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A Class of Asymptotic Orbits in the Problem of Three Bodies

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Contents.

I. *Introduction.*

If we have a system consisting of two arbitrary finite spherical bodies revolving in circles about their common center of gravity, Lagrange* has shown that there are three points on the straight line passing through the centers of the finite bodies such that if an infinitesimal body be placed at one of them and be projected so as to be instantaneously fixed relatively to the revolving system it will always remain fixed relatively to the system. These three equilibrium points, as they are called, are separated by the finite bodies, whose masses are denoted by μ and $1-\mu$, $(0<\mu\leqq\frac{1}{2})$. The point beyond the mass μ is called (a), that between the finite bodies is called (b), and that beyond the mass $1-\mu$ is called (c).

In the present paper, it is shown that in the neighborhood of each of the equilibrium points (a), (b), (c) there exists a class of orbits, in which the infinitesimal body approaches the equilibrium points as t becomes infinite. Such orbits are called asymptotic orbits.†

* Lagrange, "Collected Works," Vol. VI, pp. 229–324; Tisserand, "Mécanique Céleste," Vol. I, Chapter VIII; Moulton, "Introduction to Celestial Mechanics" (New Edition), Chapter VIII.

† Poincare, "Les Méthodes Nouvelles de la Mécanique Céleste."

In § II the equations of motion of the infinitesimal body are given with reference to the system of rotating axes, passing through the center of gravity of the finite masses, the x-axis coinciding with the line joining their centers.

In § III there is a detailed explanation of what is meant by asymptotic solutions of systems of differential equations, and asymptotic orbits of a moving body; and it is shown that in the three-body problem under consideration the only existing orbits which are asymptotic to any of the equilibrium points (a), (b), or (c) lie wholly in the plane of revolution of the finite masses.

In § IV the equations of the asymptotic orbits are developed as power series in exponential functions of the time, and in the following article it is shown that these power series expansions are convergent for t sufficiently large.

In § VI an alternative method is given for building up the equations of these orbits.

In § VII there is a discussion of some of the principal properties of these asymptotic orbits, their position relative to the rotating axes, and the change in their direction of approach to the equilibrium points as μ varies from zero to $\frac{1}{2}$.

In § VIII it is shown how the orbits can be continued by the method of mechanical quadratures beyond the range of convergence of the solutions of the differential equations. A special case $\mu=0.02$ is discussed in detail, and one of the corresponding asymptotic orbits for point (a) is continued by this process.

II. *The Differential Equations of Motion of the Infinitesimal Body.*

In the following discussion we consider a system consisting of two finite bodies revolving in circles about their common center of mass, and of an infinitesimal body subject to their attractions. Let the constant distance between the finite bodies be unity. Denote the masses of the finite bodies by μ and $1-\mu$, where $0<\mu\leqq\frac{1}{2}$, so that the sum of the masses shall be unity. Choose the unit of time so that the gravitational constant k^2 shall be unity. With the units so chosen the time of revolution of the finite bodies will be *unity*.

Take the origin of coordinates at the center of mass of the finite bodies, and refer the motion of the bodies to a system of axes rotating in the plane of motion of the finite body, in such a way that the ξ-axis always passes through the centers of the finite bodies. If ξ, η, ζ denote the coordinates of the infinitesimal body, then the differential equations of motion for the infinitesimal body are:*

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2\frac{d\eta}{dt} &= \xi - (1-\mu)\frac{(\xi-\xi_1)}{r_1^3} - \mu\frac{(\xi-\xi_2)}{r_2^3}, \\ \frac{d^2\eta}{dt^2} + 2\frac{d\xi}{dt} &= \eta - (1-\mu)\frac{\eta}{r_1^3} - \mu\frac{\eta}{r_2^3}, \\ \frac{d^2\zeta}{dt^2} \qquad\quad &= -(1-\mu)\frac{\zeta}{r_1^3} - \mu\frac{\zeta}{r_2^3}, \end{aligned}\right\} \tag{1}$$

where $(\xi_1, 0, 0)$, $(\xi_2, 0, 0)$ are the coordinates of the bodies $1-\mu$ and μ respectively; $r_1 = \sqrt{(\xi-\xi_1)^2+\eta^2+\zeta^2}$ and $r_2 = \sqrt{(\xi-\xi_2)^2+\eta^2+\zeta^2}$.

Let $(\xi_0, 0, 0)$ denote the coordinates of one of the equilibrium points on the ξ-axis. If, then, by the transformation $\xi=\xi_0+x$, $\eta=y$, $\zeta=z$, we move the origin to one of these points, the equations of motion take the form†

$$\left.\begin{aligned} x'' - 2y' &= (1+2A)x + \frac{3}{2}B[-2x^2+y^2+z^2] + 2C[2x^3-3xy^2-3xz^2]\ldots, \\ y'' + 2x' &= (1-A)y + 3Bxy + \frac{3}{2}Cy[-4x^2+y^2+z^2] + \ldots, \\ z'' \qquad &= -Az + 3Bxz + \frac{3}{2}Cz[-4x^2+y^2+z^2] + \ldots, \\ \text{where} & \\ A &= \frac{1-\mu}{[(\xi_0+\mu)^2]^{\frac{3}{2}}} + \frac{\mu}{[(\xi_0-1+\mu)^2]^{\frac{3}{2}}} = \frac{1-\mu}{r_1^{(0)^3}} + \frac{\mu}{r_2^{(0)^3}}, \\ B &= \substack{+\\ \pm}\frac{1-\mu}{r_1^{(0)^4}} \substack{\pm\\ -}\frac{\mu}{r_2^{(0)^4}}, \\ C &= \frac{1-\mu}{r_1^{(0)^5}} + \frac{\mu}{r_2^{(0)^5}}, \end{aligned}\right\} \tag{2}$$

where in the expression for B the upper, middle, or lower signs are to be taken according as the orbits in the vicinity of the point (a), (b), or (c) are being treated.

In what follows we shall have to deal chiefly with the first two equations of (2), and it will be more convenient to have them in a normal form.

The linear terms of the first two equations of (2) are

$$\left.\begin{aligned} x'' - 2y' - (1+2A)x &= 0, \\ y'' + 2x' - (1-\ A)y &= 0. \end{aligned}\right\} \tag{3}$$

* Moulton, "Introduction to Celestial Mechanics" (New Edition), §152.

† Moulton, "Periodic Orbits," pp. 156, 168.

The general solution of these equations is

$$\left.\begin{aligned} x&=K_1e^{\sigma\sqrt{-1}t}+K_2e^{-\sigma\sqrt{-1}t}+K_3e^{\rho t}+K_4e^{-\rho t},\\ y&=n\sqrt{-1}\,[K_1e^{\sigma\sqrt{-1}t}-K_2e^{-\sigma\sqrt{-1}t}]+m\,[K_3e^{\rho t}-K_4e^{-\rho t}].\end{aligned}\right\}\quad(4)$$

Where $\sigma\sqrt{-1}$, $-\sigma\sqrt{-1}$, ρ, $-\rho$ are the roots of the bi-quadratic equation

$$\lambda^4+(2-A)\lambda^2+(1+2A)(1-A)=0. \qquad (5)$$

If then we make the transformation

$$\left.\begin{aligned} x&=u_1+u_2+u_3+u_4,\\ x'&=\sigma\sqrt{-1}\,(u_1-u_2)+\rho(u_3-u_4),\\ y&=n\sqrt{-1}\,(u_1-u_2)+m(u_3-u_4),\\ y'&=-n\sigma(u_1+u_2)+m\rho(u_3+u_4),\end{aligned}\right\}\quad(6)$$

the first two equations of (3) assume the normal form,*

$$\left.\begin{aligned} u_1'&=\sigma\sqrt{-1}\,u_1+\frac{m}{2(m\sigma-n\rho)\sqrt{-1}}\Big[\frac{3}{2}B(-2x^2+y^2+z^2)\\ &\qquad\qquad+2C(2x^3-3xy^2-3xz^2)+\dots\Big]\\ &\qquad-\frac{1}{2(m\rho+n\sigma)}\Big\{3Bxy+\frac{3}{2}Cy(-4x^2+y^2+z^2)+\dots\Big\},\\ u_2'&=-\sigma\sqrt{-1}\,u_2-\frac{m}{2(m\sigma-n\rho)\sqrt{-1}}\Big[\text{“}\Big]-\frac{1}{2(m\rho+n\sigma)}\Big\{\text{“}\Big\},\\ u_3'&=\quad\rho u_3-\frac{n}{2(m\sigma-n\rho)}\Big[\text{“}\Big]+\frac{1}{2(m\rho+n\sigma)}\Big\{\text{“}\Big\},\\ u_4'&=-\rho u_4+\frac{n}{2(m\sigma-n\rho)}\Big[\text{“}\Big]+\frac{1}{2(m\rho+n\sigma)}\Big\{\text{“}\Big\}.\end{aligned}\right\}\quad(7)$$

Equation (7) may be put in the form

$$\left.\begin{aligned} u_1'&=\quad\sigma\sqrt{-1}\,u_1+H_1^{(2)}+H_1^{(3)}+H_1^{(4)}+\dots,\\ u_2'&=-\sigma\sqrt{-1}\,u_2+H_2^{(2)}+H_2^{(3)}+H_2^{(4)}+\dots,\\ u_3'&=\quad\rho u_3+H_3^{(2)}+H_3^{(3)}+H_3^{(4)}+\dots,\\ u_4'&=-\rho u_4+H_4^{(2)}+H_4^{(3)}+H_4^{(4)}+\dots,\end{aligned}\right\}\quad(8)$$

where $H_k^{(r)}$ denotes all the terms on the right-hand side of the k-th equation which are of degree r in x, y, z, and therefore of degree r in u_1, u_2, u_3, u_4.

III. *Asymptotic Orbits Defined and Proof that They all Lie in the xy-Plane.*

It has been shown by Póincaré† and Picard‡ that certain systems of differential equations of the form $\dfrac{dx_i}{dt}=X_i(x,t)\ (i=1,\dots,n)$ admit of solu-

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* Moulton, "Periodic Orbits," pp. 161–162.

† Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, Chap. VIII.

‡ Picard, "Traité D'Analyse," Vol. III, Chap. VIII, § V.

tions as power series in $A_i e^{\lambda_i t}$ $(i=1, \ldots, n)$, where the A_i are arbitrary constants, and where the fixed constants λ_i, which are the roots of the characteristic equation, are called the "characteristic exponents." It has been shown further* that if there are k of the λ_i $(i=1, \ldots, n)$ which are represented by k points on the complex plane all of which lie on the same side of a straight line passing through the origin, and which are such that none of the relations $\sum_{j=1}^{k} p_j \lambda_j - \lambda_i = 0$ $(i=1, \ldots, n)$ holds for any positive integral values of the p_j such that $\sum_{j=1}^{k} p_j \geq 2$, then the solutions as power series in $A_i e^{\lambda_i t}$ $(i=1, \ldots, k)$ will be convergent for $|A_i e^{\lambda_i t}|$ sufficiently small. In particular, if we put equal to zero the A_i corresponding to those λ_i whose real parts are zero or positive, the solutions as power series in the remaining $A_i e^{\lambda_i t}$ will be convergent for all values of t which are sufficiently great; and if the A_i involved in these latter expansions are taken sufficiently small the convergence will hold for all values of t from $t=0$ to $t=\infty$.

Again, if we build up solutions as power series in those $A_j e^{\lambda_j t}$ where the real parts of λ_j are positive, the exponentials $e^{\lambda_j t}$ approach zero as t approaches $-\infty$. Such expansions will be convergent for t sufficiently large and negative; and if the A_j are sufficiently small they will be convergent for all negative values of t. Such solutions are said to be "asymptotic" to the solutions obtained by putting all the A_i $(i=1, \ldots, n)$ equal to zero, and they are called "Asymptotic Solutions of the System of Differential Equations."

In the problem under consideration we shall show that the differential equations of motion of the infinitesimal body have asymptotic solutions such as have just been described. We shall see also that the infinitesimal body, moving in an orbit defined by one of these solutions, will approach asymptotically one of the equilibrium points (a), (b), or (c) as t becomes infinitely great.

We proceed to show, first of all, that all orbits which are asymptotic to one of the equilibrium points (a), (b), or (c) lie entirely in the plane of revolution of the finite bodies, that is, in the xy-plane.

If in the equations of motion (2), we make the transformation $x=x\varepsilon$, $y=y\varepsilon$, $z=z\varepsilon$, we obtain, on dividing through by ε,

$$\left.\begin{aligned} x''-2y'-(1+2A)x &= \varepsilon X_2(x^2, y^2, z^2)+\varepsilon^2 X_3(x^3, xy^2, xz^2)+\varepsilon^3 X_4(\ldots\ldots)+\ldots\ldots, \\ y''+2x'-(1-\ A)x &= \varepsilon y Y_2(x)+\varepsilon^2 y Y_3(x^2, y^2, z^2)+\varepsilon^3 y Y_4(\ldots\ldots)+\ldots\ldots, \\ z''+\qquad\qquad Az &= \varepsilon z Z_2(x)+\varepsilon^2 z Z_3(x^2, y^2, z^2)+\varepsilon^3 z Z_4(\ldots\ldots)+\ldots\ldots. \end{aligned}\right\} \quad (9)$$

* Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, Chap. VIII, § 105.

Since the right-hand members of (2) converge, so also will the right members of (9) converge for all ε $(0<\varepsilon\leqq 1)$. It follows, therefore, that the equations (9) have a solution of the form

$$\left.\begin{aligned} x &= \sum_{j=0}^{\infty} x_j(t)\varepsilon^j, \\ y &= \sum_{j=0}^{\infty} y_j(t)\varepsilon^j, \\ z &= \sum_{j=0}^{\infty} z_j(t)\varepsilon^j. \end{aligned}\right\} \quad (10)$$

In order that z may be zero for all t sufficiently large, and for all values of ε, it follows that $z_j(t)$ $(j=0,\ldots.\infty)$ must each be zero for all t sufficiently large. On substituting (10) in (9) and equating coefficients of like powers of ε on both sides of the resulting equations, we obtain sets of differential equations from which the values of x_j, y_j, z_j $(j=0,\ldots.\infty)$ can be obtained sequentially.

From the terms independent of ε, we have

$$\left.\begin{aligned} &x_0''-2y_0'-(1+2A)x_0=0, \\ &y_0''+2x_0-(1-\ A)y_0=0, \\ &z_0''\qquad\qquad +\ Az_0\ =0. \end{aligned}\right\} \quad (11)$$

The general solution of equations (11) is

$$\left.\begin{aligned} &x_0=K_1^{(0)}e^{\sigma\sqrt{-1}t}+K_2^{(0)}e^{-\sigma\sqrt{-1}t}+K_3^{(0)}e^{\rho t}+K_4^{(0)}e^{-\rho t}, \\ &y_0=n\sqrt{-1}[K_1^{(0)}e^{\sigma\sqrt{-1}t}-K_2^{(0)}e^{-\sigma\sqrt{-1}t}]+m[K_3^{(0)}e^{\rho t}-K_4^{(0)}e^{-\rho t}], \\ &z_0=c_1^{(0)}\cos\sqrt{A}\,t+c_2^{(0)}\sin\sqrt{A}\,t. \end{aligned}\right\} \quad (12)$$

In order that $z_0(t)$ shall approach zero for t infinite, we see that $c_1^{(0)}=c_2^{(0)}=0$, and, therefore

$$z_0(t)=0. \quad (13)$$

When we equate the terms in the first power of ε, we get

$$\left.\begin{aligned} &x_1''-2y_1''-(1+2A)x_1=X_2(x_0^2,\ y_0^2,\ z_0^2), \\ &y_1''+2x_1''-(1-\ A)y_1=y_0Y_2(x_0), \\ &z_1''\qquad\qquad +\ \ Az_1=z_0Z_2(x_0). \end{aligned}\right\} \quad (14)$$

If we substitute the value $z_0=0$ in (14) the third equation becomes $z_1''+Az_1=0$, which is of the same form as the third equation in (11). In order that $z_1(t)$ shall approach zero for t infinite we see, therefore, that $z_1(t)\underset{t}{\equiv}0$. Similarly, we can show that $z_2(t)\underset{t}{\equiv}0$. Suppose we have proved sequentially by this method that $z_j(t)\underset{t}{\equiv}0$, for $j=0, 1, \ldots, n$. Since the right-hand member of the third equation of (9) carries z as a factor, and since the factor ε has been removed from this equation, it follows that the right member of the differential equation which defines z_{n+1} will consist of terms which carry as a factor one or more of the $z_j(t)$ $(j=0, 1, \ldots, n)$ but will not contain any z_j, $j>n$. Hence, the right member of this equation is zero, and, therefore, $z_{n+1}(t) \underset{t}{\equiv} 0$. We see, therefore, that $z_j(t) \underset{t}{\equiv} 0$ for $j=0, 1, \ldots \infty$, from which it follows that $z(t) \underset{t}{\equiv} 0$. In order, therefore, that the infinitesimal body shall come to rest at one of the equilibrium points (a), (b), (c) its whole orbit must be in the xy-plane.

IV. *Formal Construction of the Solutions.*

We proceed to show that equations (7) admit of solutions defining asymptotic orbits in the xy-plane. These will be found as power series in $e^{-\rho t}$, and $e^{+\rho t}$, the former convergent for t sufficiently large and positive, the latter for t sufficiently large and negative.

(A) *Solutions as Power Series in* $e^{-\rho t}$.

If, in equations (8), we make a transformation on the independent variable by putting $\omega = e^{-\rho t}$ these equations take the form

$$\left.\begin{aligned}
-\rho\omega\frac{\partial u_1}{\partial \omega} &= \sigma\sqrt{-1}\,u_1 + H_1^{(2)} + H_1^{(3)} + \ldots, \\
-\rho\omega\frac{\partial u_2}{\partial \omega} &= -\sigma\sqrt{-1}\,u_2 + H_2^{(2)} + H_2^{(3)} + \ldots, \\
-\rho\omega\frac{\partial u_3}{\partial \omega} &= \rho u_3 + H_3^{(2)} + H_3^{(3)} + \ldots, \\
-\rho\omega\frac{\partial u_4}{\partial \omega} &= -\rho u_4 + H_4^{(2)} + H_4^{(3)} + \ldots.
\end{aligned}\right\} \quad (15)$$

It is required to find solutions of (15) as power series in ω, convergent for ω sufficiently small.

By Maclaurin's expansion

$$u_i(\omega) = u_i(0) + \omega\left(\frac{\partial u_i}{\partial \omega}\right)_{\omega=0} + \frac{\omega^2}{2!}\left(\frac{\partial^2 u_i}{\partial \omega^2}\right)_{\omega=0} + \ldots, \quad (i=1, \ldots, 4). \quad (16)$$

Since the body is to be at rest at the origin at $\omega=0$, it follows from (6) that $u_i(0)=0$, $(i=1, \ldots, 4)$. By repeated differentiation of equations (15) with regard to ω, and putting $\omega=0$, we can build up the coefficients of the successive powers of ω in the expansion (16). Since

$$\left.\begin{aligned}
x &= u_1 + u_2 + u_3 + u_4, \\
y &= n\sqrt{-1}\,(u_1 - u_2) + m(u_3 - u_4),
\end{aligned}\right\} \quad (17)$$

it follows that the coefficients of successive powers of ω in the expansions for x and y will thus be known also.

The first equation of (15) written at length is

$$-\rho\omega\frac{\partial u_1}{\partial\omega}=\sigma\sqrt{-1}\,u_1+\left[\frac{3mB}{4(m\sigma-n\rho)\sqrt{-1}}(-2x^2+y^2)-\frac{3B}{2(m\rho+n\sigma)}(xy)\right]$$
$$+\left[\frac{mC}{(m\sigma-n\rho)\sqrt{-1}}(2x^3-3xy^2)-\frac{3C}{4(m\rho+n\sigma)}(-4x^2y+y^3)\right]$$
$$+\,(\text{terms of higher degree in } x \text{ and } y).$$

On differentiating this with regard to ω, we get

$$-\rho\frac{\partial u_1}{\partial\omega}-\rho\omega\frac{\partial^2 u_1}{\partial\omega^2}=\sigma\sqrt{-1}\frac{\partial u_1}{\partial\omega}$$
$$+\left[\frac{3mB}{4(m\sigma-n\rho)\sqrt{-1}}\left(-4x\frac{\partial x}{\partial\omega}+2y\frac{\partial y}{\partial\omega}\right)-\frac{3B}{2(m\rho+n\sigma)}\left(y\frac{\partial x}{\partial\omega}+x\frac{\partial y}{\partial\omega}\right)\right]$$
$$+\left[\frac{mC}{(m\sigma-n\rho).\sqrt{-1}}\left(6x^2\frac{\partial x}{\partial\omega}-6xy\frac{\partial y}{\partial\omega}-3y^2\frac{\partial x}{\partial\omega}\right)-\frac{3C}{4(m\rho+n\sigma)}\right.$$
$$\left.\left(-8xy\frac{\partial x}{\partial\omega}-4x^2\frac{\partial y}{\partial\omega}+3y^2\frac{\partial y}{\partial\omega}\right)\right]+\ldots\ldots \qquad (18)$$

On putting $\omega=0$, and therefore $x=y=0$, this becomes

$$(-\rho-\sigma\sqrt{-1})\left(\frac{\partial u_1}{\partial\omega}\right)_{\omega=0}=0. \qquad (19)$$

Since $A>1$* for each of the equilibrium points for $0 \lesseqgtr \mu \leqq \frac{1}{2}$, it can be readily seen that two of the roots of equation (5) are real and equal numerically but opposite in sign, and that the other two are conjugate imaginaries. Further, none of the roots of (5) is zero. It follows that none of the relations,

$$p\rho=0;\quad p\rho=-\rho;\quad p\rho=\pm\sigma\sqrt{-1},$$

can hold for $p \gtreqless 1$ (p a positive integer). Hence, from (19) we see that

$$\left(\frac{\partial u_1}{\partial\omega}\right)_{\omega=0}=0. \qquad (20)$$

Similarly, by differentiating the second, third, and fourth equations of (15) with regard to ω, we obtain, respectively,

$$(-\rho+\sigma\sqrt{-1})\left(\frac{\partial u_2}{\partial\omega}\right)_{\omega=0}=0,\ -2\rho\left(\frac{\partial u_3}{\partial\omega}\right)_{\omega=0}=0,\ \text{and}\ (-\rho+\rho)\left(\frac{\partial u_4}{\partial\omega}\right)_{\omega=0}=0. \qquad (21)$$

* Moulton, "Periodic Orbits," p. 159.

It follows then, that

$$\left(\frac{\partial u_2}{\partial \omega}\right)_{\omega=0}=0;\quad \left(\frac{\partial u_3}{\partial \omega}\right)_{\omega=0}=0;\quad \left(\frac{\partial u_4}{\partial \omega}\right)_{\omega=0}=\text{arbitrary}=c.$$

Hence,

$$\left(\frac{\partial x}{\partial \omega}\right)_{\omega=0}=c;\quad\text{and}\quad \left(\frac{\partial y}{\partial \omega}\right)_{\omega=0}=-mc. \tag{22}$$

If we differentiate equations (18) and the three corresponding equations in u_2, u_3, and u_4 obtained from (15), and put $\omega=0$, we obtain in succession, by the aid of (20) and (21)

$$\left.\begin{aligned}
\left(\frac{\partial^2 u_1}{\partial \omega^2}\right)_{\omega=0}&=-\frac{1}{2\rho+\sigma\sqrt{-1}}\left[\frac{3m(m^2-2)B}{2(m\sigma-n\rho)\sqrt{-1}}+\frac{3mB}{m\rho+n\sigma}\right]c^2,\\
\left(\frac{\partial^2 u_2}{\partial \omega^2}\right)_{\omega=0}&=\frac{1}{2\rho-\sigma\sqrt{-1}}\left[\frac{3m(m^2-2)B}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3mB}{m\rho+n\sigma}\right]c^2,\\
\left(\frac{\partial^2 u_3}{\partial \omega^2}\right)_{\omega=0}&=\frac{1}{3\rho}\left[\frac{3n(m^2-2)B}{2(m\sigma-n\rho)}+\frac{3mB}{m\rho+n\sigma}\right]c^2,\\
\left(\frac{\partial^2 u_4}{\partial \omega^2}\right)_{\omega=0}&=-\frac{1}{\rho}\left[\frac{3n(m^2-2)B}{2(m\sigma-n\rho)}-\frac{3mB}{m\rho+n\sigma}\right]c^2.
\end{aligned}\right\} \tag{23}$$

On reduction, then, we obtain

$$\left.\begin{aligned}
\left(\frac{\partial^2 x}{\partial \omega^2}\right)_{\omega=0}&=c^2\left[\frac{3m(m^2-2)B\sigma}{(m\sigma-n\rho)(4\rho^2+\sigma^2)}-\frac{n(m^2-2)B}{(m\sigma-n\rho)\rho}\right.\\
&\qquad\left.-\frac{12mB\rho}{(m\rho+n\sigma)(4\rho^2+\sigma^2)}+\frac{4mB}{(m\rho+n\sigma)\rho}\right],\\
\left(\frac{\partial^2 y}{\partial \omega^2}\right)_{\omega=0}&=-mc^2\left[\frac{6n(m^2-2)B\rho}{(m\sigma-n\rho)(4\rho^2+\sigma^2)}-\frac{2n(m^2-2)B}{(m\sigma-n\rho)\rho}\right.\\
&\qquad\left.+\frac{6nB\sigma}{(m\rho+n\sigma)(4\rho^2+\sigma^2)}+\frac{2mB}{(m\rho+n\sigma)\rho}\right].
\end{aligned}\right\} \tag{24}$$

By repeating the process and applying (20) and (21) at each step, we can find in succession the values of $\left(\frac{\partial^k u_i}{\partial \omega^k}\right)_{\omega=0}$ $(i=1,\ \ldots,\ 4;\ k=3,\ \ldots\ \infty)$, and thence we can readily find $\left(\frac{\partial^k x}{\partial \omega^k}\right)_{\omega=0}$ and $\left(\frac{\partial^k y}{\partial \omega^k}\right)_{\omega=0}$. We notice from (22) and (24) that the first and second partial derivatives carry the arbitrary parameter c to the first and second powers respectively. We see further that, at each step in the differentiating, the right-hand members of the equations are homogeneous in the orders of the partial derivatives in each term. It follows then that $\left(\frac{\partial^k u_i}{\partial\omega^k}\right)_{\omega=0}$ and, therefore, also $\left(\frac{\partial^k x}{\partial\omega^k}\right)_{\omega=0}$ and $\left(\frac{\partial^k y}{\partial\omega^k}\right)_{k=0}$ each carry a factor c^k, after the values of the partial derivatives of orders lower than k have been substituted in the right-hand members. Hence, the terms of the expansion in (16) carry c and ω as factors to the same power. On substituting the values of the partial derivatives in Maclaurin's expansions for x and y, and replacing ω by its value $e^{-\rho t}$, we obtain a set of solutions of equations (8) in the form

$$\left.\begin{aligned} x= &\quad ce^{-\rho t}+\frac{1}{2\,!}\left[\frac{3m(m^2-2)B\sigma}{(m\sigma-n\rho)(4\rho^2+\sigma^2)}-\frac{n(m^2-2)B}{(m\sigma-n\rho)\rho}-\frac{12mB\rho}{(m\rho+n\sigma)(4\rho^2+\sigma^2)}\right.\\ &\qquad\left.+\frac{4mB}{(m\rho+n\sigma)\rho}\right]c^2e^{-2\rho t}+\frac{1}{3\,!}[\ldots\ldots]c^3e^{-3\rho t}+\ldots\ldots,\\ y= &-mce^{-\rho t}-\frac{m}{2\,!}\left[\frac{6n(m^2-2)B\rho}{(m\sigma-n\rho)(4\rho^2+\sigma^2)}-\frac{2n(m^2-2)B}{(m\sigma-n\rho)\rho}+\frac{6nB\sigma}{(m\rho+n\sigma)(4\rho^2+\sigma^2)}\right.\\ &\qquad\left.+\frac{2mB}{(m\rho+n\sigma)\rho}\right]c^2e^{-2\rho t}-\frac{m}{3\,!}[\ldots\ldots]c^3e^{-3\rho t}+\ldots\ldots \end{aligned}\right\} \tag{25}$$

(B) *Solutions in Powers of $e^{+\rho t}$.*

If, in equations (8), we were to transform our independent variable by writing $\omega=e^{+\rho t}$, we could build up a second set of solutions of form similar to (25), by a process exactly parallel to that used in section (A). We shall show, however, that this new set of solutions can be obtained directly from solutions (25) by changing the sign of t throughout, and changing the sign of y in the result.

The first two equations of (1), with z dropped from the right-hand members, can be written in the form

$$\left.\begin{aligned} x''-2y'&=F_1(x, y^2),\\ y''+2x'&=yF_2(x, y^2). \end{aligned}\right\} \tag{26}$$

If we suppose the initial conditions are

$$x(0)=\alpha,\quad x'(0)=\alpha_1,\quad y(0)=\beta,\quad y'(0)=\beta_1, \tag{27}$$

then the solutions of (26) have the form

$$x=f_1(t),\quad x'=\phi_1(t),\quad y=f_2(t),\quad y'=\phi_2(t). \tag{28}$$

If in equations (26) we put $x=x$, $y=-\eta$, and $t=-\tau$, we get equations of identically the same form in x, η, and τ as (26) are in x, y, and t. These equations are

$$\left.\begin{aligned} \ddot{x}-2\dot{\eta}&=F_1(x, \eta^2),\\ \ddot{\eta}+2\dot{x}&=\eta F_2(x, \eta^2), \end{aligned}\right\} \tag{29}$$

where the dot denotes differentiation with regard to τ. If we impose the same initial conditions in these new variables, viz.:

$$x(0)=\alpha,\quad \dot{x}(0)=\alpha_1,\quad \eta(0)=\beta,\quad \dot{\eta}(0)=\beta_1, \tag{30}$$

then the solutions of (29) are

$$x=f_1(\tau),\quad \dot{x}=\phi_1(\tau),\quad \eta=f_2(\tau),\quad \dot{\eta}=\phi_2(\tau); \tag{31}$$

that is x, $\dot{x}$, η, and $\dot{\eta}$ are the same functions of τ as x, x', y and y' were of t before. But the initial conditions (30) are the same as

$$x(0)=\alpha,\quad x'(0)=-\alpha_1,\quad y_0=-\beta,\quad y'(0)=\beta_1. \tag{32}$$

Again, the solutions (31) are the same as

$$x=f_1(-t),\quad x'=-\phi_1(-t),\quad y=-f_2(-t),\quad y'=\phi_2(-t).$$

It is readily seen, therefore, that for initial conditions (32) equations (8) admit of a solution which can be derived from (25) simply by changing the sign of t throughout and changing the sign of y in the result. These solutions, therefore, have the form

$$\left.\begin{aligned} x= {} & c_1e^{\rho t}+\frac{1}{2!}\left[\frac{3m(m^2-2)B\sigma}{(m\sigma-n\rho)(4\rho^2+\sigma^2)}-\frac{n(m^2-2)B}{(m\sigma-n\rho)\rho}-\frac{12mB\rho}{(m\rho+n\sigma)(4\rho^2+\sigma^2)}\right. \\ & \qquad\left.+\frac{4mB}{(m\rho+n\sigma)\rho}\right]c_1^2e^{2\rho t}+\frac{1}{3!}[\ldots\ldots]c_1^3e^{3\rho t}+\ldots\ldots, \\ y= {} & mc_1e^{\rho t}+\frac{m}{2!}\left[\frac{6n(m^2-2)B\rho}{(m\sigma-n\rho)(4\rho^2+\sigma^2)}-\frac{2n(m^2-2)B}{(m\sigma-n\rho)\rho}+\frac{6nB\sigma}{(m\rho+n\sigma)(4\rho^2+\sigma^2)}\right. \\ & \qquad\left.+\frac{2mB}{(m\rho+n\sigma)\rho}\right]c_1^2e^{2\rho t}+\frac{m}{3!}[\ldots\ldots]c_1^3e^{3\rho t}+\ldots\ldots \end{aligned}\right\} \tag{33}$$

V. *Proof of the Convergence of the Solutions.*

It is necessary to prove the convergence of the series (25), which we have deduced from equations (8), or their transformed equivalents (9). This we shall do by a method analogous to that used by Picard.* Write equations (9) in the following form

$$\left.\begin{aligned} -\rho\omega\frac{\partial u_1}{\partial\omega}-\sigma\sqrt{-1}\,u_1 &= H_1^{(2)}+H_1^{(3)}+\ldots\ldots, \\ -\rho\omega\frac{\partial u_2}{\partial\omega}+\sigma\sqrt{-1}\,u_2 &= H_2^{(2)}+H_2^{(3)}+\ldots\ldots, \\ -\rho\omega\frac{\partial u_3}{\partial\omega}-\rho u_3 \qquad &= H_3^{(2)}+H_3^{(3)}+\ldots\ldots, \\ -\rho\omega\frac{\partial u_4}{\partial\omega}+\rho u_4 \qquad &= H_4^{(2)}+H_4^{(3)}+\ldots\ldots \end{aligned}\right\} \tag{34}$$

---

* Picard, "Traité D'Analyse," Vol. III, Chap. I, § 12.

The coefficients of any of the partial derivatives of u_i $(i=1, \ldots, 4)$ for $\omega=0$ are respectively of the form

$$p(-\rho)-\sigma\sqrt{-1},\quad p(-\rho)-(-\sigma\sqrt{-1}),\quad p(-\rho)-\rho,\quad p(-\rho)-(-\rho), \tag{35}$$

all of which are, in absolute value, greater than $\varepsilon(p-1)$ where ε is a real quantity. In this case ε is a fixed quantity smaller than the absolute value of the smallest of $\pm\rho$ and $\pm\sigma\sqrt{-1}$.

It has been shown by Moulton* that the expansions in the right-hand members of (14) converge within and on the boundary of a circle of radius α $(\alpha>0)$ about each of the equilibrium points; the value of α depending upon which of the points (a), (b), or (c) is being considered.

Let M be the maximum modulus of the expressions in the right members of (34) for all values of the variables u_1, u_2, u_3, u_4, in this circle of radius α.

Consider the comparison set of differential equations

$$\varepsilon\left[\omega\frac{\partial v_i}{\partial\omega}-v_i\right]=\frac{M}{1-\dfrac{v_1+v_2+v_3+v_4}{\alpha}}-M-M\frac{v_1+v_2+v_3+v_4}{\alpha},\quad (i=1,\ldots,4). \tag{36}$$

It is evident that equations (36) dominate (34); and it can readily be shown that the solutions of (36) dominate the solutions of (34). The terms on the right of (36) are all positive, and it is obvious from the method by which the solutions are built up that all the terms of the solution of (36) are positive. In building up the solutions of (34) and (36) as power series in ω, whose coefficients are the values of the successive derivatives of the u_i and the v_i, respectively, for $\omega=0$, we see by (35) that the absolute value of the coefficient of any partial derivative on the left obtained from (34) is greater than the coefficient of the corresponding partial derivative on the left obtained from (36). But each term on the right-hand side of (36) dominates the corresponding term on the right of (34), and therefore any partial derivative of the right members of (36) is greater than the absolute value of the corresponding partial derivative of the right members of (34). It follows, therefore, that the values of the successive partial derivatives obtained from (36) are greater respectively than the absolute values of the corresponding partial derivatives obtained from (34). Hence, each term in the solutions of (36) is greater than the absolute value of the corresponding term in the solutions of (34); or the solutions of (36) dominate those of (34).

* Moulton, "Periodic Orbits," p. 154.

It remains to be shown that the solutions of (36) are convergent for ω sufficiently small. From the symmetry of (36) in the v_i ($i=1, \ldots, 4$), all the v_i are equal. We can, therefore, replace equations (36) by a single equation in one variable, viz.:

$$\varepsilon\left[\omega\frac{\partial v}{\partial\omega}-v\right]=\frac{M}{1-\dfrac{4v}{\alpha}}-M-M\frac{4v}{\alpha},$$

which reduces to the form

$$\omega\frac{\partial v}{\partial\omega}-v=\frac{M'v^2}{\alpha-4v}, \text{ where } M'=\frac{16M}{\alpha\varepsilon}; \tag{37}$$

whence,

$$\frac{\partial\omega}{\omega}=\frac{\partial v}{v}-\frac{M'\partial v}{\alpha+(M'-4)v}.$$

On integrating this equation, we have

$$\log c\omega=\log\frac{v}{[\alpha+(M'-4)v]^{\frac{M'}{M'-4}}},$$

or

$$c\omega=\frac{v}{\left(\alpha+\dfrac{M'}{\kappa}v\right)^{\kappa}}, \text{ where } \kappa=\frac{M'}{M'-4}.$$

Therefore,

$$v-c\omega\left(\alpha^{\kappa}+\kappa\alpha^{\kappa-1}\frac{M'}{\kappa}v+\ldots\right)=0.$$

From this it follows, by the theory of implicit functions, that v can be expressed as a power series in ω, vanishing with ω, and convergent for ω sufficiently small. Thus, we see that the solutions of (36) converge for ω sufficiently small; and it has been shown that the solutions of (36) dominate the solutions of (34). Hence the solutions of (34) are convergent for ω sufficiently small, that is for t sufficiently large.

VI. *Alternative Method of Constructing the Solutions.*

The solutions of equations (8) may be built up by a process quite different from that used in § IV.

If, in equations (2), we make the transformation $x=x\varepsilon$, $y=y\varepsilon$, the first two equations, considered for $z=0$, give, on dividing through by ε,

$$\left.\begin{aligned} x''-2y'&=(1+2A)x+\tfrac{3}{2}B[-2x^2+y^2]\varepsilon+2C[2x^3-3xy^2]\varepsilon^2+\ldots, \\ y''+2x'&=(1-A)y+3Bxy\varepsilon+\tfrac{3}{2}Cy[-4x^2+y^2]\varepsilon^2+\ldots. \end{aligned}\right\} \tag{38}$$

Since, as was pointed out in § V, the right-hand members of the original equations converge within and on the boundary of a circle of radius α ($\alpha > 0$) about each of the equilibrium points, it follows that the right-hand members of (38) converge for ε sufficiently small ($0 < \varepsilon \overline{\gtrless} 1$). Such a system of differential equations, with constant coefficients on the left, can be solved for x and y as power series in ε, of the form

$$\left.\begin{aligned} x &= \sum_{j=0}^{\infty} x_j(t)\varepsilon^j, \\ y &= \sum_{j=0}^{\infty} y_j(t)\varepsilon^j, \end{aligned}\right\} \quad (39)$$

where the coefficients x_j and y_j are to be determined. Since (39) are the solutions of (38), the latter must be satisfied identically when x and y are replaced by the power series in (38). When we make this substitution, from terms independent of ε, we have

$$\left.\begin{aligned} x_0'' - 2y_0' - (1+2A)x_0 &= 0, \\ y_0'' + 2x_0' - (1-\ A)y_0 &= 0. \end{aligned}\right\} \quad (40)$$

The general solution of these equations is

$$\left.\begin{aligned} x_0 &= K_1 e^{\sigma\sqrt{-1}t} + K_2 e^{-\sigma\sqrt{-1}t} + K_3 e^{\rho t} + K_4 e^{-\rho t}, \\ y_0 &= L_1 e^{\sigma\sqrt{-1}t} + L_2 e^{-\sigma\sqrt{-1}t} + L_3 e^{\rho t} + L_4 e^{-\rho t}, \end{aligned}\right\} \quad (41)$$

where the K_i and L_i are constants of integration, the K_i ($i=1, \ldots, 4$) being arbitrary, that is dependent upon the initial conditions imposed which are arbitrary, and the L_i depending on the K_i, the relations between them being*

$$\left.\begin{aligned} L_1 &= \sqrt{-1}\,\frac{\sigma^2+1+2A}{2\sigma} K_1 = \sqrt{-1}\, nK_1 = -\frac{K_1}{K_2} L_2, \\ L_3 &= \frac{\rho^2-1-2A}{2\rho} K_3 = mK_3 = -\frac{K_3}{K_4} L_4. \end{aligned}\right\} \quad (42)$$

Since the K_i are arbitrary, and since we are seeking solutions which vanish as t approaches infinity, we choose $K_1 = K_2 = K_3 = 0$, and, therefore, also $L_1 = L_2 = L_3 = 0$. Equations (40) then have a particular solution of the form

$$x_0 = K_4 e^{-\rho t}, \quad y_0 = L_1 e^{-\rho t}. \quad (43)$$

Let us impose the initial conditions that $x=b$ at $t=0$. Since

$$x(t) \underset{\varepsilon}{\equiv} x_0(t) + x_1(t)\varepsilon + x_2(t)\varepsilon^2 + \ldots,$$

then

$$x_0(0) = b, \ x_j(0) = 0, \quad (j=1, \ldots \infty). \quad (44)$$

---

* Moulton, "Periodic Orbits," p. 159, Eqns. (28).

On imposing these initial conditions and using (42), we see that (43) takes the form

$$x_0 = be^{-\rho t}, \quad y_0 = -mbe^{-\rho t}. \tag{45}$$

In finding successively the values of x_j and y_j $(j=1, \ldots . \infty)$ it is more convenient to use the normal form of equations (26).

As we saw in § 2, the transformations

$$\left.\begin{aligned} x &= u_1+u_2+u_3+u_4, \\ x' &= \sigma\sqrt{-1}\,(u_1-u_2)+\rho(u_3-u_4), \\ y &= n\sqrt{-1}\,(u_1-u_2)+m(u_3-u_4), \\ y' &= -n\sigma(u_1+u_2)+m\rho(u_3+u_4), \end{aligned}\right\} \tag{46}$$

change equations (26) into the normal form

$$\left.\begin{aligned} u_1'-\sigma\sqrt{-1}\,u_1 &= \frac{m[\ldots.]\varepsilon}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{\{\ldots.\}\varepsilon}{2(m\rho+n\sigma)}, \\ u_2'+\sigma\sqrt{-1}\,u_2 &= -\frac{m[\ldots.]\varepsilon}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{\{\ldots.\}\varepsilon}{2(m\rho+n\sigma)}, \\ u_3'-\rho u_3 &= \frac{-n[\ldots.]\varepsilon}{2(m\sigma-n\rho)}+\frac{\{\ldots.\}\varepsilon}{2(m\rho+n\sigma)}, \\ u_4'+\rho u_4 &= \frac{n[\ldots.]\varepsilon}{2(m\sigma-n\rho)}+\frac{\{\ldots.\}\varepsilon}{2(m\rho+n\sigma)}, \end{aligned}\right\} \tag{47}$$

where

$$\left.\begin{aligned} [\ldots.] &= \tfrac{3}{2}B[-2x^2+y^2]+2C[2x^3-3xy^2]\varepsilon+ \text{ terms in } \varepsilon^2, \varepsilon^3, \ldots., \\ \{\ldots.\} &= 3B\{xy\}+\tfrac{3}{2}Cy\{-4x^2+y^2\}\varepsilon+ \text{ terms in } \varepsilon^2, \varepsilon^3, \ldots.. \end{aligned}\right\} \tag{48}$$

These equations have solutions of the form

$$u_i = \sum_{j=0}^{\infty} u_i^{(j)}\varepsilon^j, \qquad (i=1, \ldots., 4).$$

To determine the $u_i^{(j)}$ we substitute (49) in (47) and equate coefficients of like powers of ε on both sides of the resulting equations. From the terms independent of ε, by applying (46) and (44) we get the value of x_0 and y_0 as given in (45). Since, by (44),

$$x_0(0)=b, \text{ and } x_j(0)=0, \qquad (j=1, \ldots . \infty), \text{ and } x_j=\sum_{i=1}^{4} u_i^{(j)}, \tag{49}$$

it follows that

$$\left.\begin{aligned} & u_1^{(0)}(0)+u_2^{(0)}(0)+u_3^{(0)}(0)+u_4^{(0)}(0)=b, \\ & u_1^{(j)}(0)+u_2^{(j)}(0)+u_3^{(j)}(0)+u_4^{(j)}(0)=0, \qquad j=(1, \ldots . \infty). \end{aligned}\right\} \tag{50}$$

On equating coefficients of the first power of ε, we have

$$\left.\begin{aligned}
u_1^{(1)\prime}-\sigma\sqrt{-1}\,u_1^{(1)} &= \frac{3mB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)\sqrt{-1}}-\frac{3Bx_0y_0}{2(m\rho+n\sigma)},\\
u_2^{(1)\prime}+\sigma\sqrt{-1}\,u_2^{(1)} &= -\frac{3mB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)\sqrt{-1}}-\frac{3Bx_0y_0}{2(m\rho+n\sigma)},\\
u_3^{(1)\prime}-\rho u_3^{(1)} &= -\frac{3nB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)}+\frac{3Bx_0y_0}{2(m\rho+n\sigma)},\\
u_4^{(1)\prime}+\rho u_4^{(1)} &= \frac{3nB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)}+\frac{3Bx_0y_0}{2(m\rho+n\sigma)}.
\end{aligned}\right\} \quad (51)$$

After substituting for x_0 and y_0 their values from (33) we have

$$\left.\begin{aligned}
u_1^{(1)\prime}-\sigma\sqrt{-1}\,u_1^{(1)} &= \left\{\frac{3mB(m^2-2)}{4(m\sigma-n\rho)\sqrt{-1}}+\frac{3mB}{2(m\rho+n\sigma)}\right\}b^2e^{-2\rho t}\equiv M_{12}^{(1)}b^2e^{-2\rho t},\\
u_2^{(1)\prime}+\sigma\sqrt{-1}\,u_2^{(1)} &= \left\{-\frac{3mB(m^2-2)}{4(m\sigma-n\rho)\sqrt{-1}}+\frac{3mB}{2(m\rho+n\sigma)}\right\}b^2e^{-2\rho t}\equiv M_{22}^{(1)}b^2e^{-2\rho t},\\
u_3^{(1)\prime}-\rho u_3^{(1)} &= \left\{-\frac{3nB(m^2-2)}{4(m\sigma-n\rho)}-\frac{3mB}{2(m\rho+n\sigma)}\right\}b^2e^{-2\rho t}\equiv M_{32}^{(1)}b^2e^{-2\rho t},\\
u_4^{(1)\prime}+\rho u_4^{(1)} &= \left\{\frac{3nB(m^2-2)}{4(m\sigma-n\rho)}-\frac{3mB}{2(m\rho+n\sigma)}\right\}b^2e^{-2\rho t}\equiv M_{42}^{(1)}b^2e^{-2\rho t},
\end{aligned}\right\} \quad (52)$$

where the $M_{i2}^{(1)}$ $(i=1,\ldots,4)$ denote the expressions in the brackets. The superscript and the first subscript on the $M_{i2}^{(1)}$ are the same as those on the corresponding $u_i^{(1)}$, while the second subscript denotes that they are coefficients of $e^{-2\rho t}$. On integrating (52), we get

$$\left.\begin{aligned}
u_1^{(1)} &= c_1^{(1)}e^{\sigma\sqrt{-1}t}-\frac{M_{12}^{(1)}b^2}{2\rho+\sigma\sqrt{-1}}e^{-2\rho t}\equiv c_1^{(1)}e^{\sigma\sqrt{-1}t}-d_{12}^{(1)}b^2e^{-2\rho t},\\
u_2^{(1)} &= c_2^{(1)}e^{-\sigma\sqrt{-1}t}-\frac{M_{22}^{(1)}b^2}{2\rho-\sigma\sqrt{-1}}e^{-2\rho t}\equiv c_2^{(1)}e^{-\sigma\sqrt{-1}t}-d_{22}^{(1)}b^2e^{-2\rho t},\\
u_3^{(1)} &= c_3^{(1)}e^{\rho t}-\frac{M_{32}^{(1)}b^2}{3\rho}e^{-2\rho t}\equiv c_3^{(1)}e^{\rho t}-d_{32}^{(1)}b^2e^{-2\rho t},\\
u_4^{(1)} &= c_4^{(1)}e^{-\rho t}-\frac{M_{42}^{(1)}b^2}{\rho}e^{-2\rho t}\equiv c_4^{(1)}e^{-\rho t}-d_{42}^{(1)}b^2e^{-2\rho t}.
\end{aligned}\right\} \quad (53)$$

If then we put $c_1^{(1)}=c_2^{(1)}=c_3^{(1)}=0$ and choose $c_4^{(1)}$ so that the initial conditions (50) are satisfied, we find that

$$c_4^{(1)}=b^2[d_{12}^{(1)}+d_{22}^{(1)}+d_{32}^{(1)}+d_{42}^{(1)}].$$

Since

$$\left.\begin{aligned} x_j&=u_1^{(j)}+u_2^{(j)}+u_3^{(j)}+u_4^{(j)},\\ y_j&=n\sqrt{-1}\,(u_1^{(j)}-u_2^{(j)})+m\,(u_3^{(j)}-u_4^{(j)}), \end{aligned}\right\}\quad(54)$$

we see that

$$\left.\begin{aligned} x_1=&[(d_{12}^{(1)}+d_{22}^{(1)}+d_{32}^{(1)}+d_{42}^{(1)})e^{-\rho t}-(d_{12}^{(1)}+d_{22}^{(1)}+d_{32}^{(1)}+d_{42}^{(1)})e^{-2\rho t}]b^2\\ &\equiv[D_2^{(1)}e^{-\rho t}-D_2^{(1)}e^{-2\rho t}]b^2,\\ y_1=&[-m(d_{12}^{(1)}+d_{22}^{(1)}+d_{32}^{(1)}+d_{42}^{(1)})e^{-\rho t}-\{n\sqrt{-1}\,(d_{12}^{(1)}-d_{22}^{(1)})\\ &+m(d_{32}^{(1)}-d_{42}^{(1)})\}e^{-2\rho t}]b^2\equiv[-mD_2^{(1)}e^{-\rho t}-\overline{D}_2^{(1)}e^{-2\rho t}]b^2. \end{aligned}\right\}\quad(55)$$

From the coefficients of ε^2 above, we see that the differential equations which define the $u_i^{(2)}$ $(i=1,\ldots,4)$ are

$$\left.\begin{aligned} u_1^{(2)\prime}-\sigma\sqrt{-1}\,u_1^{(2)}=&\ \frac{3mB(-2x_0x_1+y_0y_1)}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3B(x_0y_1+x_1y_0)}{2(m\rho+n\sigma)}\\ &+\frac{mC(2x_0^3-3x_0y_0^2)}{(m\sigma-n\rho)\sqrt{-1}}+\frac{3C(4x_0^2y_0-y_0^3)}{4(m\rho+n\sigma)},\\ u_2^{(2)\prime}+\sigma\sqrt{-1}\,u_2^{(2)}=&-\frac{3mB(-2x_0x_1+y_0y_1)}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3B(x_0y_1+x_1y_0)}{2(m\rho+n\sigma)}\\ &-\frac{mC(2x_0^3-3x_0y_0^2)}{(m\sigma-n\rho)\sqrt{-1}}+\frac{3C(4x_0^2y_0-y_0^3)}{4(m\rho+n\sigma)},\\ u_3^{(2)\prime}-\rho u_3^{(2)}\quad=&-\frac{3nB(-2x_0y_0+y_0y_1)}{2(m\sigma-n\rho)}+\frac{3B(x_0y_1+x_1y_0)}{2(m\rho+n\sigma)}\\ &-\frac{nC(2x_0^3-3x_0y_0^2)}{m\sigma-n\rho}-\frac{3C(4x_0^2y_0-y_0^3)}{4(m\rho+n\sigma)},\\ u_4^{(2)\prime}+\rho u_4^{(2)}\quad=&+\frac{3nB(-2x_0y_0+y_0y_1)}{2(m\sigma-n\rho)}+\frac{3B(x_0y_1+x_1y_0)}{2(m\rho+n\sigma)}\\ &+\frac{nC(2x_0^3-3x_0y_0^2)}{m\sigma-n\rho}-\frac{3C(4x_0^2y_0-y_0^3)}{4(m\rho+n\sigma)}. \end{aligned}\right\}\quad(56)$$

On substituting for x_0, y_0, x_1, y_1 their values from (45) and (55) these equations take the form

$$\left.\begin{aligned} u_1^{(2)\prime}-\sigma\sqrt{-1}\,u_1^{(2)}&=M_{12}^{(2)}b^3e^{-2\rho t}+M_{13}^{(2)}b^3e^{-3\rho t},\\ u_2^{(2)\prime}+\sigma\sqrt{-1}\,u_2^{(2)}&=M_{22}^{(2)}b^3e^{-2\rho t}+M_{23}^{(2)}b^3e^{-3\rho t},\\ u_3^{(2)\prime}-\rho u_3^{(2)}&=M_{32}^{(2)}b^3e^{-2\rho t}+M_{33}^{(2)}b^3e^{-3\rho t},\\ u_4^{(2)\prime}+\rho u_4^{(2)}&=M_{42}^{(2)}b^3e^{-2\rho t}+M_{43}^{(2)}b^3e^{-3\rho t}, \end{aligned}\right\}\quad(57)$$

where

$$
\left.
\begin{aligned}
M_{12}^{(2)} &= \frac{3mB(m^2-2)D_2^{(1)}}{2(m\sigma-n\rho)\sqrt{-1}}+\frac{6mBD_2^{(1)}}{2(m\rho+n\sigma)}\,;\\
M_{22}^{(2)} &= -\frac{3mB(m^2-2)D_2^{(1)}}{2(m\sigma-n\rho)\sqrt{-1}}+\frac{6mBD_2^{(1)}}{2(m\rho+n\sigma)}\,;\\
M_{32}^{(2)} &= -\frac{3nB(m^2-2)D_2^{(1)}}{2(m\sigma-n\rho)}-\frac{6mBD_2^{(1)}}{2(m\rho+n\sigma)}\,;\\
M_{42}^{(1)} &= \frac{3nB(m^2-2)D_2^{(1)}}{2(m\sigma-n\rho)}-\frac{6mBD_2^{(1)}}{2(m\rho+n\sigma)}\,;\\
M_{13}^{(2)} &= \frac{3mB(2D_2^{(1)}+m\overline{D}_2^{(1)})}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3B(mD_2^{(1)}-\overline{D}_2^{(1)})}{2(m\rho+n\sigma)}-\frac{mC(3m^2-2)}{(m\sigma-n\rho)\sqrt{-1}}\\
&\qquad+\frac{3mC(m^2-4)}{4(m\rho+n\sigma)}\,;\\
M_{23}^{(2)} &= -\frac{3mB(2D_2^{(1)}+m\overline{D}_2^{(1)})}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3B(mD_2^{(1)}-\overline{D}_2^{(1)})}{2(m\rho+n\sigma)}+\frac{mC(3m^2-2)}{(m\sigma-n\rho)\sqrt{-1}}\\
&\qquad+\frac{3mC(m^2-4)}{4(m\rho+n\sigma)}\,;\\
M_{33}^{(2)} &= -\frac{3nB(2D_2^{(1)}+m\overline{D}_2^{(1)})}{2(m\sigma-n\rho)}+\frac{3B(mD_2^{(1)}-\overline{D}_2^{(1)})}{2(m\rho+n\sigma)}+\frac{nC(3m^2-2)}{m\sigma-n\rho}\\
&\qquad-\frac{3mC(m^2-4)}{4(m\rho+n\sigma)}\,;\\
M_{43}^{(2)} &= +\frac{3nB(2D_2^{(1)}+m\overline{D}_2^{(1)})}{2(m\sigma-n\rho)}+\frac{3B(mD_2^{(1)}-\overline{D}_2^{(1)})}{2(m\rho+n\sigma)}-\frac{nC(3m^2-2)}{m\sigma-n\rho}\\
&\qquad-\frac{3mC(m^2-4)}{4(m\rho+n\sigma)}\,.
\end{aligned}
\right\}\quad(58)
$$

The solutions of (57) have the form

$$
\left.
\begin{aligned}
u_1^{(2)} &= c_1^{(2)}e^{\sigma\sqrt{-1}t}-d_{12}^{(2)}b^3e^{-2\rho t}-d_{13}^{(2)}b^3e^{-3\rho t},\\
u_2^{(2)} &= c_2^{(2)}e^{-\sigma\sqrt{-1}t}-d_{22}^{(2)}b^3e^{-2\rho t}-d_{23}^{(2)}b^3e^{-3\rho t},\\
u_3^{(2)} &= c_3^{(2)}e^{\rho t}-d_{32}^{(2)}b^3e^{-2\rho t}-d_{33}^{(2)}b^3e^{-3\rho t},\\
u_4^{(2)} &= c_4^{(2)}e^{-\rho t}-d_{42}^{(2)}b^3e^{-2\rho t}-d_{43}^{(2)}b^3e^{-3\rho t},
\end{aligned}
\right\}\quad(59)
$$

where

$$
\left.
\begin{aligned}
d_{12}^{(2)} &= \frac{M_{12}^{(2)}}{2\rho+\sigma\sqrt{-1}}\,; & d_{22}^{(2)} &= \frac{M_{22}^{(2)}}{2\rho-\sigma\sqrt{-1}}\,; & d_{32}^{(2)} &= \frac{M_{32}^{(2)}}{3\rho}\,; & d_{42}^{(2)} &= \frac{M_{42}^{(2)}}{\rho}\,;\\
d_{13}^{(2)} &= \frac{M_{13}^{(2)}}{3\rho+\sigma\sqrt{-1}}\,; & d_{23}^{(2)} &= \frac{M_{23}^{(2)}}{3\rho-\sigma\sqrt{-1}}\,; & d_{33}^{(2)} &= \frac{M_{33}^{(2)}}{4\rho}\,; & d_{43}^{(2)} &= \frac{M_{43}^{(2)}}{2\rho}\,.
\end{aligned}
\right\}\quad(60)
$$

If we put $c_1^{(2)}=c_2^{(2)}=c_3^{(2)}=0$ and determine $c_4^{(2)}$ to satisfy the initial conditions (50), we have

$$c_4^{(2)}=[\sum_{i=1}^{4} d_{i2}^{(2)}+\sum_{i=1}^{4} d_{i3}^{(2)}]b^3.$$

On substituting these values for $c_i^{(2)}$ $(i=1,\ldots,4)$ in (59), and writing

$$\left.\begin{aligned}&\sum_{i=1}^{4} d_{i2}^{(2)}=D_2^{(2)};\quad n\sqrt{-1}(d_{12}^{(2)}-d_{22}^{(2)})+m(d_{32}^{(2)}-d_{42}^{(2)})=\bar{D}_2^{(2)},\\&\sum_{i=1}^{4} d_{i3}^{(2)}=D_3^{(2)};\quad n\sqrt{-1}(d_{13}^{(2)}-d_{23}^{(2)})+m(d_{33}^{(2)}-d_{43}^{(2)})=\bar{D}_3^{(2)},\end{aligned}\right\}\qquad(61)$$

we have, by (54),

$$\left.\begin{aligned}x_2&=[(D_2^{(2)}+D_3^{(2)})e^{-\rho t}-D_2^{(2)}e^{-2\rho t}-D_3^{(2)}e^{-3\rho t}]b^3,\\y_2&=[-m(D_2^{(2)}+D_3^{(2)})e^{-\rho t}-\bar{D}_2^{(2)}e^{-2\rho t}-\bar{D}_3^{(2)}e^{-3\rho t}]b^3.\end{aligned}\right\}\qquad(62)$$

If we proceed in this way, we can build up in succession the values of x_j and y_j for $j=3, 4, \ldots \infty$. By induction we can get the form of the general term. From (45), (55), (62) we notice that the x_j and y_j are sums of powers of $e^{-\rho t}$, the highest power of $e^{-\rho t}$ occurring being $j+1$. The equations defining $u_i^{(\nu)}$ $(i=1,\ldots,4)$ have in their right-hand members only sums of powers of $e^{-\rho t}$, the lowest power being $e^{-2\rho t}$ and the highest $e^{-(\nu+1)\rho t}$. When we integrate these equations we will have terms in powers of $e^{-\rho t}$ the lowest power being the first and the highest the $(\nu+1)$-th. These solutions will have the form

$$u_i^{(\nu)}=c_i^{(\nu)}e^{\lambda t}-[\sum_{j=2}^{\nu+1} d_{ij}^{(\nu)}e^{-j\rho t}]b^{\nu+1},\qquad(i=1,\ldots,4;\ \lambda=\sigma\sqrt{-1},\ -\sigma\sqrt{-1},\ \rho,\ -\rho).\qquad(63).$$

If we put $c_1^{(\nu)}=c_2^{(\nu)}=c_3^{(\nu)}=0$, then to satisfy the initial conditions we must put $c_4^{(\nu)}=b^{\nu+1}\sum_{i=1}^{4}\sum_{j=2}^{\nu+1} d_{ij}^{(\nu)}$. If then we put

$$\left.\begin{aligned}&D_j^{(\nu)}=\sum_{i=1}^{4} d_{ij}^{(\nu)},\qquad(j=2,\ldots,\nu+1),\\&\bar{D}_j^{(\nu)}=n\sqrt{-1}(d_{1j}^{(\nu)}-d_{2j}^{(\nu)})+m(d_{3j}^{(\nu)}-d_{4j}^{(\nu)}),\quad(j=2,\ldots,\nu+1),\end{aligned}\right\}\qquad(64)$$

it follows that

$$\left.\begin{aligned}x_\nu&=[\sum_{j=2}^{\nu+1} D_j^{(\nu)}e^{-\rho t}-\sum_{k=2}^{\nu+1} D_k^{(\nu)}e^{-k\rho t}]b^{\nu+1},\\y_\nu&=[-m\sum_{j=2}^{\nu+1} D_j^{(\nu)}e^{-\rho t}-\sum_{k=2}^{\nu+1}\bar{D}_k^{(\nu)}e^{-k\rho t}]b^{\nu+1}.\end{aligned}\right\}\qquad(65)$$

When we substitute the results of (45), (55), (62), (65) in (39), we have the values of x and y which are solutions of (38). The x and y belonging to the physical problem as defined by (8) are ε times the x and y, respectively, which we have just obtained from (38). On multiplying the values of each x_j and y_j by ε and substituting in (39), we see that the resulting expressions carry b and ε

as factors to the same power in each term. Hence $b\varepsilon$ is equivalent to a single parameter and may be replaced by β. Our solutions then may be written.

$$\left.\begin{aligned} x &= e^{-\rho t}\beta + [D_2^{(1)}e^{-\rho t} - D_2^{(1)}e^{-2\rho t}]\beta^2 + [\sum_{k=2}^{3} D_k^{(2)}e^{-\rho t} - \sum_{k=2}^{3} D_k^{(2)}e^{-k\rho t}]\beta^3 + \ldots \\ &\qquad + [\sum_{k=2}^{\nu+1} D_k^{(\nu)}e^{-\rho t} - \sum_{k=2}^{\nu+1} D_k^{(\nu)}e^{-k\rho t}]\beta^{\nu+1} + \ldots, \\ y &= -me^{-\rho t}\beta - [mD_2^{(1)}e^{-\rho t} + \bar{D}_2^{(1)}e^{-2\rho t}]\beta^2 - [m\sum_{k=2}^{3} D_k^{(2)}e^{-\rho t} + \sum_{k=2}^{3} \bar{D}_k^{(2)}e^{-k\rho t}]\beta^3 \\ &\qquad - \ldots - [m\sum_{k=2}^{\nu+1} D_k^{(\nu)}e^{-\rho t} + \sum_{k=2}^{\nu+1} \bar{D}_k^{(\nu)}e^{-k\rho t}]\beta^{\nu+1} - \ldots \end{aligned}\right\} \quad (66)$$

If, instead of taking $x(0)=b$ as our initial conditions, we had taken

$$x(0) = b - D_2^{(1)}b^2\varepsilon - D_3^{(2)}b^3\varepsilon^2 - D_4^{(3)}b^4\varepsilon^3 - \ldots,$$

we would have had

$$x_0(0) = b,\ x_j(0) = -D_{j+1}^{(j)}b^{j+1}.$$

Then at the successive steps the constants of integration

$$c_4^{(1)} = c_4^{(2)} = c_4^3 = \ldots c_4^{(j)} = \ldots = 0.$$

The $u_i^{(j)}$ ($i=1, \ldots, 4$; $j=0, \ldots \infty$), instead of being expressed as sums of powers in $e^{-\rho t}$, would each be expressed as a single term in $e^{-(j+1)\rho t}$. The solutions (66) would then have the form

$$\left.\begin{aligned} x &= \quad e^{-\rho t}\beta - D_2^{(1)}e^{-2\rho t}\beta^2 - D_3^{(2)}e^{-3\rho t}\beta^3 - \ldots - D_{\nu+1}^{(\nu)}e^{-(\nu+1)\rho t}\beta^{\nu+1} - \ldots, \\ y &= -me^{-\rho t}\beta - \bar{D}_2^{(1)}e^{-2\rho t}\beta^2 - \bar{D}_3^{(2)}e^{-3\rho t}\beta^3 - \ldots - \bar{D}_{\nu+1}^{(\nu)}e^{-(\nu+1)\rho t}\beta^{\nu+1} - \ldots \end{aligned}\right\} \quad (68)$$

If we compute the values of the coefficients in these expansions in (68) and write c for β, we find that solutions (68) are identical with solutions (25) of § IV.

By a method exactly analogous to that just used we could build up a set of solutions arranged in ascending powers of $e^{+\rho t}$, and with a proper choice of initial conditions it could be shown that they were identical with solutions (33) of § IV.

VII. *Properties of the Orbits.*

When we consider the solutions of the differential equations as given by equations (25) of § IV, which are in the form of power series in $e^{-\rho t}$, we see that the values of x and y continually decrease as t becomes greater and greater, and finally x and y approach the value zero as t becomes infinitely great. Since x and y are the coordinates of the infinitesimal body referred to an equilibrium point as origin, it follows that it will approach nearer and nearer to one of the equilibrium points as t increases. Such orbits are said to be asymptotic to these points.

Similarly, we see that if the infinitesimal body were moving on one of the orbits which are given by equations (33) of § IV, where x and y are expressed in the form of power series in $e^{+\rho t}$, it would approach one of the equilibrium points when t became infinitely large and negative. In other words, if we imagine the infinitesimal body to be placed at one of the equilibrium points, it would gradually leave it on one of these orbits, requiring, however, an infinite time to describe the first small part of the orbit.

(A) *Meaning of the Parameter c.*

When the masses of the finite bodies are given, the only arbitrary in solutions (25) is c, where $c=\left(\frac{\partial x}{\partial \omega}\right)_{\omega=0}$. If c is fixed, the x and y are determined uniquely by (25) for all values of t so large that the convergence of the solutions as power series holds. Now the slope of the orbit at any time t, for t sufficiently large, is given by

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{m\rho c e^{-\rho t}+2\rho\bar{D}_2^{(1)}c^2e^{-2\rho t}+3\rho\bar{D}_3^{(2)}c^3e^{-3\rho t}+\ldots}{-\rho c e^{-\rho t}+2\rho D_2^{(1)}c^2e^{-2\rho t}+3\rho D_3^{(2)}c^3e^{-3\rho t}+\ldots}\,. \tag{69}$$

$=-m+$ a power series in $e^{-\rho t}$ beginning with the term in $e^{-\rho t}$. Therefore,

$$\left(\frac{dy}{dx}\right)_{t\doteq\infty}=-m. \tag{70}$$

Hence the direction of approach to the equilibrium point is independent of the value of the arbitrary parameter c.

The position of the body in its orbit at the initial time, $t=0$, is given as a power series in c with constant coefficients; so that if c is fixed the position of the body in its orbit at the initial time is determined. It follows therefore that the direction of approach to the equilibrium points is independent of the initial position of the infinitesimal body in its orbit at the initial time. Since c is arbitrary, nothing of generality will be lost in the actual construction of such orbits if we take c equal to unity, and the numerical computation will be simplified. This we do in the numerical computation of an orbit in § VIII.

(B) *The Number and Position of the Asymptotic Orbits.*

In equations (42) of § VI the quantity m is defined

$$m=\frac{\rho^2-1-2A}{2\rho}.$$

It will be shown later* that for each of the equilibrium points (a), (b), (c) the quantity m is negative for all values of μ, where $0\overline{\gtrless}\mu\leqq\frac{1}{2}$. Therefore, in the case of the orbits given by (25), it follows that $\left(\frac{dy}{dx}\right)_{t\doteq\infty}=$ a positive quantity. Hence equations (25) represent two orbits, one in the first and one in the third

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* See § VII (C).

quadrant in the neighborhood of each of the equilibrium points. For any preassigned value of μ these two orbits are equally inclined to the x-axis in the neighborhood of the origin.

Similarly, it may be shown that equations (33) represent two orbits leaving each of the equilibrium points, one in the second and one in the fourth quadrant. For any given value of μ, these orbits are, in the neighborhood of the origin, the images by reflection on the x-axis of the orbits given by (25).

Near the points of equilibrium, that is for t very large, the first terms in the expansion of (25) are the most important and determine the sign of the right-hand members. It follows therefore from (25), since m is negative, that if c is positive x and y are both positive, but if c is negative x and y are both negative. Hence for positive values of c the orbit is in the first quadrant, and for negative values of c the orbit is in the third quadrant in the neighborhood of the equilibrium points.

Similarly, when t is very large and negative, the first terms in the expansion in (20) determine the sign of the right-hand members. For c positive x is positive and y negative, and for c negative x is negative and y positive. Hence for positive values of c the orbit is in the fourth quadrant, and for negative values of c the orbit is in the first quadrant.

The value of $\dfrac{dy}{dx}$ for a point near the origin is given by (69). If we take the second derivative

$$\frac{d^2y}{dx^2}=\frac{1}{2}\frac{d\left(\frac{dy}{dx}\right)^2}{dt}\bigg/\frac{dy}{dt}$$

we find that

$$\frac{d^2y}{dx^2}=-2[\bar{D}_2^{(1)}+mD_2^{(1)}]+\text{ a power series in } e^{-\rho t}.$$

Hence the value of $\left(\dfrac{d^2y}{dx^2}\right)_{t=\infty}$ is independent of c, and therefore does not change sign with c. Therefore, near the equilibrium points, the orbits in the first and third quadrants lie on the same side of their common tangent line $y+mx=0$. Similarly, it can be shown that the orbits in the second and fourth quadrants lie upon the same side of their common tangent, $y-mx=0$.

(C) *Variation in the Direction of Approach as μ Varies from Zero to 1/2.*

Now it has been shown* that if r_1 and r_2 denote the distances of the infinitesimal body from the finite masses $1-\mu$ and μ respectively, the values of r_1 and r_2 for the equilibrium points can be expressed, for μ sufficiently small,

* Moulton, "Introduction to Celestial Mechanics" (New Edition), § 158.

as convergent power series in $\mu^{1/3}$ in the case of the points (a) and (b), and in μ in the case of the point (c).

For the point (a),

$$r_2=\left(\frac{\mu}{3}\right)^{1/3}+\frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3}-\frac{1}{9}\left(\frac{\mu}{3}\right)^{3/3}\cdots,$$

$$r_1=1+r_2.$$

For the point (b),

$$r_2=\left(\frac{\mu}{3}\right)^{1/3}-\frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3}-\frac{1}{9}\left(\frac{\mu}{3}\right)^{3/3}\cdots,$$

$$r_1=1-r_2.$$

For the point (c),

$$r_2=2-\frac{7}{12}\mu-\frac{23\times 7^2}{12^4}\mu^3-\cdots,$$

$$r_1=r_2-1,$$

$$A=\frac{1-\mu}{r_1^{(0)3}}+\frac{\mu}{r_2^{(0)3}}\ ;\quad m=\frac{\rho^2-1-2A}{2\rho}, \tag{72}$$

we have seen further that $-m$ gives the value of the tangent made by the curve with the positive x-axis in the neighborhood of one of the equilibrium points.

The following table has been constructed to show how the elements r_1, r_2, A, and $-m$ vary as the ratio of the finite masses changes from zero to one.

μ	$\mu \doteq 0$			$\mu \doteq 0.5$		
Point	(a)	(b)	(c)	(a)	(b)	(c)
r_2	$\doteq\left(\frac{\mu}{3}\right)^{1/3}$	$\doteq\left(\frac{\mu}{3}\right)^{1/3}$	$2-\frac{7}{12}\mu$	0.63273	0.4308	1.700
r_1	$\doteq 1$	$\doteq 1$	$1-\frac{7}{12}\mu$	1.63273	0.5692	0.700
A	$\doteq 4$	$\doteq 4$	$1+\frac{7}{8}\mu=1+\varepsilon$	2.0886	8.9654	1.5595
ρ	$\doteq 2.5083$	$\doteq 2.5083$	$\sqrt{2\varepsilon}$	1.5564	4.0305	1.1469
$-m$	$\doteq 0.56224$	$\doteq 0.56224$	$\doteq\infty$	0.88498	0.3324	1.2232
ϕ	$\doteq 29°\ 20'$	$\doteq 29°\ 20'$	$=90°$	$41°\ 30'$	$18°\ 23'$	$50°\ 44'$

From the above table we see that as μ increases from zero to 1/2
for (a), ϕ increases from 29° 20′ to 41° 30′ approximately;
for (b), ϕ decreases from 29° 20′ to 18° 23′ approximately;
for (c), ϕ decreases from 90° 00′ to 50° 44′ approximately.

VIII. *Continuation of the Orbits beyond the Range of Convergence of the Solutions of the Differential Equations.*

It has been pointed out that the expansions given in (25) and (33) define orbits asymptotic to the equilibrium points for t sufficiently large. It remains to be shown what becomes of the infinitesimal body for smaller values of t.

(A) *Method of Mechanical Quadratures.*

If the x- and y-coordinates of the moving body be computed for a sufficient number of equidistant values of the time then the coordinates of the body for the next equidistant value of the time may be computed by the method of mechanical quadratures. If, for example, we know the values of x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5 at five equidistant values of the time t_1, t_2, t_3, t_4, t_5 then we shall show how the value of x_6 and y_6 for the time t_6 can be computed, where

$$t_6-t_5=t_5-t_4=\ldots\ldots=t_2-t_1.$$

Form a table of the values of x and their successive differences for the successive values of the time. Form also similar tables for y, x', y', F_2, F_4 where from (1),

$$\left.\begin{aligned}
&\frac{dx}{dt}=x'\\
&\frac{d^2x}{dt^2}=\frac{dx'}{dt}=F_2(x,y,y')=2y'+x-\frac{(1-\mu)(x-x_1)}{r_1^3}-\frac{\mu(x-x_2)}{r_2^3},\\
&\frac{dy}{dt}=y'\\
&\frac{d^2y}{dt^2}=\frac{dy'}{dt}=F_4(x,x',y)=-2x'+y-\frac{(1-\mu)y}{r_1^3}-\frac{\mu y}{r_2^3}.
\end{aligned}\right\}\quad(73)$$

It is necessary to have computed the values of these quantities at sufficient dates that some order of differences obtainable from them will be small. Suppose that such tables of values have been set up for each of the quantities x, y, x', y', F_2, F_4 for the dates small t_1, t_2, t_3, $\ldots$, t_n, and suppose that in these tables the fourth differences are small and approximately constant. In the F_2 and F_4 tables we assume the next fourth difference about equal to the previous ones, and from the tables compute $F_2^{(n+1)}$ and $F_4^{(n+1)}$. Then the values of the next first differences of x' and y' can be readily computed by means of the formulæ:

$$\left.\begin{aligned}
\Delta_1^{(n+1)}x'&=(t_{n+1}-t_n)\Big[F_2^{(n+1)}-\frac{1}{2}\Delta_1F_2^{(n+1)}-\frac{1}{12}\Delta_2F_2^{(n+1)}-\frac{1}{24}\Delta_3F_2^{(n+1)}\\
&\qquad\qquad-\frac{1}{36}\Delta_4F_2^{(n+1)}\ldots\Big],\\
\Delta_1^{(n+1)}y'&=(t_{n+1}-t_n)\Big[F_4^{(n+1)}-\frac{1}{2}\Delta_1F_4^{(n+1)}-\frac{1}{12}\Delta_2F_4^{(n+1)}-\frac{1}{24}\Delta_3F_4^{(n+1)}\\
&\qquad\qquad-\frac{1}{36}\Delta_4F_4^{(n+1)}\ldots\Big].
\end{aligned}\right\}\quad(74)$$

Having found $\Delta_1^{(n+1)}x'$ and $\Delta_1^{(n+1)}y'$ we can at once compute $x'(t_{n+1})$ and $y'(t_{n+1})$, and complete the table of differences for this date. We can then compute $\Delta_1^{(n+1)}x$ and $\Delta_1^{(n+1)}y$ from $x'(t_{n+1})$, $y'(t_{n+1})$ and their successive differences, by formulæ exactly similar in form to (74), namely,

$$\left.\begin{aligned}\Delta_1^{(n+1)}x &= (t_{n+1}-t_n)\Big[x'^{(n+1)}-\frac{1}{2}\Delta_1x'^{(n+1)}-\frac{1}{12}\Delta_2x'^{(n+1)}-\frac{1}{24}\Delta_3x'^{(n+1)}\\ &\qquad -\frac{1}{36}\Delta_4x'^{(n+1)}\dots\Big],\\ \Delta_1^{(n+1)}y &= (t_{n+1}-t_n)\Big[y'^{(n+1)}-\frac{1}{2}\Delta_1y'^{(n+1)}-\frac{1}{12}\Delta_2y'^{(n+1)}-\frac{1}{24}\Delta_3y'^{(n+1)}\\ &\qquad -\frac{1}{36}\Delta_4y'^{(n+1)}\dots\Big];\end{aligned}\right\} \quad (75)$$

whence we obtain at once $x(t_{n+1})$ and $y(t_{n+1})$.

Having found the values of x, y, x', y' at the date t_{n+1} we compute $F_2^{(n+1)}$ and $F_4^{(n+1)}$ by the second and fourth relations of (73). If the results so obtained agree with the results we already have for $F_2^{(n+1)}$ and $F_4^{(n+1)}$, we assume that the fourth differences which we guessed are correct. But if the computed values of $F_2^{(n+1)}$ and $F_4^{(n+1)}$ differ from the assumed values we replace the latter by the computed values, and repeat the process as before from this point on. We keep on repeating this process, which generally has to be done only once, until the computed value is the same as the value from which it is computed. Having thus found the coordinates of the body for the date t_{n+1}, we assume another set of fourth differences for F_2 and F_4 for the date t_{n+2} and repeat the process. Thus any number of points on the orbit may be determined, and the orbit can thus be continued beyond the range of convergence of the solutions of the differential equations.

(B) *Jacobi's Constant.*

The question arises how we know that the orbit so continued is really the orbit on which the infinitesimal body would move when forming a part of the system in question.

It has been shown* that in the case of the infinitesimal body

$$V^2=x^2+y^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}-C, \quad (76)$$

or

$$C=x^2+y^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}-x'^2-y'^2. \quad (77)$$

This gives us a relation that must hold between the coordinates and the x- and y-components of the velocity of the infinitesimal body at all positions of its orbit.

* Moulton, "Introduction to Celestial Mechanics" (New Edition), § 153.

The constant C, which is called *Jacobi's Constant*, can be computed from the initial configuration of the system. In the present case the value of C would be the value of the right-hand member of (77) when the body is at rest at the equilibrium point in question. At such points $x'=y'=x=y=0$; r_2 and r_1 are readily determined from (72).

Formula (77) may then be used as a check on the computation at each step, that is, for each new date. If the integral will not check, when the computation has been accurately made, it is usually necessary to divide the time-interval, and take shorter steps. It then becomes necessary to interpolate values of the coordinates for intermediate values of the time. After obtaining the intermediate values for F_2 and F_4 by Lagrange's *Interpolation Formula*, the corresponding x, x', y, y' can be computed out of these by the process described. Thus the orbit of the infinitesimal body can be continued as far as may be desired.

(C) *Specific Form of the Solutions for* $\mu=0.02$.

We proceed now to assign to μ the arbitrary value $\mu=0.02$, and find the specific form of the solutions and the shape of the orbit. We will consider the orbits in the neighborhood of the equilibrium point (a), and consider those given by equations (25), where the expansion is made in powers of $e^{-\rho t}$.

By (72) we have r_2 for the point (a) given by

$$r_2=\left(\frac{\mu}{3}\right)^{1/3}+\frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3}-\frac{1}{9}\left(\frac{\mu}{3}\right)^{3/3}\cdots\cdots$$

On neglecting all after the third term in the expansion, we find, for $\mu=0.02$ that $r_2=0.1993$. But because of the neglected terms in (72) this value of r_2 does not satisfy with sufficient accuracy the quintic equation of which (72) is the solution, namely,

$$r_2^5+(3-\mu)r_2^4+(3-2\mu)r_2^3-\mu r_2^2-2\mu r_2-\mu=0. \tag{78}$$

If we start with the value $r_2=0.1993$ and apply the method of differential corrections we find $r_2=0.200078$, a value which makes the left member of (78) differ from zero by approximately 0.00000005. Hence, for the point (a) we take

$$r_2=0.200078,\quad r_1=1.200078; \tag{79}$$

whence, by (2), (5), and (42) we find in succession

$$\begin{aligned}
&A=\ 3.064095;\quad \rho=2.098701;\quad \sigma=\ 1.827794;\\
&m=-0.648885;\quad n=2.863898;\quad B=12.952988;\\
&m\rho+n\sigma=\ 3.872514;\quad m\sigma-n\rho=\ \ 7.196426;\\
&4\rho^2+\sigma^2=20.958701;\quad m^2-2=-1.578948.
\end{aligned}$$

On substituting these values in equations (25) and putting $c=+1$,* we have

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* See § VII (A).

for the equation of the orbit in the first quadrant in the neighborhood of the origin,

$$\left.\begin{aligned} x &= e^{-2.098701t} - 2.944651\, e^{-4.197402t} \ldots, \\ y &= 0.648885\, e^{-2.098701t} + 0.0251562\, e^{-4.197402t} \ldots. \end{aligned}\right\} \quad (80)$$

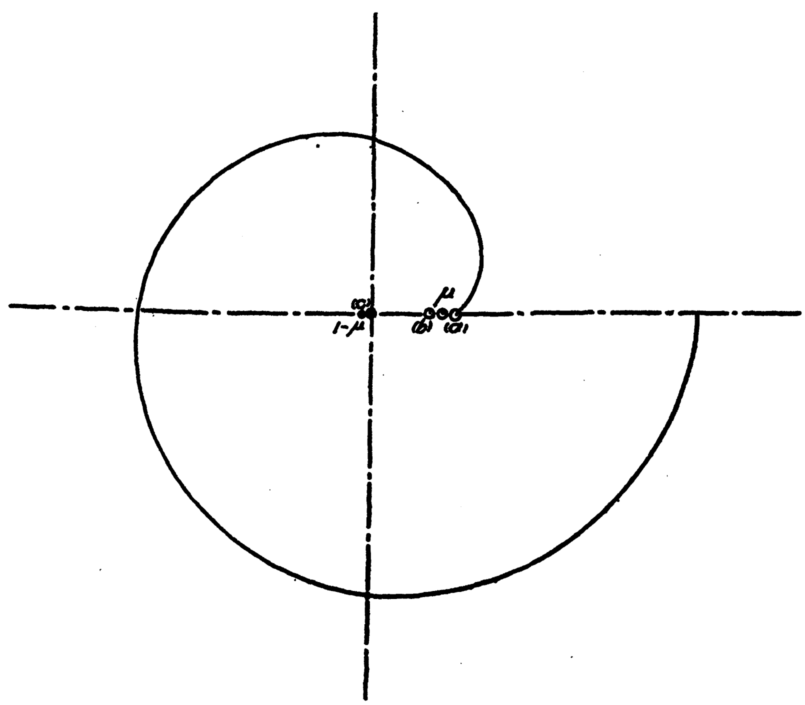
For the infinitesimal body at rest at the equilibrium point (a), Jacobi's constant C has the value

$$C = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = 3.225734.$$

Hence, at all points on the orbits asymptotic to (a), Jacobi's constant must have the value 3.225734.

The values of x, y, x', and y', were computed for the times $t=3$, 2.75, 2.50, 2.25, and 2.00 and the tables of values of x, y, x', y', F_2, F_4 and their differences built up. On proceeding as outlined in subsection (A) of this article, the table of values (on p. 248) was obtained. For the first few computations the interval of time taken was one-fourth of the unit of time, and afterwards one-eighth; but in the table are given only the values corresponding to intervals of one-fourth, except in the case of a few dates near which one of the elements changed sign.

A drawing of the orbit represented by the data in the following table is appended.



t	x	y	x'	y'
3.00	1.18191	0.00120	—0.00383	—0.00251
2.75	1.18316	0.00202	—0.00642	—0.00424
2.50	1.18526	0.00342	—0.01071	—0.00717
2.25	1.18874	0.00577	—0.01773	—0.01210
2.00	1.19447	0.00974	—0.02896	—0.02038
1.75	1.20373	0.01643	—0.04633	—0.03434
1.50	1.21830	0.02766	—0.07175	—0.05741
1.25	1.24034	0.04631	—0.10604	—0.09469
1.00	1.27190	0.07674	—0.14716	—0.15243
0.75	1.31392	0.12474	—0.18816	—0.23661
0.50	1.36500	0.19749	—0.21733	—0.35027
0.25	1.42037	0.30217	—0.21993	—0.49123
0.00	1.47154	0.44471	—0.18149	—0.65126
—0.25	1.50674	0.62827	—0.09068	—0.81669
—0.50	1.51198	0.85206	0.05873	—0.96995
—0.75	1.47247	1.11060	0.26679	—1.09141
—1.00	1.3742	1.3935	0.5269	—1.1614
—1.25	1.2057	1.6856	0.8262	—1.1621
—1.50	0.9594	1.9677	1.1459	—1.0796
—1.75	0.6331	2.2176	1.4628	—0.9050
—2.00	0.2305	2.4122	1.7508	—0.6362
—2.125	0.0037	2.4815	1.8756	—0.4681
—2.25	—0.2377	2.5284	1.9830	—0.2780
—2.375	—0.4912	2.5504	2.0699	—0.0705
—2.50	—0.7542	2.5454	2.1333	0.1542
—2.75	—1.2957	2.4468	2.1794	0.6420
—3.00	—1.8338	2.2220	2.1041	1.1580
—3.25	—2.3369	1.8680	1.8979	1.6706
—3.50	—2.7717	1.3899	1.5593	2.1452
—3.75	—3.1060	0.8015	1.0961	2.5468
—4.00	—3.3107	0.1254	0.5255	2.8422
—4.125	—3.3567	—0.2364	0.2077	2.9409
—4.25	—3.3619	—0.6083	—0.1265	3.0027
—4.50	—3.2435	—1.3629	—0.8263	3.0062
—4.75	—2.9482	—2.0971	—1.5344	2.8394
—5.00	—2.4792	—2.7680	—2.2081	2.4989
—5.25	—1.8507	—3.3327	—2.8035	1.9927
—5.50	—1.0875	—3.7521	—3.2789	1.3399
—5.75	—0.2243	—3.9929	—3.5975	0.5701
—8.875	0.2316	—4.0383	—3.6886	0.1532
—6.00	0.6958	—4.0307	—3.7303	—0.2778
—6.25	1.6237	—3.8514	—3.6580	—1.1578
—6.50	2.5069	—3.4534	—3.3729	—2.0189
—6.75	3.2927	—2.8479	—2.8803	—3.9747
—7.00	3.9313	—2.0593	—2.1988	—4.2632
—7.25	4.3790	—1.1239	—1.3597	
—7.50	4.6016	—0.0894	—0.4070	
—7.625	4.4446	0.4475	0.0947	

UNIVERSITY OF MANITOBA, *August* 20, 1913.

VITA.

I, Lloyd Arthur Heber Warren, was born at Balderson, Ontario, Canada, on the eighteenth day of November, 1879. My early education was obtained at the Balderson public school, and the Perth Collegiate Institute from which I matriculated in 1897. On continuing my work at Queen's University, Kingston, Ontario, I won the Chancellor's Scholarship in Mathematics in 1899, and graduated with the degree of Master of Arts, with honors in mathematics, in 1902. I remained at Queen's University as tutor in mathematics from 1902–1904.

The years 1904 and 1906 I spent in graduate study in mathematics and physics at Clark University, Worcester, Mass., and studied under Professors Story, Taber, Webster, Allen, and Perrot. For the year 1904–1905 I held a Junior Fellowship, and for 1905–1906 a Senior Fellowship in mathematics at Clark University. Since then I have held the following positions: Lecturer in Mathematics, Queen's University and the School of Mining, 1906–1910; Lecturer in Mathematics, University of Manitoba, 1910–1912; Assistant Professor of Mathematics and Astronomy, University of Manitoba, 1912—.

I have spent five quarters in residence at the University of Chicago: Spring and Summer, 1911 and 1912, and Summer, 1913; one quarter having been spent at Yerkes Observatory. Here I have studied under Professors Moulton, Moore, Bliss, Bolza, Lunn, Slocum, Laves, and MacMillan, and I wish to thank my instructors for the inspiration I have received from them. Particularly do I wish to express my deepest gratitude to Professor Moulton whose invaluable suggestions and assistance have made it possible for me to carry on this investigation.