimetric problems, for the meridian curves,

$$r^2 = a \cos \theta, \quad \text{.........................(5)}$$

(e). Finally, *Kneser* (1900), in his *Lehrbuch der Variationsrechnung*[2], reduces the problem to a non-isoperimetric problem by means of Euler's artifice, which we have discussed in the preceding chapter. Starting from polar co-ordinates, and assuming the admissible curves in the form, $r = f(\theta)$, he transforms the integrals, (4), by the substitutions,

$$u = \cos \theta, \; v = \int_0^{\theta_0} r^3 \, . \sin \theta \, . \, d\theta \quad \text{.........................(6)}$$

Under this transformation,

$$I = \int_{u_0}^{l} \left(\frac{dv}{du}\right)^{\prime/} u \, . \, du \quad \text{.........................(7)}$$

and the isoperimetric condition reduces to an identity. He finds as the solution in the (u, v) plane,

$$v = c \, . \, u^{5/2} \quad \text{.........................(8)}$$

to which corresponds in the (r, θ)—plane the curve, (5).

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[1] Moigno-Lindelöf—Calcr d. Var., p. 244; reproduced by Dienger (1867) Variationsrechnung, s. 61), who also discusses the Legendre condition.

[2] Kneser, l. c., s. 28.

(f). Kneser does not go beyond the consideration of the first variation, and a discussion of the sufficient conditions has never been given by means of the Calculus of Variations.[1] Moreover, the previous treatments of the problem are open to certain minor objections. The assumption of the admissible curves in one of the forms:

$$r = f(\theta) \text{ or } y = f(x),$$

involves a restriction which is not justified by the nature of the problem. Again, the conditions at the end-points, one of which is variable, have not been discussed. Further, the extremal, (3), furnished by the method of Airy, ceases to satisfy at the two points at which it meets $0x$ the conditions of continuity under which the general theory can be applied to the solution. The same remark applies to the solution (5), since at the attracted particle $\dfrac{dr}{d\theta} = \infty$. Kneser's solution is not open to this objection, but here another difficulty arises. A close analysis shows that the passage from the (r, θ) to the (u, v) plane involves a number of restrictions upon the slope of the curves in the (u, v) plane. As we have remarked in Chap. I, §6 (a), the question arises whether it is possible to secure a variation of the type required from this restricted set, a question which must be discussed in some detail; which we proceed to do.

§ 2. *Detailed Formulation of the Problem.*

(a). We use parameter representation to obtain the desired degree of generality. The attracted particle being chosen as origin for a system of rectangular co-ordinates, of which Ox is the axis of revolution, we suppose that the meridian curve, $\mathfrak{L}$, is given in the form

$$\mathfrak{L} : x = \varphi(\tau),\ y = \psi(\tau), \text{ for } \tau_0 \leqq \tau \leqq \tau_1,$$

where φ and ψ are of class D′, meeting Ox at $\tau = \tau_0$, and $\tau = \tau_1$; and at these points only. We also assume that $(yx' - xy')$ does not change sign an infinitude of times; or in polar co-ordinates, that $\dfrac{d\theta}{d\tau}$ does not change sign an infinitude of times. In other words, $\mathfrak{L}$, can be divided into a finite number of arcs on which,

i) θ is an increasing function of τ, or
ii) θ is a decreasing function of τ, or
iii) θ is constant.

[1] A sufficiency proof has, however, been given by *Schellbach*, (1851, Journal für Math., XLI, s. 343), by a method not belonging to the calculus of variations. By means of the force of attraction resolved along the axis of revolution a one-parameter set of surfaces is obtained, each dividing space into a region of greater and less resolved attraction. For an absolute maximum the bounding surface must be one of these, the parameter being determined by the mass condition.

The last occurs when $xy' - xy' \equiv 0$ on a certain interval. This arc, if produced, passes through O. We also suppose that lim. $\frac{x}{y}$ exists[1] as we approach the attracted particle; this is equivalent to assuming the existence of the tangent at this particle. We do not, however, assume the existence of a tangent at any other point. We assume also that the meridian curve has no double points. This last hypothesis, as remarked in Chap. I, §6 (b), enables to distinguish arithmetically the interior and exterior of the bounding surface. Without it we may indeed show that the attraction integral may be made infinite without violating the isoperimetric condition.[2]

(b). Inasmuch as the matter is homogeneous, the mass will be constant if the volume is. A convenient method of obtaining the latter, and also the attraction integral, is to find the effect of small conical shells, vertex the origin, cut out by right circular cones about the axis of revolution. If A denote the area and V the volume, $\frac{dA}{d\tau} = yx' - xy'$, x' and y denoting derivatives with respect to. Hence by Guldin's theorem,

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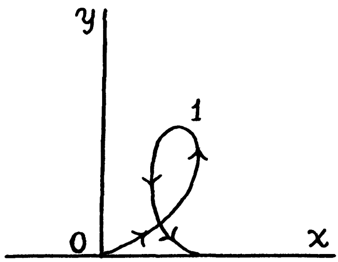
[1] By "exists" is meant that $\frac{\varphi}{\psi}$ approaches a determinate value, whether finite or infinite.



(Fig. 5)

[2] When we are unable to distinguish the interior from the exterior, the attraction of part of the matter as given by A (see b), may become negative; e.g., the contribution of the loop, 1, in fig. 5; a state of affairs that may be realized physically in statical electricity, at least in the imagination. In such a case, it is evident that we can make the attraction as great as we please by placing the positive electricity in sufficiently great quantities near the negative particle, and enough negative electricity to satisfy the isoperimetric condition at such a great distance to render its attraction infinitesimal. More explicitly, suppose that a charge, e_1 and e_1 are to satisfy the isoperimetric condition, $e_1 - e_2 = \omega > 0$. Select any positive r, $r_2 > \frac{\omega}{A}$ and any R, $R > 2r$; an $e_1 = 2r^2 A$, and $e_2 = 2r^2 A - \omega$. Then charges, e_1, and $- e_2$ on two spheres whose centres are collinear with the attracted particles and at distances, r, R, from it, exert an attraction

$$F = \frac{e_1}{r^2} - \frac{e_2}{R_2} = \frac{2r^2 A}{r^2} - \frac{2r^2 A - \omega}{R^2} = 2A - \frac{\omega}{R^2} > A.$$

Now, A is any arbitrary quantity. Hence F can be made as great as we please. The absence of such loops is therefore an essential condition for the existence of a maximum.

$\frac{dV}{d\tau} = \frac{2\pi}{3}(yx' - xy')$, whence the volume may be taken as defined by :

$$V = \frac{2\pi}{3} \int_{\tau_0}^{\tau_1} . \; y \; (yx' - xy') \; . \; d\tau.$$

Summing in a similar fashion the attraction integral takes the form :

$$A = \mu \int_{\tau 0}^{1} \frac{xy}{r^3} (yx' - xy') \; . \; d\tau,$$

where $r = \sqrt{x^2 + y^2}$, and μ is a constant factor depending upon the constant of gravitation. Dropping constant factors, we have to find a

maximum value for $I = \int_{\tau_0}^{\tau_1} \frac{xy}{r^3} \; (yx' - xy') \, d\tau$ ....(9)

subject to the condition,

$$K = \int_{\tau 0}^{\tau_1} y \; (yx' - xy') \, d\tau = \omega \quad \text{......................(10)}$$

ω being a certain positive constant.

(c). Stating these hypotheses arithmetically, and collecting them for purposes of reference, we propose to find a maximum for the

integral I, $\qquad I = \int_{\tau_1}^{\tau_1} \frac{xy}{r^2} \; (yx' - xy') \, d\tau$ for the admissible curves,

$$\mathfrak{L} : x = \varphi\;(\tau), \; y = \varphi\;(\tau) \text{ in } (\tau_0 \; \tau_1,$$

where I : *General Characteristics :*

(a) φ and ψ are of class D′ on $(\tau_0 \; \tau_1)$;

(b) $\varphi\;(\tau_2) = \varphi\;(\tau_3)$, and $\psi\;(\tau_2) = \psi\;(\tau_3)$, cannot both be true if $\tau_2 \pm \tau_3$;

II : *Slope Condition :* $yx' - xy' = 0$,

(a) at a finite number of points, $\tau = \gamma$;

(b) on a finite number of segments, $\kappa_\iota \leq \tau \leq \lambda_\iota$, $\iota = 1, 2, 3 \ldots$,

III : *Initial Conditions :*

(a) $\varphi\;(\tau_0) = 0, \; \psi\;(\tau_0) = 0, \; \left. \frac{\varphi\;(\tau)}{\psi\;(\tau)} \right|_{\tau 0 + 0}$ exists ;

(b) $\varphi\;(\tau_1) = 0, \; \psi\;(\tau_1) = 0$;

IV: *Regional Conditions:*

$$\begin{rcases} \text{(a)}\ \varphi(\tau) > 0 \\ \text{(b)}\ \psi(\tau) > 0 \end{rcases} \text{ for } \tau_0 < \tau < \tau_1;$$

V: *Isoperimetric Condition:*

$$K = \int_{\tau_0}^{\tau_1} y\,(yx' - xy')\,.\,d\tau = \omega.$$ [1]

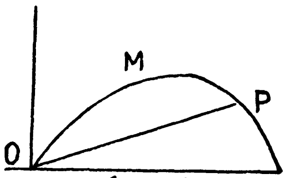
§3. *Reduction to an Non-isoperimetric Problem.*

(a): We are now going to reduce our problem to the non-isoperimetric type by means of the transformation:

$$U(\tau) = \frac{x(\tau)}{y(\tau)},\ V(\tau) = \int_{\tau_0}^{\tau} y\,(yx' - xy')\,d\tau \ldots\ldots\ldots\ldots\ldots(11)$$

The quantities, U and V, have a simple geometric meaning, viz.,

$$U = \cos\theta.$$



(Fig. 6)

(see fig. 6) and V is the volume obtained by revolving the sector, OMP, about the x axis. The transformation, (12), co-ordinates with every curve, $\mathfrak{L}$, in the (x, y) plane a curve, $\mathfrak{L}'$, in the (U, V) plane, whose properties we have to study.[2]

(b). *The function, $r\,(t)$*

By (IV : b), and (III : b), we have

$$r(\tau) > 0 \text{ for } \tau_0 < \tau < \tau_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Since φ and ψ are of class D′, and $r > 0$ for $\tau_0 < \tau \leq \tau_1$, we have that $r(\tau)$ is of class D' for $\tau_0 < \tau \leq \tau_1$, and we have

$$r' = \frac{x}{r}\,x' + \frac{y}{r}\,r' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

an equation that remains true at the points, δ, if we understand by these the progressive (regressive) derivatives. Now, from (III : a), as $\tau \doteq \tau_0$, $\frac{x}{y}$ approaches a definite limit, finite or infinite. Since $y > 0$ for τ, $\tau_0 < \tau < \tau_1$, we may write,

$$\frac{x}{r} = \frac{\frac{x}{y}}{\sqrt{1 + \left(\frac{x}{y}\right)^2}} \qquad \frac{y}{r} = \frac{1}{\sqrt{1 + \left(\frac{x}{y}\right)^2}}$$

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[1] The discontinuities of φ' and ψ' we call the points, δ.

[2] The results of this discussion will be found tabulated on p. 59.

Hence as $\tau \doteq \tau_0$, $\frac{x}{r}$ and $\frac{y}{r}$ approach positive limits, ≤ 1. Hence it follows that the progressive derivative exists at τ_0, and is equal to $r'(\tau_0 + 0)$.[1] Similarly at τ_1. Hence $r(\tau)$ is of class D′ on $(\tau_0\ \tau_1.)$

(c). *The function,* $U(\tau)$

The function, $U(\tau)$, is not defined for τ_0 as $x(\tau_0) = 0$ and $r(\tau_0) = 0$. But we have just seen that $\frac{x}{r}\Big/_{\tau_0 + 0}$ exists and ≤ 1. We define $U(\tau)$ as this limit, U_0. With this agreement, $U(\tau)$ is determinate and continuous on $(\tau_0\ \tau_1)$. By (IV : a, b)

$$0 \leq U < 1 \text{ for } \tau_0 < \tau < \tau_1 \quad \text{......(14)}$$

and by (III: b), $U(\tau_1) = 1$. Since $r(\tau) > 0$ for $\tau_0 < \tau \leq \tau_1$, the derivative U', exists and has the value,

$$U' = \frac{y\,(yx' - xy')}{r^3} \quad \text{......(15)}$$

for $\tau_0 < \tau \leq \tau_1$, understanding by U' the progressive (regressive) derivatives at the points, δ. As $\tau = \tau_0$, $U'(\tau_0 + 0)$ is in general indeterminate. From II, $U' = 0$ at a finite number of points, $\tau = \lambda$, and $\equiv 0$ on a finite number of segments, $\kappa_\iota \leq \tau \leq \lambda_\iota$, $\iota = 1, 2, 3$.

(d). *The function* $V(\tau)$.

According to (I : a), $y\,(yx' - xy')$ is integrable,[2] and has at every point of $(\tau_0\ \tau_1)$ finite and determinate right and left hand derivatives. Hence the function, $V(\tau)$, is of class D′ on $(\tau_0\ \tau_1)$ and

$$V'(\tau) = y\,(yx' - xy'), \quad \text{......(16)}$$

with an agreement as to the points of discontinuity similar to that in (b). From (15) and (16), it follows that

$$\frac{V'^{\frac{1}{3}} \cdot U'^{\frac{2}{3}}}{\sqrt{1 - U^2}} = \frac{y}{r} \cdot x' - \frac{x}{r} \cdot y', \text{ for } \tau_0 < \tau < \tau_1 \quad \text{......(17)}$$

As $\tau \doteq \tau_0 + 0$, and $\tau \doteq \tau_1 - 0$, the right-hand side of (17), and therefore also the left-hand side, approaches finite determinate limits.[3] We have from V that $V(\tau_1) = \omega$. Further as for U' in (c), $V' = 0$ at a finite number of points, and $\equiv 0$ on a finite number of segments.

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[1] See Dini, Grundlagen, etc., §68.

[2] c.f. Dini, l. c., §181 (2).

[3] Dini, l.c., § 191 (2).

(e). *The slope, P, of the (U, V) curve.*

From (c) and (d), the slope of the curve,

$$\mathcal{L}' : u = U(\tau),\ v = V(\tau) \quad \text{.......................(18)}$$

i.e., the function, $P = \dfrac{V'}{U'}$ ....(19)

has a determinate finite value, viz.,

$$P = r^3 \quad \text{...(20)}$$

on $(\tau_0\ \tau_1)$ except at $\tau = \tau_1$, $\tau = \gamma$ and the segments, $\kappa_i\ \lambda_i$, and, it may be, $\tau = \tau_0$. We define $P(\tau)$ at these points as $r(\tau)$. With this agreement, the slope, P, is continuous, even at the points, δ, and, where the latter is defined, P coincides with V'/U'. Since $r(t) > 0$ for $\tau_0 < \tau \leq \tau_1$, (see 12), $P(\tau) > 0$ for $\tau_0 < \tau \leq \tau_1$. Hence $P(\tau)$, $= r^{1/3}$, is of class D′ for $\tau_0 \leq \tau \leq \tau_1$. Now for $\tau_0 < \tau \leq \tau_1$, $r' = \frac{1}{3}\ P \,.\, P'$. Hence $P.P'/_{\tau_0+0}$ exists and is finite, being equal to $r'(\tau_0 + 0)$.

(f). *The double point condition.*

From (I : b), if r and θ be polar co-ordinates, we have $r(\tau_2) = r(\tau_3)$, and $\theta(\tau_2) = \theta(\tau_3)$ cannot both be true if $\tau_2 \neq \tau_3$. Since $P = r^3$, and $U = \cos\theta$, for $0 \leq \theta \leq \pi/_2$, we have $P(\tau_2) = P(\tau_3)$, and $U(\tau_2) = U(\tau_3)$ cannot both be true if $\tau_2 \neq \tau_3$.

(g). Collecting these results, our family of admissible curves, $\mathcal{L}$, in the (x, y) plane transforms into the totality, $\mathcal{L}'$, in the (u, v) plane :

$$\mathcal{L}' : u = U(\tau),\ v = V(\tau) \text{ for } \tau_0 \leq \tau \leq \tau_1.$$

with the following properties :

I′ : *General Characteristics* :

(a) $V(\tau)$ is of class D′ on $(\tau_0\ \tau_1)$;

(b) $U(\tau)$ is of class C on $(\tau_0\ \tau_1)$, and D′ for $\tau_0 < \tau \leq \tau_1$.

II′ : *Slope Conditions :*

(a) V' and U' vanish at most at a finite number of points, $\tau = \gamma$, and on a finite number of segments, $\kappa_i\ \lambda_i$, $= 1, 2, 3$.

(b) P is of class C on $(\tau_0\ \tau_1)$, and D′ for $\tau_0 < \tau \leq \tau_1$, and $P = V'/_{U'}$ when this quotient is defined.

(c) $P(\tau_2) = P(\tau_3)$ and $U(\tau_2) = U(\tau_3)$, cannot both be true if $\tau_2 \neq \tau_3$;

III′ : *Initial Conditions:*

(a) $V(\tau_0) = 0$, and $V(\tau_1) = \omega$;

(b) $U(\tau_0) = U_0 \leq 1$, $U(\tau_1) = 1$;

(c) $\dfrac{V'^{1/3}\ U'^{2/3}}{\sqrt{1-U}}$ $\dfrac{V'^{1/3}\ U'^{2/3}}{\sqrt{1-U^2}}$ and $P^{2/3} . P'^{1/3} \Big/_{\tau_0 + 0}$ exist, and are finite.

IV′ : *Regional Conditions:*

(a) $0 \leq U \leq 1$ for $\tau_0 < \tau < \tau_1$;

(b) $P(\tau) > 0$ for $\tau_0 < \tau \leq \tau_1$.

The isoperimetric condition becomes:

$$K = \int_{\tau_0}^{\tau_1} V' . dt = V(\tau_1) - V(\tau_0) = \omega.$$

by (III′ : a), and is therefore satisfied by all the curves, $\mathfrak{L}'$. The integral, I, takes the form

$$I' = \int_{\tau_0}^{\tau_1} V'^{1/3} . U'^{2/3} . U . dt \quad \text{.......................................} (21)$$

Our isoperimetric problem is thus reduced to the non-isoperimetric problem of finding a maximum value of I' among the totality, $\mathfrak{L}'$, one end-point, $(1, \omega)$, being fixed, and the other end-point, $(U_0, 0)$, being free to move on the axis, $v = 0$.

(h). Conversely if any curve,

$$\mathfrak{L}' : u = U(\tau), v = V(\tau), \quad \text{..................................} (22)$$

is given, satisfying the conditions set down in (i), and we define:

$$\varphi(\tau) = P^{1/3} . U, \ \psi(\tau) = P^{1/3} . \sqrt{1 - U^2},$$

it is not difficult to prove that the curve:

$$x = \varphi(\tau), y = \psi(\tau),$$

belongs to the set of curves, $\mathfrak{L}$, given in (c) and furnishes for the integral,

$$I = \frac{xy}{r^3} (yx' - xy') . dr$$

the same value as the curve, $\mathfrak{L}'$, does for the integral,

$$I' = \int_{\tau_0}^{\tau_1} V'^{1/3} . U'^{2/3} . U . dr$$

Hence *the problems of finding the isoperimetric maximum of I for the totality of curves, £, and of finding the ordinary maximum of I′ for the totality of curves, £′, are equivalent.*

§4. *Removal of Stationary Points.*[1]

(a). We have remarked in §3 (a) that the segments, on which $yx' - xy' = 0$ are straight lines whose direction passes through the origin. From the definitions of $U(\tau)$ and $V(\tau)$ given by the equations (11), it follows that these segments in the parameter, τ, correspond to stationary points on the curve, $£'$, given by (22). In order to eliminate these we introduce a new parameter, t, by means of the following substitutions:

$\tau = t$, for τ on the first interval not a $\kappa\lambda$—interval,

$\tau = t + d_1$, for τ on the second interval not a $\kappa\lambda$—interval,

$\tau = t + d_1 + d_2$, for τ on the third interval not a $\kappa\lambda$—interval,

where $d_i = \lambda_i - \kappa_i$, i being determined as follows:

i) when τ_0 is not an end-point of a $\kappa\lambda$—interval, $i = 1, 2, 3 \ldots$;

ii) when τ_0 is not an end-point of a $\kappa\lambda$—interval, $i = 1, 2, \ldots$

Geometrically, we delete the segments of the parameter, τ, corresponding to the stationary points of $£'$, and in order to remove the resulting gaps we make a simple translation of the parameter. We suppose that the new parameter, t, has the range, t_0, t_1, and denote by k_1, k_2, k_3, the values of t at the points of junction, viz:

(i) $k_1 = \kappa_1$; $k_2 = \kappa_2 - d_1$; $k_3 = \kappa_3 - d_1 - d_2$;

(ii) $k_1 = \kappa_2$; $k_2 = \kappa_2 - d_1$; $k_3 = \kappa_3 - d_1 - d_2$;

With these relations we define:

$$u(t) = U(\tau),\ v(t) = V(\tau),\ p(t) = P(\tau), \ldots\ldots\ldots\ldots \quad (23)$$

It follows that the curves,

$$\left.\begin{array}{ll} & u = U(\tau),\ v = V(\tau),\ \text{for } \tau \text{ on } (\tau_0\ \tau_1 \\ \text{and} & u = u(t),\ v = v(t),\ \text{for } t \text{ on } (t_0\ t_1 \end{array}\right\}$$

are identical as locus curves, and $p(t)$ is the slope of the latter. In case (ii), and this case only, $p(t_0) = P(\tau_1) \neq 0$ by (IV′: b). Further, $p(k_i - 0) = P(\kappa_i)$, or $P(\kappa_{i+1})$, and $p(k_i + 0) = P(\lambda_i)$ or $P(\lambda_{i+1})$ in cases (i) and (ii) respectively. Since $V'(\tau) \equiv 0$ on $(\kappa_1\ \lambda_1)$ in the last case from (III′: a), $V(\lambda_1) = v(t_0) = 0$.

It is to be noted further from the values of U' and V' given by equations (15) and (16) that the segments, $(\kappa_i\ \lambda_i)$, contribute nothing to the integral,

$$I' = \int_{\tau_0}^{\tau_1} V'^{\frac{1}{3}} .\ U'^{\frac{2}{3}} .\ U .\ dr.$$

[1] The results of this discussion are given on p. 14.

It follows that

$$I' = \int_{t_0}^{t_1} v'^{\,1/3} \, . \, u'^{\,2/3} \, . \, u \, . \, dt.$$

It remains to enunciate the conditions given in (g) of the preceding paragraph in terms of $u\ (t)$ and $v\ (t)$.

(b). In the first place we have from (I') that $v\ (t)$ is of class D′ on $(t_0\ t_1)$, and $u\ (t)$ is of class C on this interval, and except perhaps for $t = t_0$, of class D′ on the same interval. From (II′ : a), $u'\ (t)$ and $v'\ (t)$ can vanish at only a finite number of points, $t = g$. The function, $p\ (t)$, will not, in general, be continuous; from (II′ : b), however, it must be of class D′ on the intervals, $(t_0\ k_1)$, $(k_1\ k_2)$, $(k_n\ t_1)$, except perhaps for $t = t_0$. In (ii) of the preceding paragraph, as τ covers the interval, $\kappa_1\ \lambda_1$, $P\ (\tau)$ takes all values from 0 to $P\ (\lambda_1) = p\ (t_0)$, inclusive. Hence (II″ : c_3) is the equivalent of (II′ : c) in this case. In the remaining cases as τ describes the interval, $(\kappa_i\ \lambda_i)$, $P\ (\tau)$ takes all values between $p\ (k_{i-1} - 0)$ and $p\ (k_{i-1} + 0)$ or $p\ (k_i - 0)$ and $p\ (k_i + 0)$ according to whether it belongs to (i) or (ii); a remark which makes (II″ : c_1, c_2) the equivalent of (II′ : c) in these cases. If we write $u\ (t_0) = u_0$, it follows from (IV′) that in (ii) of the preceding paragraph, *i.e.*, when $p\ (t_0) \neq 0$, $u_0 < 1$. In this case also it follows that from (I′ : b) that $u\ (t)$ is of class D′ up to and including $t = t_0$. The remaining conditions are translated easily.

(c). Collecting these results, the family of admissible curves, $\mathfrak{L}'$ in the (u, v)—plane transforms into a set of curves included in the totality, $\mathfrak{L}''$, with the following properties:

I″: *General Characteristics :*

(a) $v\ (t)$ is of class D′ on $(t_0\ t_1)$;

(b) $u\ (t)$ is of class C on $(t_0\ t_1)$, of class D′ for $t_0 < t \leq t$, and if $p\ (t_0) \neq 0$, for $t = t_0$;

II″: *Slope Conditions :*

(a) v' and u' vanish at only a finite number of points, $t = g$;

(b) $p\ (t)$ is of class D′ on $(t_0\ k_1)$, $(k_1\ k_2)$ . . . . . . $(k_n\ t_1)$;

$$p = \frac{v'}{u'}, \text{ where this quotient is defined;}$$

(c_1) $p\ (t_2) = p\ (t_3)$ and $u\ (t_2) = u\ (t_3)$ cannot both be true if $t_2 \neq t_3$;

(c_2) if $u\ (t_2) = u\ (k_i)$, then $p\ (t_2)$ cannot lie between $p\ (k - 0)$ and $p\ (k + 0)$;

(c_3) if $u\ (t_2) = u_0$, then $p\ (t_2) > p\ (t_0)$ unless $t_2 = t_0$;

III$''$: *Initial Conditions :*

(a) $v\ (t_o) = 0,\ v\ (t_1) = \omega$;

(b) $u\ (t_o) = u_o,\ u\ (t_1) = 1,\ u_o < 1$ unless $p\ (t_o) = 0$;

(c) $\left.\dfrac{v'^{\frac{1}{3}}\ u'^{\frac{2}{3}}}{\sqrt{1-u^2}}\right/_{t_1 - o}$ and $\left.\dfrac{v'^{\frac{1}{3}}\ u'^{\frac{2}{3}}}{\sqrt{1-u^2}}\right/_{t_o + o}$ exist and are finite;

IV$''$: *Regional Conditions :*

(a) $0 \leq u \leq 1$ for $t_o < t < t_1$;

(b) $p\ (t) > 0$ for $t_o < t \leq t_1$.

The integral to be maximised,

$$I'' = \int_o^{t_1} v'^{\frac{1}{3}} \cdot u'^{\frac{2}{3}} \cdot u \cdot dt.$$

Conversely, if any curve, $\mathcal{L}''$, of the totality just defined, be given, and we transform the parameter, t, in a manner entirely the inverse of that used in (i), it is not difficult to see that we obtain a curve of the totality $\mathcal{L}'$, such that $\int_{\tau_o}^{\tau_1} V'^{\frac{1}{3}} \cdot U'^{\frac{2}{3}} \cdot U \cdot d\tau$ along the curve, $\mathcal{L}'$, is equal to $\int_{t_o}^{t_1} v'^{\frac{1}{3}} \cdot u'^{\frac{2}{3}} \cdot u \cdot dt$ along the given curve. It follows that the problems of finding a maximum for I$'$ among the curves, $\mathcal{L}'$, and a maximum for I'' among the curves, $\mathcal{L}''$, are equivalent.

§ 5. *Decomposition of $\mathcal{L}''$ into arcs of type, $v = f_i\ (u)$.*

(a). We denote the points, g, at which $v' = 0$, or $u' = 0$, the points, k_i, at which $p\ (t)$ is discontinuous, and the points, δ, at which p', v' and u' are discontinuous, collectively by $[\alpha]$; $\alpha_o\ (=t_o) < \alpha_1 < \alpha_2 \ldots < \alpha_{r+1}\ (= t_1)$. The corresponding values of u are $a_o\ (=u_o)$, $a_1, a_2 \ldots a_r\ 1$ (see III$''$: b). We decompose any curve, $\mathcal{L}''$, into arcs, $\mathcal{L}_i''$, and the interval, $(t_o\ t_1)$, into sub-intervals at these points, $i = 0, 1, 2, \ldots r$. Then on $\mathcal{L}_i''$ $u'\ (t)$ has a fixed sign, and $u'\ (t) \neq 0$ except perhaps at the end points. We may therefore solve for t as a function of u of class C on and C$'$ within $\mathcal{L}_i''$.[1] Substituting in $v\ (t)$ and $u\ (t)$, we obtain v and p as functions of u of class C on and C$'$ within $\mathcal{L}_i''$. Now within $\mathcal{L}_i''$ since $u' \neq 0$ and is continuous,

$$\frac{dv}{du} = {v'}/{u'}$$

[1] Dini, l. c. s.

Hence if $u' > 0$ on $\mathcal{L}_i$ as $t \doteq k + 0$, and $t \doteq ki + 1 - 0$, we have $u \doteq u_i + 0$, and $u \doteq u_{i+1} - 0$, while $\frac{dv}{du}$ approaches finite limits, viz., $p(u_i - 0)$ and $p(u_{i+1} - 0)$. It follows that the progressive derivatives of $v(u)$ at u_i, and regressive at u_{i+1} exist, and are equal to these limits. Hence $v'(u)$ is of class C' on $\mathcal{L}_i''$.[1] If we write

$$v = f_i(u), \text{ then } p = f_i'(u)$$

(h). The curve, $\mathcal{L}''$, then consists of a finite number of arcs, $\mathcal{L}_i''$, such that

(A). *Conditions on Single Arc.*

$$\mathcal{L}_i'' : v = f_i(u);$$

(1) $f_i(u)$ is of class C' in $(u_i\, u_{i+1})$, $i = 0, 1, 2, \ldots r$;

(2) $f_r(1) = \omega$;

(3) $f_i''(u) > 0$ in $(u_i\, u_{i+1})$ except that it may be that $f_0''(u_0) = 0$;

(4) $\left.\dfrac{p^{1/3}}{\sqrt{1-u^2}}\right|_{u \doteq u_0}$ and $\left.\dfrac{p^{1/3}}{\sqrt{1-u}}\right|_{u \doteq 1}$ exist and are finite,[2]

(B). *Conditions for Composition of the Arcs.*

(1) $0 \leq u \leq 1$ except on $\mathcal{L}_0''$ where, if $p(u_0) = 0$, it may be that $u_0 = 1$, and on $\mathcal{L}_r''$ where always, $u_{r+1} = 1$;

(2) (a) if $f'_i(u) = f_j''(u)$, then $i = j$, except possibly at u_{i+1} where $j = i + 1$;

(b) if $f_j''(u_{j+1}) \neq f'_{j+1}(u_{j+1})$, then $f_i''(u_{i+1})$ does not lie between them;

(c) if $f_0''(u_0) > 0$, then $f_i''(u_0) > f_0(u_0)$ for $i \neq 0$;

(3). The compound curve is continuous.

Conversely, if any curve be made up of the finite number of arcs, $v' = f_i(u)$, subject to the conditions just enunciated, it is not difficult to show that we may select a suitable parameter, (e.g. the arc-length) so as to exhibit the curve as a member of the totality, $\mathcal{L}''$. It follows that *the problems of finding a maximum of I'' taken along a set of arcs, $\mathcal{L}_i''$ and the curve, $\mathcal{L}''$, are identical.*

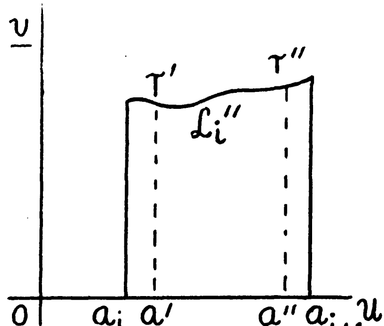
---

[1] Dini, l. c., §68.

[2] These follow from (III'' c).

§ 6. *The Extremal.*

(a). Let us suppose that some curve $\mathfrak{C}$, of the set, $\mathfrak{L}''$, (fig. 6), furnishes a maximum for I'', and that $\mathfrak{C}$ is divided into arcs, $\mathfrak{L}_0''$, $\mathfrak{L}_1''$, ... , $\mathfrak{L}_r''$, of type, $v = f_i(u)$ as in the preceding paragraph. Let τ', τ'', be any two points in the interior of $\mathfrak{L}_i''$, and $u(\tau') = a'$, $u(\tau'') = a''$. Choose a variation, $\bar{\mathfrak{L}}'' : \bar{u} = u(t)$, $\bar{v} = f_i\left\{u(t)\right\} + \epsilon\,\eta\left\{u(t)\right\}$, where



(Fig.6)

i) $\eta(u)$ is of class C′ in $(a'\ a'')$;

ii) $\eta(a') = 0$, $\eta''(a') = 0$, $\eta''(a') = 0$,
$\eta(a'') = 0$ $\eta'(a'') = 0$ $\eta''(a'') = 0$;

iii) $\eta(t) = 0$ outside of $(\tau'\ \tau'')$. Then the curve, $\bar{\mathfrak{L}}''$ $(\bar{u}, \bar{v})$, is an admissible variation; *i.e.*, belongs to the totality, $\mathfrak{L}''$, described in § 4 (c). To show this, since $\mathfrak{L}_i''$ is the only arc affected by the variation, we need only show that (non-zero) limits of $|\epsilon|$ can be fixed so small as to satisfy A (3), and B (2), of the preceding paragraph.

(b). To obtain these limits, we observe that since $\eta''(t)$ is continuous, there is an upper limit, m', for $|\eta'|$ [1]. Since $(\tau'\ \tau'')$ lies within $\mathfrak{L}_i''$, $v' \neq 0$ and $u' \neq 0$ on $(\tau'\ \tau'')$. Since these are continuous, there is a positive, v, $v < |u'|$, and $v < |v'|$, on this interval. Further, if there are any points t', on $\mathfrak{L}_j'$, $j \neq i$, such that $u(t') = u(t)$ for $\tau' \leq \tau \leq \tau''$, then the difference $|f_i''(u) - f_j''(u)|$ is a continuous function of u on $(a'\ a'')$ and $\neq 0$. Hence there exists a ρ such that

$$|f_i''(u) - f_j(u)| > \rho > 0.^{2}$$

If now ξ be any positive quantity,

$$\xi < \frac{\rho\, v}{m'},\quad \xi < \frac{v}{m'},$$

then for any ϵ, $0 < |t| < \xi$, $\mathfrak{L}''$ belongs to the set of admissible curves.

(c). Since (a) $|\epsilon| < \eta < \dfrac{v}{m'}$

(b) $|u'| < v$, $|v'| < v$

(c) $m' > |\eta'|$

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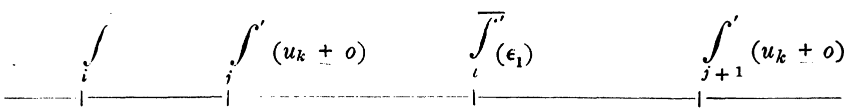
[1] Osgood Funktionentheorie, §13.

[2] Dini Grundlagen, etc., s 68, 70.

it follows that $\bar{v}' = \bar{v}' + \epsilon\,\eta'(t)$, and v' have the same sign, and do not vanish on $(\tau'\,\tau'')$. Since $\bar{f_i}'(u) = \dfrac{\bar{v}'}{\bar{u}'}$, and $f_i'(u) = \dfrac{v'}{u'}$,[1] (3) is satisfied. Since $\epsilon < \xi < \dfrac{\rho v'}{m'}$ we have $|\bar{f_i}' - f_i'| = \dfrac{\epsilon\,\eta''}{u'} < \rho$. As $\bar{f_j}' = f_j''$ for $j \neq i$, and $|f_j' - f_i'| > {}^2$, we have $|\bar{f_j} - \bar{f_i}| > 0$, whence B : 2 (a) is satisfied on the interior of $\mathfrak{L}_i''$.

(d). Let us suppose, if possible, that B : 2 (b) is violated for some ϵ_1, $0 < |\epsilon_1| < \xi$ and some particular u_k. From the equation, $\bar{f_i}'' = \dfrac{v' + \epsilon\,\eta''}{u'}\,\bar{f_i}''$ is certainly a continuous function of ϵ for $u = u_k$. For $\epsilon = 0$, *i.e.*, for f_0, f_i'' does not lie between $f''_{j+1}\,(u_k \pm 0)$ and $f_j''\,(u_k \pm 0$, while for $\epsilon = \epsilon_1$; it does. Hence for some



(Fig. 7).

intermediate value of ϵ, $f_i'(\epsilon) = f_j'\;u_k \pm 0)$, or $f'_{j+1}\,(u_k \pm 0$. This, however, contradicts what we have proved in §6 (c). Similarly for B : 2 (c). We have therefore finally that the variations of §6 (a) are admissible variations for every $0 < |\epsilon| < \xi$.

(e). It follows in the usual way[3] that $\mathfrak{L}_i''$ from τ' to τ'' satisfies the equation,

$$\frac{u'^{\,2/3}}{v'^{\,2/3}}\,u = \text{const.}$$

Since $u' \neq 0$ on $(\tau'\,\tau'')$, we may solve for t as a function of u, and substitute obtaining the relation

$$\frac{dv}{du} = c_i\,u\,,$$

whence
$$v = c''_{i+1}\,u^{\frac{5}{2}} + c''_{i+1}.$$

Since τ' and τ'' may be selected as near as we please to the end-points,

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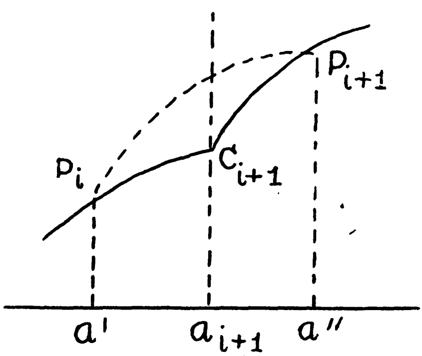
[1] See §5 (a).
[2] See §6 (b).
[3] See Bolza, Variations, p. 22.

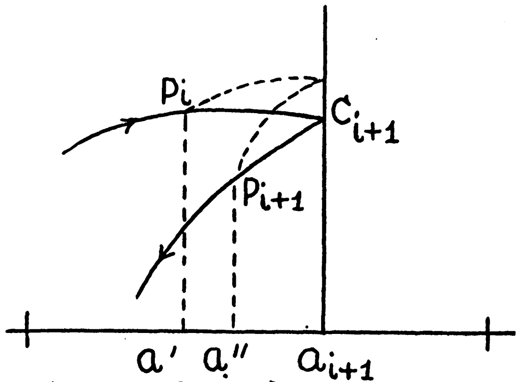
a_i and α_{i+1}, of $\mathcal{L}_i''$, it follows from the continuity of $\mathcal{L}_i''$ that this relation holds up to and including these end points.

(f). It remains to show that there are no corners.[1] For this purpose we select points, $P_i\ (\tau')$ and $P_{i+1}\ (\tau'')$ within the arcs, $\mathcal{L}_i''$, and $\mathcal{L}''_{i+1}$, and let $u\ (\tau') = a''$, $u\ (\tau'') = a''$. Consider the variation,

$$\bar{v} = f\ (u) + \epsilon\ \eta_j\ (u),\ \bar{u} = u,$$



(Fig. 7)



(Fig. 8)

where: (1) $\eta_j\ (u)$ is of class C' in $(a'\ a_{i+1})$, and $(a_{i+1}\ a'')$,

(2) $\eta_i\ (a') = 0,\ \eta_i'\ (a') = 0,\ \eta_i''\ (a') = 0,$
$\eta_{i+1}\ (a'') = 0,\ \eta''_{i+1}\ (a'') = 0,\ \eta''_{i+1}\ (a'') = 0,$
$\eta_i\ (a_{i+1}) = 0,\ \eta_i'\ (a_{i+1}) = 0,\ \eta_i''\ (a_{i+1}) = 0,$
$\eta_{i+1}\ (a_{i+1}) = 0,\ \eta''_{i+1}\ (a_{i+1}) = 0,\ \eta''_{i+1}\ (a_{i+1}) = 0;$

(3) $\eta_j\ (u) = 0$ for $j \neq i, i+1$, $\eta_i\ (u) \equiv 0$, for u in $(a_i\ a')$, and $\eta\ (u) \equiv 0$ for u in $(a''\ \alpha_{i+1})$[2]

Since such a variation is made only at the corners common to two successive arcs, condition A (4) is not affected thereby. It follows therefore exactly as in §6 (b) . . . . . (e) that for $|\epsilon| < \xi$, this variation is amissible, whence from the result in the general case.[3]

$$C'_{i+1} = C_i'.$$

There are therefore no corners.

[1] See Bolza, Variations, p. 68.

[2] $\eta_i\ (u) = (u - a')^3\ (u - a_{i+1})^3$, and $\eta_{i+1}\ (u) = (u - a'')^3\ (u - a_{i+1})^3$ satisfy these conditions.

[3] See Bolza, Variations, p. 38.

(f) Since $\mathfrak{L}''$ is continuous, we have

$$v = c' \, u^{\frac{5}{2}} + c''$$

throughout. Since it must pass through $(u_0, 0)$, and $(1, \omega)$, its equation is

$$v \, (u^{\frac{5}{2}} - 1) = \omega \, (u_0^{\frac{5}{2}} - u^{\frac{5}{2}}).$$

Since $\dfrac{dv}{du} > 0$, and is finite, $u_0 \doteq 1$. Computing I'', we get

$$I'' = \sqrt[3]{\frac{5\omega}{2}} \left(1 - u_0^{\frac{5}{2}} \right)^{\frac{2}{3}}.$$

This is evidently a maximum when $u_0 = 0$; in this case

$$I'' = \sqrt[3]{\frac{5\,\omega}{2}}$$

The maximising curve, if any exist, is therefore given by:

$$\mathfrak{C}_0 : v = \omega \, u^{\frac{5}{2}}.$$

§ 7. *Slope Properties of Curves, $\mathfrak{L}''$.*

(a) In order to prove that the curve, $\mathfrak{C}$, obtained at the conclusion of the preceding paragraph, actually furnishes a maximum, we shall need certain results with reference to the slope properties of the arcs, $\mathfrak{L}''$. These are connected with certain "outside, inside and outside," properties of the original solid, and are most easily deduced by returning to the original set of curves, $\mathfrak{L}$, (see § 2 (c)). We represent these in an (r, θ) plane by means of the transformation,

$$r = \sqrt{\varphi^2 + \psi^2}, \quad \theta = \cot^{-1} \frac{\varphi}{\psi}, \quad 0 \leq \theta \leq \frac{\Pi}{2}, \quad \text{..................(1)}$$

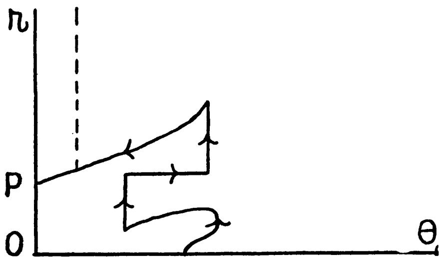
$0\,r$ and $0\,\theta$ being a pair of rectangular axes. Since $\psi > 0$, (see IV : b)[2] for any τ, $\tau_0 < \tau < \tau_1$, θ is uniquely determinate for any such τ. As $\tau \doteq \tau_1$ $\dfrac{\varphi}{\psi}$ approaches a determinate limit, (see III:a)*, positive if $\left.\dfrac{\varphi}{\psi}\right/_{\tau_1 + 0}$ be finite, and 0 if infinite. As $\tau \doteq \tau_1$, since $\varphi\,(\tau_1) > 0$, and $\psi\,(\tau_1) = 0$, $\dfrac{\varphi}{\psi}$ approaches infinity (III : b), and therefore $\theta \doteq 0$. For the point,

[1] See § 5 : b (A : 3).

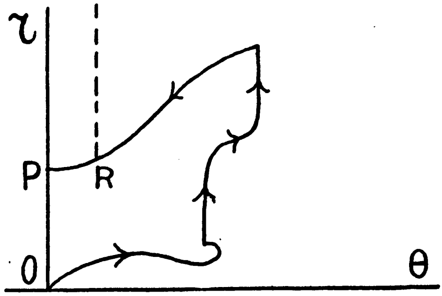
[2] These references are to the tabulated results, §4 (c.)

τ_1, and for τ_0 when it is not already determinate we define $\theta(\tau_1)$ and $\theta(\tau_0)$ as the limits thus approached. Then the arc (r, θ), joining $P\left\{r = \varphi(\tau_1), \theta = 0\right\}$, and $Q\left\{r = 0, \theta = \theta(\tau_0)\right\}$, is continuous (see III: a). If $\theta(\tau_0) > 0$, the curve made up of this arc from P to Q, the axis, $r = 0$, from Q to 0, and the axis, $\theta = 0$, from 0 to P, (fig 9) is closed. It is also *simple*. For since $\psi > 0$, (IV : b) for $\tau_0 < \tau < \tau_1$, and $\theta = \cot^{-1}\frac{\varphi}{\psi}$, $0 \leq \theta \leq \frac{\pi}{2}$ we have $\theta(\tau) > 0$ for $\tau_0 < \tau < \tau_1$, ...(2)

Hence $P\,Q$ cannot meet the axis, $O\,P$, $(\theta = 0)$, for any τ, $\tau_0 < \tau < \tau_1$. By hypothesis at Q (τ_0), $\theta(\tau_0) > 0$. Hence $P\,Q$ meets $O\,P$ at P (τ_1) only. Again by equations (1) since $\psi(\tau) > 0$ for $\tau_0 < \tau < \tau_1$ (IV : b), while $\varphi(\tau_1) > 0$, we have $r(\tau) > 0$ for $\tau_0 < \tau \leq \tau_1$. Hence that arc,



(Fig 9)



(Fig.10)

$P\,Q$, meets the axis, $O\,Q$, $r = 0$, at Q only. Further, the arc, $P\,Q$, can have no double points on its interior. For if there were such a point, $\tau = \tau_2, \tau_3$, we have from (1),

$$\varphi^2(\tau_2) + \psi^2(\tau_2) = \varphi^2(\tau_3) + \psi^2(\tau_3), \text{ and } \frac{\varphi(\tau_2)}{\psi(\tau_2)} = \frac{\psi(\tau^3)}{\psi(\tau^3)}$$

whence $\varphi(\tau_2) = \varphi(\tau_3)$, and $\psi(\tau_2) = \psi(\tau_3)$, contrary to (I : b). Similarly if $\theta(\tau_0) = 0$, we may show that *the arc, PQ, and the axis, $\theta = 0$ from Q to P form a simple closed curve.*

(c). Since $\psi < 0$ on the interior of $\mathcal{L}$ (IV : b), we have

$$\theta' = \frac{\psi\,\varphi' - \varphi\,\psi'}{\psi^2}$$

Hence $\theta' \equiv 0$ on the segments, $\kappa_i\,\lambda_i$, and vanishes at only a finite number of other points, (II: a, b). On the former, $\theta =$ const. Entirely after the manner of § 5, we may exhibit the remaining arcs as a finite number of continuous arcs of the type $r = f(\theta)$. It follows that *if any ordinate be* *drawn not meeting an arc, θ = const., of this curve, then the points of intersections of the ordinate with the curve are finite in number, and if these intersection be arranged according to increasing values of r, θ is alternately a decreasing and increasing function of τ. Further for all values of θ, θ is a decreasing function of τ for the intersection for which r is the greatest.*

(d). To see the truth of the last statement, we observe that P (τ_1) cannot be the end-point of a segment on which $\theta' \equiv 0$. For if it were, since θ (τ_1) = 0, we should have $\theta \equiv 0$ on this segment, contrary to (2). Hence P is an end-point of an arc of type, $r = f_1$ (θ). Let $\tau = \tau_1$ be the other end-point. Since θ (τ_1) = 0, we have from (2)

$$\tau_1' < \tau_1, \ \theta\ (\tau_1') > \theta\ (\tau_1')$$

Hence θ (τ) is a decreasing function of τ on $r = f_1$ (θ). In the same way, if θ (τ_0) = 0, (fig. 10), we may show that Q is an end-point of an arc on which $r = f_0$ (θ); let τ_0' be the other end-point. If θ $(\tau_0) \neq 0$, the distances, θ (τ), of the arc, $P\ Q$, for $\tau_0 \leq \tau \leq \tau_1'$ from $0\ r$ have a minimum.[1] Since $P\ Q$ meets $0\ r$ only at P, $(\tau > \tau_1')$, in the first case, and only at P and Q in the second, $(\tau > \tau_1', \ \tau < \tau_0')$, these minima are positive, = $2\ m$ say. Then if $0 < \varepsilon \leq m$, the line, $x = \varepsilon$, for $y \geq f_1$ (ε), (drawn from R in figs. 9, 10), meets our curve at $\left\{ \varepsilon, f_1\ (\varepsilon) \right\}$, and at this point only. For since $\theta = f_1$ $(\varepsilon) > 0$, (see eqn. 2), it does not meet $0\ r$ or $0\ \theta$. Since $\varepsilon \leq m$, it does not meet any arc of $P\ Q$ other than $r = f_1$ (θ), and possibly $r = f_0$ (θ) if θ (τ_0) = 0. It can meet $r = f_1(\theta)$ at but one point, viz., R. If it meet $r = f_0$ (θ), we should have f_0 (θ) − f_1 $(\theta) \geq 0$ for $\theta = \varepsilon$. Now f_0 (θ) − f_1 $(\theta) < 0$ for $\theta = 0$. Hence since f_0 (θ) and f_1 (θ) are continuous, (see (c)) f_0 (θ) − f_1 (θ) =0 for some θ, $0 < \theta \leq \varepsilon$[2]. This contradicts what we have proved in (b), viz., that the arc, $P\ Q$, is simple. Hence the given half-line, $\theta = \varepsilon, r > f_1$ (ε), does not meet our curve. Hence for $0 < \varepsilon \lesssim m$, the greatest value of r for the intersections of $\theta = \varepsilon$ is on $r = f_1$ (θ). We have shown that on this arc, θ is a decreasing function of τ. We shall prove that the sense of θ for the greatest r is independent of θ.[3] Hence θ is always a decreasing function of for the intersection for which r is that the greatest.

(e) In § 3 (a), we have transformed the curves, $\mathcal{L}$, into the curves, $\mathcal{L}''$, and have the relations,

$$u\ (\tau) = \cos\theta, \ P\ (\tau) = r^3 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

(see § 3 (a), and § 3 (e), eqn. 20), where P (τ) is the slope of the new curve. The arcs, θ = const., (*i.e.*, $y\ x' - x\ y' \equiv 0$), become stationary

[1] Dini Funktionen, s. 68.

[2] Dini, l.c., s 70.

[3] These results of Analysis Situs are proved in the third chapter.

points on $\mathcal{L}''$ (see § 4 (a)). Points of regression, *i.e.*, points at which θ'' and therefore U' change sign, will correspond in the two curves. An ordinate, θ = const., of the (r, θ) plane, by (3), maps into an ordinate, U = const., in the (U, V) – plane. In § 4, we have merely made a translation of the parameter to eliminate the stationary points, and have the relations

$$u(t) = U(\tau), \quad p(t) = P(\tau).$$

Again when θ is a decreasing function of τ, $U(\tau)$, ($= \cos\theta$) is an increasing function of τ, and hence also $u(t)$ of t (see § 4a), and *vice versa*. Hence from (e) if any ordinate, u = const., be drawn, not through a point of regression or through the homologue of a stationary point, the points of intersection of the ordinate with the curve are finite in number, and if these be arranged according to increasing values of the slope, p, u is alternately an increasing and decreasing function of t. It is not difficult to prove from the continuity of the slope on each arc, $\mathcal{L}_i''$, that the same holds at the homologues of the stationary points which are not also points of regression. We have finally therefore, *if any ordinate, $u = v$, be drawn, not through a point of regression, the points of intersection of the ordinate with the curve are finite in number; and if these be arranged according to increasing values of the slope, p, u is alternately an increasing and decreasing function of t; and for all values of u, u is an increasing function of t for the intersection with the greatest slope, p.*

(f). Suppose now that we have drawn the ordinates, $u = u_0$, and $u = 1$, and also through the points of regression of $\mathcal{L}''$. The number of such points of regression $\leq r + 1$, (see § 5a); and is therefore finite. Let $u = v_1$ and $u = v_2$ be any two adjacent ordinates among these, $v_1 < v_2$, and let $u = v$, $v_1 < v < v_2$, be any ordinate not meeting a discontinuity of p. Then by (II″ : c, p. 13), the value of p at the intersection of $u = v$, and $\mathcal{L}''$ are all unequal. We name the arcs of $\mathcal{L}''$ between v_1 and v_2, 1, 2, 3 . . . . in such a way that

$$p_1 > p_2 > p_3 > p_4 > p_5 > \dots > \qquad (4)$$

Then *this naming is independent of v for $v_1 < v < v_3$.* For suppose, if possible, that for any other ordinate, $u = v'$, $v_1 < v' < v_2$, not meeting a discontinuity of p, the order of magnitude of $p_1, p_2, p_3, \dots$ is different from that given by (4). We have then for some i and j.

$$p_i(v) > p_j(v), \text{ and } p_i(v') < p_j(v') \qquad (5)$$

We suppose that $v < v''$. Since by hypothesis, p_i and p_j are continuous in the vicinity of v and v'', we have:

$$\left.\begin{array}{l} p_i(u) - p_j(u) > 0 \text{ for } u > v \text{ in some vicinity of } v \\ p_i(u) - p_j(u) < 0 \text{ for } u < v' \text{ in some vicinity of } v' \end{array}\right\} \qquad (6)$$

Consider the set of points on $(v\ v'')$ such that there is no point, u, between them and v for which $p_i(u) - p_j(u) < 0$. From (5), these have v'' for an upper bound, and hence have an upper limit, v''.[1] By (5), $v < v'' < v''$. Since v'' is a limit-point for a set, $[u]$, for which ${}_i p(u) - p_j(u) < 0$, we have, remembering that $p(u)$ is class D,

$$p_\iota(v'' - 0) - p_j(v'' - 0) > 0.$$

From (II$''$: c, p. 13), $\quad p_\iota(v'' - 0) \mp p(v'' - 0)$. Hence

$$p_\iota(v'' - 0 - p_j(v'' - 0) < 0 \quad \ldots\ldots\ldots\ldots\ldots (7)$$

On the other hand, if $p_i(v'' + 0) - p_j(v'' + 0)$ 0, since p_i and p_j are of class C in some vicinity of v'' for $u > v''$, $p_\iota(u) - p_j(u) < 0$ in some vicinity of v'' for $u > v''$. Hence v'' is not an upper limit for the set for which $p_i(u) - p_j(u) < 0$. By II$''$: c, $p_\iota(v'' + 0) \mp p_j(v'' + 0)$. Hence

$$p_i(v'' + 0) - p_j(v'' + 0) > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Now the intervals, $\left\{p_i(v'' + 0) \quad . \quad . \quad . \quad . \quad . \quad p_j(v'' + 0)\right\}$, and $\left\{p_j(v'' + 0) \quad . \quad . \quad . \quad . \quad . \quad p_j(v'' - 0)\right\}$ cannot have any point in common. For if they did, either $p_\iota(v'' + 0)$, or $p_i(v'' - 0)$ must lie on $\left\{p_j(v'' + 0) \quad . \quad . \quad . \quad . \quad p_j(v'' - 0)\right\}$, contrary to (II$''$: c).

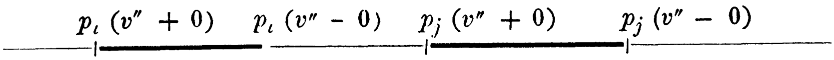


Fig. 11.

The inequalities, (7) and (8), are evidently incompatible with this con dition (see fig. 11). Hence the order of magnitude of p_1. p_2 p_3, ... is independent of v when $u = v$ passes through no discontinuity of p.

(g). From this it follows that at any point, k, of discontinuity of p_i or p_j, the slopes, p_1, p_2, p_3, ... are in the same order of inequality, where by $p_i(k)$ either of the values, $p_\iota(k + 0)$ or $p_i(k - 0)$, is meant, and similarly for $p_j(k)$. To see this, we select any v' and v'', $v' < k < v''$, such $p_i(u)$ and $p_j(u)$ are continuous on (v', k) and (k, v''). Then, as in (b), we derive $p_i(k - 0) > p_j(k - 0)$, and $p_i(k + 0) < p_j(k + 0)$. (9) Since the intervals, $p_i(k - 0) \ . \ . \ . \ p_i(k + 0)$ and $p_j(k - 0) \ . \ . \ p_j(k + 0)$, can have no point in common, it follows from (9) that $p_i(k - 0)$ and $p_j(k + 0)$ are both greater than $p_j(k - 0)$ and $p_j(k + 0)$. Hence *the inequalities*, (4), hold for every v, $v_1 < v < v_2$. Further, since $p(u)$ is continuous on each $\mathfrak{L}_\iota''$ (see §5), at the points v_1. v_2, we have

$$p_1 \geq p_2 \geq p_3 \geq p_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

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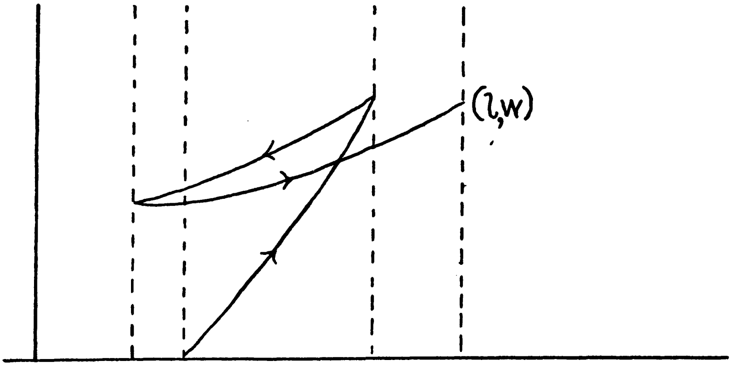
[1] Dini, l.c., s. 57.

§ 8. *Reduction to an Integral on a Path without Points of Regression.*

(a). *Construction.*



(Fig. 12)

We now proceed to show that the value of I'' taken along $\mathfrak{C}_o$, (see §6) exceeds that along any other path of the set, $\mathcal{L}''$. For this purpose consider any particular curve, $\overline{\mathcal{L}}$. If it have a point of regression we proceed to construct an associated curve, C, without such points, which gives I'' a greater value. We then show that I'' along $\mathfrak{C}_o$ exceeds I'' I along C or along any curve, $\overline{\mathcal{L}}$, without points of regression. Since $u\ (t)$ is continuous (I'' : b),* there must be a point, t, on at which $u\ (t)$ reaches a minimum value, u_{-1} say. If $u_{-1} < u_o$, we adjoin to $\overline{\mathcal{L}}$ the line, $u_{-1}\ u_o$ on $0\ u$. The resulting curve we name $\mathcal{L}^*$, and the parameter of the end-point we still call t_o. Since on $u_{-1}\ u_o\ u' \equiv 0$ if $u_{-1} < u_o$, it follows from §6 that the integral, I'', taken along $\mathcal{L}^*$ is equal to I'' along $\overline{\mathcal{L}}$, although if $u_{-1} < u_o$, $\mathcal{L}^*$ does not belong to the set, $\mathcal{L}''$.

As in §5 (a) we divide $\mathcal{L}^*$ into arcs on which $v = f_i\ (u)$. The number of arcs and points of regression on $\mathcal{L}^*$ can be made to exceed the number on $\mathcal{L}''$ by not more than one. We draw ordinates, $u = u_{-1}$, $u = 1$, and through each point of regression of $\mathcal{L}^*$ after the manner of the preceding paragraph, and propose to consider the contributions to I'' of the arcs between two successive ordinates, v_1 and v_2.

(b). *Intersections of an Ordinate with $\mathcal{L}^*$.*

Consider any ordinate, $u = v$, where $v_1 < v < v_2$. Since $u'\ (t) \neq 0$, within $\mathcal{L}_{i_1}$ and we can certainly choose a parameter, t, on $u_{-1}\ u_o$ so that $u'\ (t) > 0$ on $\overline{u_{-1}\ u_o}$ u is either an increasing function of t, or a decreasing function of t in some vicinity of $(\tau'\ \tau)$, $\tau' < \tau$; and likewise in some

[1] See p. 14.

vicinity $(\tau\ \tau'')$, $\tau'' > \tau$. If τ is not a point of regression or an end-point, $u\ (t)$ is an increasing (decreasing) function of t at τ; *i.e.*, it is increasing (decreasing) in some vicinity $(\tau''\ \tau'')$, $\tau'' < \tau < \tau''$. Since $v_1 < v < v_2$, $u = v$ does not meet a point of regression on $\mathfrak{L}^*$ or an end-point. Hence $u\ (t)$ is an increasing (decreasing) function at each intersection of $u = v$ and $\mathfrak{L}^*$. The number of these is finite. *If τ be the least among their paramrters, then $u\ (t)$ is an increasing function of t at τ.* For $u_{-1} \leq v_1 < v$; it follows that $v = u\ (\tau) > u_{-1} = u\ (t_0)$. If $u\ (t)$ were a decreasing function at τ, there would exist a t', $t' < \tau$, such that the difference, $u\ (\tau) - u\ (t)$, positive at t_0, and negative at t', must vanish for some τ'', $t_0 < \tau' < t'$. This contradicts the hypothesis that τ is the least among the values of t for which $u\ (t) = v$. In the same way we have that at the intersection with the greatest parameter, since $v < v_2 \leq 1$, $u\ (t)$ is an increasing function of t; a d that, arranged in order of their parameters, at the intersections $u\ (t)$ is alternately increasing and decreasing. Hence *the total number of intersections of $\mathfrak{L}^*$ with $u = v$ must be odd.*

(c). *Slopes on $\mathfrak{L}^*$.*

We have seen, (§ 7 *f*, *g*), that if we number the arcs between v_1 and v_2, 1, 2, 3 . . . . . according to magnitudes of the slopes at the inter sections of $\bar{\mathfrak{L}}$ and $u = v$, the numbering is independent of v. The same is true of $\mathfrak{L}^*$. For since $\bar{\mathfrak{L}}$ and $\mathfrak{L}^*$ coincide except for the straight line $u_{-1}\ u_0$, this will certainly be true for any interval $(v_1\ v_2)$ that does not contain points of $(u_{-1}\ u_0)$ on its interior. If it does contain such, the order of the slopes on the arcs of $\mathfrak{L}^*$ other than $(u_{-1}\ u_0)$ will be fixed. Since $\bar{p}^* = 0$ on $\overline{u_{-1}\ u_0}$ and $\bar{p} > 0$ within $(v_1\ v_2)$, (IV'' : b, p. 15), the slope will be less on $\overline{u_{-1}\ u_0}$ than on any other arc, and the order is still fixed. We have then between v_1 and v_2.

$$p_1 > p_2 > p_3 \cdot \cdot \cdot \cdot \cdot > p_{2n+1} > \cdots\cdots\cdots\cdots (1)$$

At the points, v_1 and v_2, some of the signs of inequality may be replaced by signs of equality if we understand by $p_r\ (v_1)$ the value of $p_r\ (v_1 + 0)$ and by $p_r\ (v_2)$, the value, $p_r\ (v - 0)$.

(d). *Construction of C.*

We now construct a curve, C, whose slope, p, is given by

$$\tilde{p} = p_1 - p_2 + p_3 - \cdot \cdot \cdot \cdot \cdot - p_{2n+1} \cdots\cdots\cdots\cdots (2)$$

on $(v_1\ v_2)$, which shall be continuous, and pass through $(u_{-1}\ , 0)$; *i.e.*,

$C\ (\tilde{v}, u)$, where: $$\tilde{v} = \int_{u-1}^{u} \tilde{p}\ .\ d\tilde{u}.$$

Then C also passes through $(1, \omega)$. For

$$\tilde{v}\ (1) = \int_{u-1}^{1} \hat{p}\ .\ du = \sum_{v_1 v_2} \int_{v_1}^{v_2} \tilde{p}\ .\ du$$

$$\sum_{v_1 v_2} \int_{v_1}^{v_2} (p_1 - p_2 + \ .\ .\ .\ .\ .\ - p_{2n+1})\ du \qquad (3)$$

Now we have seen that u is an increasing function of t on the arc, 1, (see §7, (e) and equation, 1), and that it is alternately decreasing and increasing on the arcs, 1, 2, 3 . . . . $(2n + 1)$. Hence u is a decreasing function of t on the arcs, 2, 4, 6 . . . . $2n$, between v_1 and v_2. Now by (III″ : a, II″ : c),

$$v\ (1) = \int_{t_0}^{t_1} v'\ .\ dt = \int_{t_0}^{t_1} p\ .\ u'\ .\ dt \qquad (4)$$

We consider the contributions of the arcs of $\mathcal{L}^*$ from v_1 to v_2 to this integral. On the arcs, 1, 3, 5 . . . . . $2n + 1$), since $u\ (t)$ is an increasing function of t, if we denote by τ, τ', $\tau < \tau'$, the parameters of the end-points, then $u\ (\tau) = v_1$, and $u\ (\tau') = v_2$, whence

$$\int_{\tau}^{\tau 1} p_{2n+1}\ u'\ .\ dt = \int_{v_1}^{v_2} p_{2n+1}\ .\ du \qquad (5)$$

expressing as a function of u. On an arc, 2, 4, 6 . . . . $2n$ from τ to τ'', $\tau < \tau$, since $u\ (t)$ is a decreasing function of t, $u\ (\tau) = v_2$, and $u\ (\tau') = v_1$. Hence

$$\int_{\tau}^{\tau} p_{2r}\ .\ u'\ .\ dt = \int_{v_2}^{v_1} p_{2r}\ .\ du = - \int_{v_1}^{v_2} p_{2r}\ .\ du \qquad (6)$$

Rearranging and combining the arcs between v_1 and v_2 we have:

$$v\ (1) = \sum_{v_1 v_2} \int_{v_1}^{v_2} (p_1 - p_2 + p_3 - \ .\ .\ .\ .\ .\ .\ - p_{2n+1})\ .\ du \qquad (7)$$

From (3), (4), (5), and III″ : a), we have:

$$\tilde{v}\ (1) = v\ (1) = \omega \qquad (8)$$

(e). *Lemma.*

To prove that I'' taken along C exceeds I'' along $\mathcal{L}^*$, we observe that *if we have any four not-negative quantities, a, b, c, d, $c > a > b$, and such that $a - b = c - d$, that $a^a - b^a < b^a - d^a$ if $0 < \alpha < 1$.* For consider the integrals

$\frac{1}{a}\int_b^a x^{a-1} \,.\, dx$, and $\frac{1}{a}\int_d^c y^{a-1} \,.\, dy$. They are integrated over equal intervals. At x in (ab) and y in (cd) such that $x - b = y - d$, we have $0 \leq x < y$. Hence since $\alpha < 1$, $x^{a-1} < y^{a-1}$. Integrating,

$$a^a - b^a < c^a - d^a.$$

(*f*). *Proof that I'' along C exceeds I'' along $\mathcal{L}^*$.*

To show that the integral, I'', along C exceeds I'' along $\mathcal{L}^*$, we have

$$I^* = \int_{t_1}^{t_1} v'^{1/3} \,.\, u'^{2/3} \,.\, u \,.\, dt$$

$$= \int_{to}^{t_1} p^{1/3} \,.\, u' \,.\, u \,.\, dt, \quad (\text{II}'' : \text{c}).$$

Rearranging as for equation (7) in (d),

$$I^* = \sum_{v_1 v_2} \int_{v_1}^{v_2} (p_1^{1/3} - p_2^{1/3} + p_3^{1/3} - \dots\dots - p_{2n+1}^{1/3})\, u \,.\, du \quad \dots(9)$$

whereas $\tilde{I} = \sum_{v_1 v_2} \int_{v_1}^{v_2} \tilde{p}^{1/3} \,.\, u \,.\, du$ .... (10)

From (9) and (10), to show that $\tilde{I} > I^*$, since the ordinates, $u = v_1, v_2$, are finite in number, it is sufficient to show that between v_1 and v_2,

$$\tilde{p}^{1/3} > p_1^{1/3} - p_2^{1/3} + \dots\dots - p_2{}^{1/3}{}_{2n+1}) \quad \dots\dots(11)$$

(h). We introduce the intermediate quantities, by

$$\begin{aligned} \pi_1 - p_{2n+1} &= p_{2n-1} - p_{2n}, \\ \pi_2 - \pi_1 &= p_{2n-3} - p_{2n-2}, \\ \pi_3 - \pi_2 &= p_{2n-5} - p_{2n-4}, \\ &\text{— — — — — — —} \\ \pi_{n-1} - \pi_{n-2} &= p_3 - p_4, \end{aligned} \quad \dots\dots(12)$$

whence $\tilde{p} - \pi_{n-1} = p_1 - p_2$ by (3).

The quantities, $p_{2n+1}, p_{2n} \ldots \ldots p_1$, are in ascending order of magnitude between v_1 and v_2. Hence from (12)

$$p_{2n-1} > \pi_1 > p_{2n+1}.$$

It follows that $\pi_1 < p_{2n-3}$, and therefore,

$$p_{2n-3} > \pi_2 > \pi_1$$

and so on. Putting $\alpha = \frac{1}{3}$ in (c), we have therefore

$$\begin{aligned} \pi_1^{1/3} - p_{2n+1}^{1/3} &> p_{2n-1}^{1/3} - p_{2n-2}^{1/3} \\ \pi_2^{1/3} - \pi_1^{1/3} &> p_{2n-3}^{1/3} - p_{2n-4}^{1/3} \\ \tilde{p}^{1/3} - \pi_{n-1}^{1/3} &> p_1^{1/3} - p_2^{1/3} \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Adding and transposing p_{2n+1}, we have,

$$\tilde{p}^{1/3} > p_1^{1/3} - p_2^{1/3} + p_3^{1/3} - \ldots\ldots - p_{2n+1}^{1/3}.$$

Hence by (11), $\tilde{I} < I^*$.

(i). *Properties of C.*

Since $p_1, p_2 \ldots\ldots$ are in descending order of magnitude except at the (finite number of) points v_1, v_2 (see (6), where some of them become equal, we have from (3) that $p > 0$ except at a finite number of points, v_1, v_2. Since $p_1, p_2 \ldots\ldots$ are of Class D′, (II″ : c, p. 14), p is of class D. Hence the set of reduced curves is included in the totality of curves, C, with the following properties:

$$C: \; v = f(u),$$

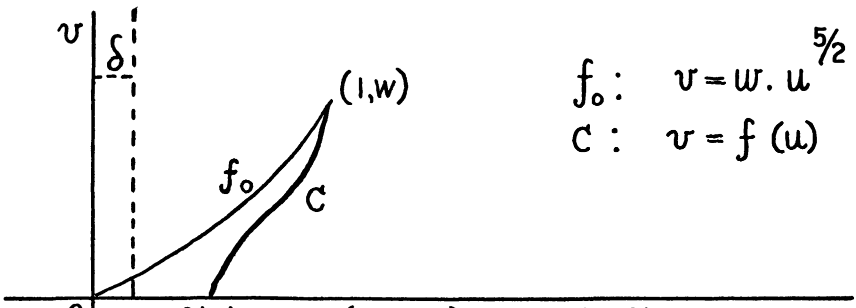
where (1) $f(u)$ is of class D′ for $u_{-1} \leq u \leq 1$;

(2) $v(u_{-1}) = 0, \; v(1) = \omega$;

(3) $f'(u) > 0$, except at a finite number of points.

This also includes those members of the totality, $\mathfrak{L}$, for which $u'(t) \neq 0$; *i.e.*, which have no points of regression, (see I″ —— V″, p. 14). Then if $\mathfrak{C}$ furnishes a maximum for the set, C, it certainly furnishes one for the totality, $\mathfrak{L}$.

§ 9. *Sufficiency Proof by Taylor's Theorem.*



(Fig.13)

In the present case, we can prove that $\mathfrak{C}$ furnishes a maximum for the totality, $\mathfrak{L}^*$, by the remainder formula in Taylor's Theorem. As in § 7, if $u_{-1} < 0$, we adjoin to C the line, $0\ u_{-1}$, and denote the resulting curve by C^*. Since $\frac{dv}{du} \equiv 0$ in $0\ u_{-1}$, if $u_{-1} > 0$, it follows from the definition, (see p.) that I'' along C is equal to I'' along C^*. Now

$$\begin{aligned} \triangle I &= I_{f_0} - I_C \quad \text{..............................} (1) \\ &= I_{f_0} - I_{C^*} \end{aligned}$$

We write $\tilde{v} = v + \eta$. Then η is of class D′ in u, (see §8, i : 1). We denote by η' the derivative of η with respect to u. Expressing I'' as an integral,

$$\triangle I = \int_0^1 \left\{ v'^{1/3} - (v' + \eta')^{1/3} \right\} u \,.\, du \quad \text{..............} (2)$$

Now if $u_{-1} > 0$, $\tilde{v}' \equiv 0$ on $0\ u_{-1}$, and $\tilde{v}' = f'(u) \geq 0$ from u_{-1} to 1 (§8, : i 3). Hence $\tilde{v}' \geq 0$. Again $v' = \frac{5}{2}\ \omega,\ u,^{\frac{3}{2}}$ and therefore, $v' > 0$ for $u > 0$. We surround the origin by a small δ – interval, $0 < \delta < 1$. Then for $\delta \leq u \leq 1$, $v' > 0$, and $\tilde{v}' \geq 0$. Hence if $0 < \theta < 1$,

$$v' + \theta\, \eta' > 0 \quad \text{..............................} (3)$$

If now we write: $V = (v' + h)^{1/3}$, we have

$$\frac{\delta V}{\delta h} = \frac{1}{3}\, \frac{1}{(v' + h)^{2/3}}, \quad \frac{\delta^2 V}{\delta h^2} = -\, \frac{2}{9}\, \frac{1}{(v' + h)^{\frac{5}{3}}} \quad \text{..............} (4)$$

From (4), $\frac{\delta V}{\delta h}$ and $\frac{\delta^2 V}{\theta h^2}$ exist, and are continuous functions of h on 0 . . . . η'' except perhaps for $h = \eta'$. It follows[1] that

$$(v' + h)^{1/3} = v'^{1/3} + \frac{1}{3}\, \frac{\eta'}{v'^{2/3}} - \frac{1}{9}\, \frac{\eta'^2}{(v' + \theta\, \eta'')^{\frac{5}{3}}},\ 0 < \theta < 1 \quad \text{..............} (5)$$

[1] Stolz, Diff. u. Integr. - rechnung, s. 97.

for any u, $\delta \leq u \leq 1$. If now we write

$$\triangle I_\delta = \int_\delta^1 \left\{ v'^{1/3} - (v' + \eta')^{1/3} \right\} u.\, du \quad \ldots\ldots\ldots(6)$$

then from (2), $$\lim_{\delta \doteq o} \triangle I_\delta = \triangle I \quad \ldots\ldots\ldots(7)$$

and from (5), $$\triangle I_\delta = -\frac{1}{3}\int_\delta^1 \frac{\eta' \,.\, u \,.\, du}{v'^{2/3}} + \frac{1}{9}\int_\delta^1 \frac{\eta'^2 \,.\, u}{(v'' + \theta\,\eta')^{5/3}}\, du$$

$$= -\frac{1}{3\left(\frac{5\,\omega}{2}\right)^{2/3}} \int_\delta^1 \eta'\, du + \quad \text{“} \quad \ldots\ldots\ldots(8)$$

from the equation of $\mathfrak{C}$.

$$= -\frac{1}{3\left(\frac{5\,\omega}{2}\right)^{2/3}}\, \eta\,(\delta) + \quad \text{“} \quad \ldots\ldots\ldots(9)$$

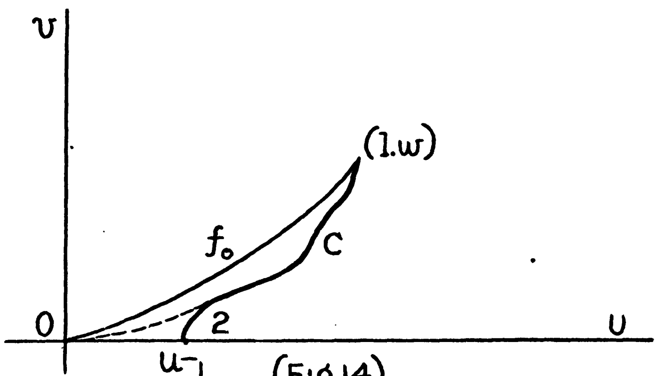
Since $\lim_{\delta \doteq o} \triangle I_\delta = \triangle I$ by (7), and $\lim_{\delta \doteq o} \eta\,(\delta) = 0$, the integral,

$$\int_o^1 \frac{\eta'^2 \,.\, u}{(v' + \theta\,\eta')^{5/3}}\, du$$

exists and we have :

$$\triangle I = \frac{1}{9}\int_o^1 \frac{\eta'^2 \,.\, u}{(v' + \theta\,\eta')^{5/3}}\, du \quad \ldots\ldots\ldots(10)$$

§10. *The Weierstrass' Sufficiency Proof.*[1]



(Fig.14)

Since by (3), $v' + \theta\,\eta' > 0$ within (0, 1), and $u > 0$ for $\mathfrak{C}$, the integral on the right hand of equation (10) must be positive unless $\eta' \equiv 0$ within $(0 \,.\,.\,.\,.\, 1)$ *i.e.*, since $\eta\,(t_0) = 0$, unless $\eta \equiv 0$, or C coincides with $\mathfrak{C}$. Hence $\mathfrak{C}$ actually does furnish a maximum.

[1] Bolza, Variations, p. 74.

Since $\frac{d\tilde{v}}{du} > 0$ except at a finite number of points, (§8, i : 3), $\tilde{v} > 0$ except at u_{-1}. There is therefore one and but one extremal joining 0 to a point, 2, on C, $2 = u_{-1}$, viz:

$$v = \frac{\tilde{v}}{u_2^{\frac{5}{3}}} \quad \text{.......................................} \quad (1)$$

With the usual notation,[1] we write,

$$S\ (u_2) = \int_0^{u_2} v'^{\,1/3} \ .\ u\ .\ du + \int_{u_2}^{1} \bar{v}'^{\,2/3} \ .\ u\ .\ du \quad \text{.........} (2)$$

for $u_{-1} < u_2 < 1$. Then $S\ (1) = \int_0^1$ along $\mathfrak{C}$. Again, as $u_2 \doteq u_1$,

$S\ (u_2) \doteq \int_{u-1}^{1}$ along C. We define $S\ (u_1)$ as this limit. Then $S\ (u_2)$ is continuous on $(u_{-1}\ .\ .\ .\ .\ 1)$. Again, except for $u_2 = u_{-1}$,

$$\frac{dS}{du_2} = \left(\frac{2}{5}\right)^{2/3} \left\{ \frac{5}{3} v_2^{1/3} \ .\ u_2^{2/3} + \frac{1}{3} v_2^{2/3} \ .\ \tilde{v}'_2 \ .\ u_2^{\frac{5}{3}} \right\} - \tilde{v}_2^{1/3} \ .\ u_2 \ . (3)$$

As $u_2 \doteq u_{-1}$, this approaches a definite finite limit, viz., $-\tilde{v}'_{-1}\ .\ u_{-1}$. Hence $\frac{dS}{du_{-1}}$ exists progressively, and is equal to this limit.[2] Then

$$\begin{aligned} \triangle I &= \int_0^1 v'^{1/3} \ .\ u\ .\ du - \int_{u-1}^{1} \tilde{v}'^{1/3} \ .\ u\ .\ du, \\ &= S\ (1) \text{ ——— } S\ (0), \\ &= \int_{u-1}^{1} \frac{dS}{du_2}\ .\ du_2 \quad \text{..} (4) \end{aligned}$$

Now from (3), if $\tilde{v}'_2 = 0$, $\frac{dS}{du_2} = \left(\frac{2}{3}\right)^{2/3} \frac{5}{3}\ .\ v_2^{1/3}\ .\ u_2^{2/3}$, when $\frac{dS}{du_2} > 0$ within C. If $\tilde{v}'_2 \neq 0$, consider the values of the right-hand of (3) as a function of $\tilde{v}'_2$. It has a maximum or minimum when

$$\tilde{v}'_2 = \sqrt{\frac{5}{2}\ \frac{v_2}{u_2}} \quad \text{.......................................} (5)$$

the latter since $\frac{d^2S}{du_2^2} = \frac{2}{9}\ \frac{u_2}{v_2} < 0$. This minimum value is 0.

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[1] Bolza, Variations, p. 87.

[2] Dini, l.c., § 68.

Hence for $\tilde{v}'$ other than that given by putting the right side of (3) $= 0$, $\frac{dS}{du_2} > 0$. Now the slope, $\tilde{v}$, given by this is exactly the slope of the extremal through 2 by (1). Hence C at all interior points has a tangent in common with the extremal through that point $\frac{dS}{du_2}$ is positive, and therefore from (4) $\triangle\, I > 0$. Since

$$\frac{d\tilde{v}}{du} = \sqrt{\frac{5}{2}\,\frac{\tilde{v}_2}{u}}$$

has only one solution passing through $(1, \omega)$[1], viz. f_0, we have that $\triangle\, I > 0$. Hence again $\mathfrak{E}$ actually furnishes a minimum.

§ 11. *Conclusion.*

The extremal, $\mathfrak{E}$, $v = \omega u^{\frac{5}{2}}$, therefore furnishes a maximum. Translating into the (x, y)—plane, we have

$$\frac{dv}{du} = r^3,\; u = \cos\theta,$$

where r and θ are the polar o co-ordinates. The extremal therefore has

for polar equation, $r = \sqrt[3]{\frac{5\,\omega}{2}} \,.\, \cos\theta,$

or in Cartesians, $x^2 + y^2 = \sqrt[3]{\frac{5\,\omega}{2}} \,.\, x.$

It is to be noted furthermore that these sufficiency proofs establish the fact that not only does f_0 furnish a maximum for curves in some vicinity[2] of f_0, but for all curves of the set; $\mathfrak{L}''$, into which the admissible curves in the (x, y)—plane transform. Hence $x^2 + y^2 = \sqrt[3]{\frac{5\omega}{2}} \,.\, x$ furnishes a maximum for the totality of curves in the (x, y)—plane.

CHAPTER III.

AUXILIARY THEOREMS OF ANALYSIS SITUS.

§ 1. (a). In § 7 of the preceding chapter, we have used certain results dependent upon the division[3] of the plane into two continua by

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[1] Picard, Traité d'analyse, II, pp. 314-5.

[2] See Bolza, l.c., § 19.

[3] See Jordan, Cours d'analyse, 2nd edn., vol. 1, 96-103 ; Schönfliess Gött. Nachr. Math. Phys., Kl. 1896, p. 79 ; Veblen, Trans. Amer. Math. Soc., vol. 6, No. 1, Jan., 1905, p. 83 ; Bliss, Bull. Amer. Math. Soc., ser. 2, vol. 10, p. 398, and vol. 12, p. 336 ; Osgood, Funktionentheorie, p. 130 ; Ames, Thesis, 1905.

a particular class of simple closed curve. Using Cartesian co-ordinates, our simple closed curve, C, given in the form,

$$x = \varphi(t), y = \psi(t),$$

can be divided into a finite number of arcs.

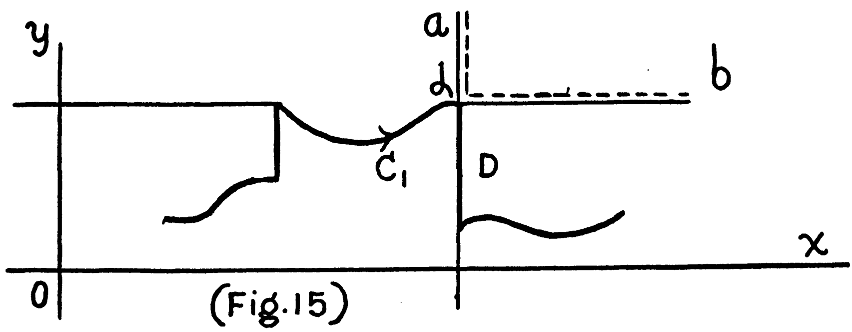
(a) of type $y = f(x)$, f denoting a continuous function, and

(b) straight lines parallel to $0\ y$.

The method used by Bliss in his first article (see below) for proving the section of the plane into two continua by a simple closed s curve consisting of a finite number of arcs of type (a) may be readily extended to the present case. Through the end of each arc of type (a), (the arc, C_1 of fig. 15), we draw half-lines parallel to $0\ x$. These with C_1 divide the plane into two regions. As in the article referred to, we construct a continuous function, $g_v\ (x, y)$, which vanishes on these lines and these lines only, and takes different signs at points (x, y), in these different regions. If a line, D, of type, (b), have for equation,

$$x = x_1,$$

the function, $$h_\mu(x, y) = x - x_1$$



(Fig. 15)

has the same properties with reference to this line. At the intersection of an arc of type (a) with an arc of type (b) (see fig. 15) at (a), we construct the function,

$$k_\lambda(x, y) = (x - x_1)\sqrt{2} \text{ for } x - x_1 \geq y - y_1,$$
$$(y - y_1)\sqrt{2} \text{ for } x - x_1 \leq y - y_1,$$

supposing that the auxiliary half-lines already drawn from this corner are in the positive directions with reference to $0\ x$, $0\ y$ as in the figure; a similar function may be constructed by appropriate change of sign if in other directions. This function is continuous, vanishes only at the broken lines of the figure, and takes opposite signs at the points in the two regions into which the auxiliar lines, $\alpha\, a$ and $\alpha\, b$, divide the plane. The product,

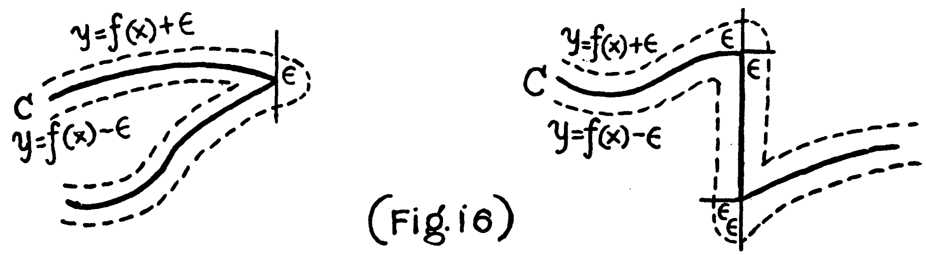
$$G\,(x,\,y) = \prod_{\lambda\mu,v} g_v\,(x,\,y)\,.\,h_\mu\,(x,\,y)\,\,.\,\,k_\lambda\,(x,\,y),$$

for the whole plane then has the essential property of the G-function in the article referred to, viz., that in the neighbourhood of the zeros, the factors change signs in pairs except at points on the curve itself. Proceeding in a fashion entirely similar to that there given, we may prove that the totality of points in the plane fall into three classes :[1]

(1) points, (x, y), such that $G\,(x,\,y)$ has both signs in every vicinity of $(x\;\;y)$, however small;

(2) points, $(x,\,y)$, such that $G\,(x,\,y)$ takes positive but not negative values in every vicinity (sufficiently small) of $(x,\,y)$;

(3) points, $(x,\,y)$ such that $G\,(x,\,y)$ takes negative but not positive values in every vicinity (sufficiently small) of $(x,\,y)$.



(Fig. 16)

The points of the first class turn out to be identical with the curve, C. Using the auxiliary lines,

$$y = f\,(x) \pm \epsilon, \text{ and } x = x_1 \pm \epsilon,$$

and joining them up by arcs of circles of radius, ε, we may construct two auxiliary curves[2] consisting of points of classes (2) and (3) respectively as near as we please to C.[3] By means of these we may join any two points of class (2), or any two points of class (3) without meeting C, showing that there are just two continua.

§ 2 (a). An ordinate can meet an arc of type, $y = f\,(x)$, at most once. Since there are a finite number of such arcs, it follows that *any ordinate not through a straight line parallel to* 0 y *must meet the curve at a finite number of points only.* Consider any ordinate through a point, 1 $(\xi_1\;\eta_1)$, (see fig. 17), interior to an arc, C_2, $y = f_2\,(x)$, end-points $(x_0,\;y_0)$ and

[1] For details, c.f. Bliss, l.c., vol. 10.

[2] See fig. 16, and for details compare, Bliss, l.c., vol. 10.

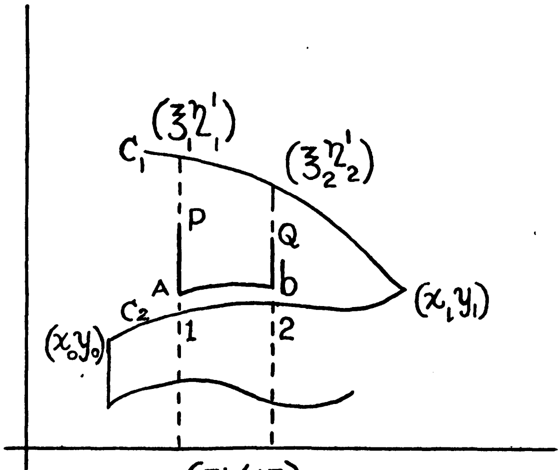
[3] This construction applies equally to any curve consisting of a finite number of arcs of type, $y = f\,(x)$, or $x = f\,(y)$, including curves of class D'. (See Ames, Thesis, already referred to.)

(x_1, y_1). Then $x = \xi_1$, meets only a finite number of arcs of type, $y = f(x)$. If there be any such that the ordinate $> \eta_1$, let η_1' be the least of these; then $\eta_1' > \eta_1$. The point-set, (ξ_1, y), such that $\eta_1' > y > \eta$ for any ξ_1 *within* (x_0, x_1). we call the *points immediately above* C_2. Similarly we define the *points immediately below* C_2 by

$$\eta'' < y < \eta_1 \text{ for any } \xi_1 \textit{ within } (x_0, x_1),$$

where (η_1'', ξ_1) is the intersection of $x = \xi_1$, of greatest ordinate less than η_1. If there are no intersections, (ξ_1, y) such that $y > \eta_1$ (or $y < \eta_1$), we take $\eta_1' = \infty$ (or $\eta_1'' = -\infty$).

(b). *The points immediately above* C_2 *are all of the same class.* For let P (ξ_1, y_1) and Q (ξ_2, y_2) be any two points immediately above C_2. Then, from the definition, if (ξ_1, η_1) and (ξ_2, η_2) be the points at which $x = \xi_1$



(Fig. 17)

and $x = \xi_2$ meet C_2, these points are *interior* to C_2. Let m be the least distance between C_2 from P to Q (inclusive) and the remainder of the curve, C_2 excluded. Then $m > 0$.[1] Select any positive δ, $\delta < m$, $y_1 - \eta_1$, $y_2 - \eta_2$, and construct the curve, $y = f_2(x) + \delta$, between a where $x = \xi_1$, and b where $x = \xi_2$. Since $\delta < m$, this cannot meet the curve C. Then Pab Q forms a continuous curve not meeting C and joining P and Q. These, therefore, belong to the same class. Similarly *points immediately below* C *are all of the same class.* Since there must be points of both classes in every vicinity of points on C, (see § 1), *the points immediately above* C, *and immediately below* C *are of different classes, 2 and 3.*

(c). We say that an arc, C_2, of type $y = f(x)$ is of *species 2 or 3 according as the points immediately above it are of class 2 or 3.* Returning

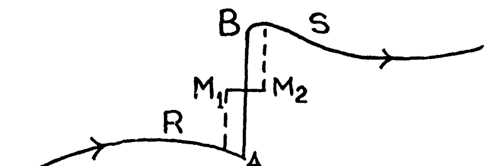
[1] Dini, Grundlagen, etc., s. 68.

to the representation of the curve, C, as a function of t we wish to show that *the senses of two arcs are the same or different according as the species are the same or different.* To see this it will only be necessary to show that as we go from one arc, C_1, of type, $y = f_1(x)$, to the succeeding one, C_2, $y = f_2(x)$ neither sense nor species changes, or both. If x is an increasing function of t on C_1 and C_2, C_1 and C_2 cannot have a common end-point; for if they did, C_1 and C_2 together would form a single arc of type, $y = f(x)$. Hence C_1 and C_2 must be joined by a straight line parallel to $0\,y$; from $A\,(x_1, y_1)$ say to $B\,(x_1, y_1')$ (fig. 18). Select any points, $R\,(\xi, \eta)$, and $S\,(\xi'\ \eta')$ on C_1 and C_2 respectively such that

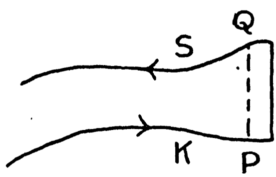
$$\left| f_1(x) - f_1(x_1) \right| < \frac{y_1' - y_1}{2} \text{ for } x_1 > x > \xi,$$

$$\left| f_2(x) - f_2(x_1) \right| < \frac{y_1' - y_1}{2} \text{ for } x_1 < x < \xi^*$$

Let m be any positive quantity less than the least distance of $RABS$ from the curve C with C_1, AB, C_2 deleted. Let ε be some positive quantity, $\varepsilon < m$, $x_1 - \xi$, $\xi' - x_1$. Then the line $x = x_1 - \varepsilon$ between $\left\{ x_1 - \varepsilon,\ f_1(x_1 - \varepsilon) \right\}$ and $\frac{y_1 + y_1'}{2}$ does not meet C except at $\left\{ x_1 - \varepsilon, f_1(x_1 - \varepsilon) \right\}$. Similarly the line, $x = x_1 + \varepsilon$ between



(Fig. 18)



(Fig. 19)

$\left\{ x_1 + \varepsilon,\ f_1(x_1 + \varepsilon) \right\}$ and $\frac{y_1 + y_1}{2}$ does not meet C except at $\left\{ x_1 + \varepsilon, f_0(x_1 + \varepsilon) \right\}$. Since

$$\xi < x_1 - \varepsilon < x_1, \text{ and } x_1 < x_1 + \varepsilon < \xi'$$

the straight line joining $M_1\ (x_1 - \varepsilon,\ \frac{y_1 + y_1')}{2}$ and $M_2\,(x_1 + \varepsilon,\ \frac{y_1 + y^{1''})}{2}$ does not meet C_1 or C_2, and since $\varepsilon < m$, it does not meet any arc of C other than AB. The middle point of $M_1 M_2$ being on AB, must have points of classes 2 and 3 in every vicinity, however small. It follows easily that

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* Dini, l.c , s 65.

M_1 and M_2 must be of different classes. Hence by the results at the end of (b) the points immediately above C_1 and the points immediately above C_2 are of the same class. Hence by the definition neither the class nor the species changes, as we go from C_1 to C_2. Similarly if x is a decreasing function of t on C_1 and C_2.

(d) If the senses on C_1 and C_2 are different, let us suppose in the first place that x is increasing on C_1 and decreasing on C_2. As in (c) select an R and S, interior to C_1 and C_2, and a positive m less than the least of the distance from $RABS$, (see fig. 19) : A and B coincide if C_1 and C_2 are contiguous arcs. Then let ε be any positive quantity .

$$\varepsilon < m, \ \varepsilon < x_1 - \xi, \ \varepsilon < \xi' < x_1.$$

Then the line, $x = x_1 - \varepsilon$, meets C_1 at $P \left\{ x_1 - \varepsilon, f_1(x_1 - \varepsilon) \right\}$, and C_2 at $Q \left\{ x_1 - \varepsilon, f_2(x_1 - \varepsilon) \right\}$ and since $\varepsilon < m$, it does not meet the curve otherwise. Points $(x_1 - \varepsilon, y)$, within PQ give $f_1(x_1 - \varepsilon) - y$, $f_2(x_1 - \varepsilon) - y$, different signs, and hence are above one and below the other of C_1 and C_2. These are of the same class. Hence the points immediately above C_1 and the points immediately above C_2 are of different classes. Similarly if x is a decreasing function of t on C_1 and increasing on C_2. Hence when the sense of description with reference to $0\,x$ changes, the species changes. We have thus the result of (b).

(e) Draw any line, $x = \xi$, meeting C but none of the straight lines parallel to $0\,y$. Since the curve is simple, the intersections of $x = \xi$ with the arcs of type $y = f(x)$ cannot coincide. Let them be arranged according to the increasing magnitude of their ordinates. Let R and S on C_1 and C_2 be two successive intersections for this arrangement. Since R and S are *successive* intersections, RS can meet no arc of C on its interior. Hence points within RS are of the same class by the results of § 1. As in (c), we may show that C_1 and C_2 are of different species. It follows from (c) that the senses of description of C_1 and C_2 are different. Further, since $\psi(t)$[1] is continuous, we can obtain a b such that $b > \left| \psi(t) \right|$ on C.[2] If then η be the greatest of the ordinates among the intersections of $x = \xi$ with C, the ordinate, $x = \xi$, does not meet C for $y < \eta$. Since $b > \left| \psi(t), \right|$ this half-ordinate meets $y = b$. Similarly for any other value of x, $x = \xi'$. Since $y = b$ does not meet C, points above the arcs C_1, C_2, on which (η, ξ) and (η', ξ') lie, which are joined by $y = b$ not meeting C are of the same class. Hence from the definition

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[1] See p. 26.

[2] Osgood Funktionen theorie, s. 13.

C_1 and C_2 are of the same species, and by the senses of description are the same. Hence:

Theorem:—

If any simple closed curve, $x = \varphi(t)$, $y = \psi(t)$, *can be divided into a finite number of arcs, of type* $y = f(x)$ $f(x)$ *being continuous, and of lines parallel to* Oy, *and if intersections of* $x = \xi$ *not meeting an end point of an arc* $y = f(x)$ *but meeting* C *be arranged according to the magnitude of their ordinates, the senses of description of the arcs at these points of intersection are alternately positive and negative, and the sense at the intersection of greatest ordinate is independent of* ξ.

This is the result that we have made use of in the second chapter.

In accordance with custom, the writer of this dissertation appends the following account of his academic course. His elementary training was received in private schools and the Collegiate Institute of Cobourg, Canada. He matriculated into the University of Toronto, in 1895, receiving his B.A. degree from that institution in 1899, and his M.A. in 1901; thesis "Simple Groups of Order less than 2000." He acted as Lecturer in Mathematics in the Royal Millitary College, Kingston, for 1899-1900 and for part of the following year. Since 1900, he has been on the Staff of Wesley College, of the University of Manitoba, Winnipeg. His graduate work has consisted of a summer session at Cornell (1901), and two summer quarters with the year 1904-5 at Chicago. The writer's age is 27, and place of birth, Cobourg, Canada.

The writer wishes to thank the instructors under whom he has worked, for assistance and encrouagement; Professors Baker and, De Lury of the University of Toronto; G. A. Miller, then of Cornell University; Maschke and Dickson, of the University of Chicago; Bliss and Veblen, then of the same institution; and in particular Professor Moore of the University of Chicago; but most of all he wishes to express his thanks to Professor Oskar Bolza of that University, under whose direction most of the writer's work has been done.

Winnipeg, February, 1907.