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A Contribution to the Foundations of Fréchet's Calcul Fonctionnel

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*A Contribution to the Foundations of Fréchet's Calcul Fonctionnel.**

BY T. H. HILDEBRANDT.

PREFACE.

During the last few years there has been manifest a tendency in the direction of generalization of current analysis. Two memoirs bearing on this subject have recently made their appearance: in the first instance, M. Fréchet's Paris thesis, *Sur quelques Points du Calcul Fonctionnel;* † and secondly, E. H. Moore's *Introduction to a Form of General Analysis.* ‡ We consider briefly the contents and direction of generalization of each.

Fréchet's work may be divided into two parts: (1) a theory of continuous functions on an abstract set, and (2) a generalization of the theory of linear point sets. The first of these was no doubt suggested by the analogies between theorems on continuous function of a single variable, of n variables, of lines, of curves, etc. The element of generality enters in the consideration of a class or set $\mathfrak{Q}$ of elements q, which are not specifically defined. For the class there is postulated the existence of a notion of limit of a sequence of elements, satisfying a number of conditions which are properties of the limit of a sequence of real numbers. In terms of such a limit, it is possible to define a sequentially continuous function, and hence to construct a theory of sequentially continuous functions. To attain the second end, there is postulated for the class $\mathfrak{Q}$ the existence of a *voisinage* or distance function δ of pairs of elements, there being a value of δ for every pair of elements of the class. This distance function δ is subjected to a number of conditions, generalizations of properties of its real variable analogue, the absolute value of the difference between two numbers. In terms of such a δ, a limit is definable, and a theory of sets, concerning derived, closed, etc., sets, is obtainable.

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* This paper is in the form sent to the editors in April, 1910, with the addition of § 15 (6) and (7) and a few changes in § 17, due to the article by Fréchet: Rend. di Pal., XXX, 1-26. Cf. also Hedrick: *Trans. Am. Math. Soc.*, XII, 285-294, which contains some more general and some less general theorems than the present paper.

† Reprinted in *Rendiconti del Circulo di Palermo*, Vol. XXII, pp. 1-64.

‡ Published in *The New Haven Mathematical Colloquium*, New Haven, 1910.

The *General Analysis* of Moore may be termed "*a theory of classes of functions on an unconditioned range.*" The subject under consideration is a system $(\mathfrak{A}; \mathfrak{P}; \mathfrak{M})$, consisting of the class $\mathfrak{A}$ of real numbers a, the class $\mathfrak{P}$ of unconditioned elements p, and the class $\mathfrak{M}$ of real-valued, single-valued functions μ of p, p ranging over the class $\mathfrak{P}$; *i. e.*, of functions on $\mathfrak{P}$ to $\mathfrak{A}$. The theory treats of properties of the classes $\mathfrak{M}$, the properties in question being common to the following special classes of functions:

(1) The class of all unipartite numbers x, *i. e.*, a function of the variable p having only one value.

(2) The class of all n-partite numbers $(x_1, \ldots, x_n)$, functions of the variable index p; $p = 1, 2, \ldots, n$.

(3) The class of all absolutely convergent series $x_1, x_2, \ldots, x_n, \ldots$, where $\Sigma |x_n|$ is convergent, functions of the variable index p; $p = 1, 2, 3, \ldots, n, \ldots$.

(4) The class of all continuous functions on the interval $0 \leq p \leq 1$ of the real number system.

The first part of the memoir treats of certain closure and dominance properties of general reference, *i. e.*, independent of the nature of the parameter p. The second part treats of properties of special reference, in connection with the question of composition of classes of functions, one of the classes being on an unconditioned range. In particular, three properties, K_1, K_2, and Δ, are treated, K_1 and K_2 relating to the relations

$$K_{pm} \text{ and } K_{p_1 p_2 m},$$

which in turn depend upon a development* of the class $\mathfrak{P}$. The property Δ also relates to the development of $\mathfrak{P}$. For the real variable p, the first of these two relations is the inequality $p \geqq m$, and the second the inequality $|p_1 - p_2| \leqq \dfrac{1}{m}$.

The present work concerns itself with the Fréchet point of view. It had its inception in an attempt to replace the distance function δ of Fréchet by a weaker condition on the class $\mathfrak{Q}$. The fact that in most instances the δ appears in connection with an inequality of the type

$$\delta_{q_1 q_2} \leqq \frac{1}{m}$$

suggested the adoption of the second K-relation of Moore, $K_{q_1 q_2 m}$, in the place of

* Real variable analogue, the set of divisions of an interval into n equal parts, $n = 1, 2, 3, \ldots$.

the δ. By stating, in the case of every theorem, the precise conditions on K sufficient to carry the argument, and extending this idea to the case in which the class $\mathfrak{Q}$ is subjected only to the condition of the existence of a limit, a two-fold result was obtained: (a) *that an unconditioned limit suffices for the theorems on sequentially continuous functions obtained by Fréchet, and* (b) *that it is possible to obtain the theory of sets of elements with a distance function δ, subjected to weaker conditions than those imposed by Fréchet.* To show that the conditions in question were weaker, the complete existential theory,* of the properties of the K-relation, and as a consequence of the corresponding properties of the δ, was constructed.

The first part treats of the limit L, the K-relation, and their properties. Instead, however, of considering the existence of an L and a K-relation as a property or condition of the class $\mathfrak{Q}$, we use the notion of a system, the class $\mathfrak{Q}$ together with L being a system $(\mathfrak{Q}; L)$, and, with the K-relation, being a system $(\mathfrak{Q}; K)$. In the second part the most important theorems of Fréchet are taken up. It might be regarded in the light of a proof of the above results.

I.

Systems $(\mathfrak{Q}; L)$ $(\mathfrak{Q}; K)$ $(\mathfrak{Q}; \delta)$: Definitions, Properties and Interrelations.

1. *Introductory.* We consider in this paper properties of and functions on a class $\mathfrak{Q}$ of *general* elements q. The elements q are general in that nothing is specified as to their nature, that is, as to whether they be numbers, points on a line, points in n-dimensional space, sequences, real-valued functions, etc. However, we suppose the elements q of the class $\mathfrak{Q}$ to be well-defined, individually and in their totality. Further, that there exists at least one element in the class. These suppositions do not limit the generality of the class $\mathfrak{Q}$.

The class $\mathfrak{Q}$ enters the theory through two properties L and K, which together with the class $\mathfrak{Q}$ form systems $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$ respectively, the nature of which is to be specified in the sequel.

2. *Notational.*† We shall denote throughout this paper, *classes*, that is collections, sets, aggregates, ensembles, etc., by capital German letters, *e. g.* $\mathfrak{Q}$; $\mathfrak{R}$; $\mathfrak{M}$, etc., in particular, $\mathfrak{Q}$ a class of elements q, $\mathfrak{R}$ a class of elements r,

* Cf. Moore, *loc. cit.*, p. 82. Also below, § 9.

† For an extensive treatment of the ideas underlying subjects touched on in this paragraph see Moore, *loc. cit.*, §2, pp. 15 ff.

$\mathfrak{M}$ a class of functions μ, etc. The notation for the class always corresponds to the notation for the elements.

The brackets [] will be used to denote *a class of*, and in particular a class of the elements included within. Thus $[q]$ stands for a class of elements q, and we have therefore

$$\mathfrak{Q} = [q].$$

As special classes of frequent occurrence we have the classes of *all real numbers; all numbers greater than unity; all positive numbers; all positive integers; all positive and negative integers;* denoted variously by:

$$\mathfrak{A},\ [a];\quad [c];\quad [d],\ [e];\quad [n],\ [l],\ [k];\quad [m],$$

respectively.

Real-valued single-valued functions are denoted by Greek letters.

The fact that an element x *belongs to the class* $\mathfrak{Q}$, is expressed:

$$x^{\mathfrak{Q}}.$$

Since we have $\mathfrak{Q} = [q]$, evidently:

The statement $x^{\mathfrak{Q}}$ is equivalent to the statement x is a q.

This abbreviation is extended to the case of classes of elements, and we have:

$$\mathfrak{N}^{\mathfrak{Q}},$$

denoting the fact that *the class* $\mathfrak{N}$ *is a subclass of the class* $\mathfrak{Q}$.

To state that the element x *has the property* P, we use the notation:

$$x^P.$$

The concept of belonging to a class may be said to be a special case of such a general property, and from this point of view, the notations are in agreement. The notation is also extended to the case of classes, and we have:

$$\mathfrak{Q}^P,$$

denoting that *the class* $\mathfrak{Q}$ *has the property* P. If the property P is not holding we prefix the negative sign. Thus

$$\mathfrak{Q}^{-P}$$

specifies that *the class* $\mathfrak{Q}$ *does not have the property* P.

Finally, in the statement of propositions and proofs it has been found convenient for the sake of clearness and brevity to use some of the symbols used be Peano* and Moore,† in particular the following:

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* Peano: Formulario Matematico, Editio V[1], 1906. † *Loc. cit.*, p. 150.

$=$ to denote *logical equality,*
$\neq$ " *logical diversity,*
$\equiv$ " *definitional identity,*
$\supset$ " *it is true that; ( ) implies ( ); if ( ), then ( ),*
$\subset$ " *( ) is implied by ( ),*
$\backsim$ " *( ) is equivalent to ( ); ( ) implies and is implied by ( ),*
$\exists$ " *there exists,*
$\ni$ " *such that,*
$\cdot$ " *and,*
$\cdot$, $:$, $\because$ *as signs of punctuation, the largest number of punctuation dots being around the principal implication,*
[ ] to denote *a class of,*
{ } " *the sequence of.*

3. *Definition of Limit. The Systems* ($\mathfrak{Q}$; L). *Properties of Limit in* ($\mathfrak{Q}$; L). We assume the following definition of limit:

Limit is a relation between sequences of elements and single elements.

The nature of the relation is not specified permanently; it may vary with the type of element considered. If such a relation is holding between a sequence of elements, and a single element, the single element is said to be the limit of the sequence of elements.

In case there is defined a limit relation for the class $\mathfrak{Q}$, we say that the class $\mathfrak{Q}$ has the property L, and in this way we obtain a system ($\mathfrak{Q}$; L). The limit L in such a system might be considered as drawn up in the form of a table which for every sequence of the class specifies the corresponding single element, if such an one exists. We shall suppose that: *

$$\underset{n}{L} q_n = q$$

states the fact that the limit relation L is holding between the sequence $\{q_n\}$ and the element q.

The limit L in the class $\mathfrak{Q}$ may have one or more of the following properties:

(1) *Limit is unique.*

(2) *If a sequence has a limit, any subsequence of the sequence taken in the same order has the same limit.*†

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* Throughout this paper we denote $\underset{n=\infty}{L}$ by $\underset{n}{L}$.

† This is a statement of two distinct facts: (1) that the limit of the subsequence exists, and (2) that it is the same as the limit of the original sequence. Similarly below, in the case of properties (3), (4), (5).

(3) *If a sequence has a limit, any sequence obtained by removing a finite number of elements at the beginning of the sequence, has the same limit.*

(4) *If a sequence has a limit, any sequence obtained by prefixing a finite number of terms to the sequence, has the same limit.*

(5) *If a sequence has a limit, any sequence obtained by a rearrangement of the sequence has the same limit.*

(6) *If all of the terms of a sequence are identical, the sequence has a limit, which is the repeated element.*

We denote by L^1; L^{23}; L^{-34}; etc., a limit L having the property (1); the properties (2), and (3); not the property (3), but the property (4); etc., respectively. The limit of the real number system is an L^{123456}. The limit used by Fréchet is an L^{126}. We lay down permanently no properties of L, but in the various theorems in a system $(\mathfrak{Q}; L)$ specify explicitly sufficient properties of the L.

There exists but one relation between these properties of L. We have:

$$L^2 \cdot \supset \cdot L^3,$$

i. e., if L has the property (2) then it also has the property (3). That this relation is holding is at once evident. That it is the only relation between these properties may be shown by the exhibition of limits having the properties (1), (2), (3), (4), (5), (6), or their negatives, in all the 2^6 conceivable combinations, not excluded by the above relation. By employing the method of § 10 for the construction of limits, it is possible with little difficulty to obtain the various combinations desired.

4. *The K-relation.* We shall subsequently show that it is possible to obtain, relative to the system $(\mathfrak{Q}; L)$, the L being unconditioned, a theory of sequentially continuous functions. However, some of the theorems in the theory of point sets are not holding, even though we suppose the limit L to have the six properties of § 3. The necessity of a system more restricted, and therefore less general, than the system $(\mathfrak{Q}; L)$ is thus apparent.

We define:

The K-relation is a relation between pairs of elements, and positive and negative integers.

The nature of the K-relation will evidently depend upon the nature of the elements considered, and the situation in which it is to be employed. Relative to a class $\mathfrak{Q}$, we may consider the K-relation as drawn up in the form of a table which specifies for every combination of a pair of elements q_1, q_2 with an integer m, *i. e.*, for every $q_1 q_2 m$, whether or not the relation holds. We denote the fact that the K-relation is holding (not holding) between $q_1 q_2 m$ by

$$K_{q_1q_2m} \quad (-K_{q_1q_2m}).$$

If we join the K-relation to a class $\mathfrak{Q}$, we obtain a system $(\mathfrak{Q}; K)$. We shall, however, make a restrictive postulate relative to this system, viz.:

In the system $(\mathfrak{Q}; K)$, *for every pair of elements* q_1, q_2 *of the class* $\mathfrak{Q}$ *there exists at least one integer* m *such that*

$$K_{q_1q_2m}.\text{*}$$

The K-relation may have one or more of the following properties: †

(1) $$K_{q_1q_2m'} \cdot m \leqq m' \cdot \supset \cdot K_{q_1q_2m},$$

i. e., if the K-relation holds between q_1, q_2, and m', then it also holds between q_1, q_2, and m, where m is any integer less than m'.

(2) $$K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1m},$$

i. e., if the K-relation holds between q_1, q_2, and m, then it also holds if the q's are interchanged, *i. e.*, between q_2, q_1, and m. The relation $K_{q_1q_2m}$ is symmetrical in the arguments q_1, q_2.

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* Fréchet (*loc. cit.*, p. 18) introduces in his work a "voisinage" or distance function on pairs of elements, and supposes that to every pair of elements q_1, q_2 of the class $\mathfrak{Q}$ there corresponds a real number a, the value of the function. We shall denote this function by δ, so that δ is a function on $\mathfrak{Q}\mathfrak{Q}$ to $\mathfrak{A}$. The presence of a δ in the class $\mathfrak{Q}$ gives us a system $(\mathfrak{Q}; \delta)$. The idea of the above relation was suggested by the frequent occurrence of inequalities of the form

$$\delta_{q_1q_2} \leq \frac{1}{m}.$$

However, to permit the use of quantities on the right-hand side of this inequality which are greater than unity, it has been found convenient to substitute the inequality

$$\delta_{q_1q_2} \leq \frac{1}{c^m},$$

where c is a real number greater than unity, and m takes on both positive and negative integral values. We take up the relation between the Fréchet δ, and the K-relation in §6.

† These properties are analogues of the properties which Fréchet presupposes relative to the δ, viz.:

$$\delta_{q_1q_2} \geq 0; \quad \delta_{q_1q_2} = \delta_{q_2q_1}; \quad \delta_{q_1q_2} = 0 \cdot \backsim \cdot q_1 = q_2;$$

i. e., if $\delta_{q_1q_2}$ is zero, then $q_1 = q_2$, and if $q_1 = q_2$, then $\delta_{q_1q_2}$ is zero:

$$\exists\, \phi_e \ni (\underset{e=0}{L}\, \phi_e = 0 \cdot \delta_{q_1q_2} \leq e \cdot \delta_{q_2q_3} \leq e \cdot \supset \cdot \delta_{q_1q_3} \leq \phi_e),$$

that is, there exists a real-valued single-valued function of e which approaches zero as e approaches zero, and is such that if $\delta_{q_1q_2} \leq e$ and $\delta_{q_2q_3} \leq e$, then $\delta_{q_1q_3} \leq \phi_e$.

(3) $$K_{q_1q_2m}(m) \cdot \supset \cdot q_1 = q_2,$$

i. e., if the relation $K_{q_1q_2m}$ holds for all values of m*, then $q_1 = q_2$.

(4) $$q_1 = q_2 \cdot \supset \cdot K_{q_1q_2m}(m),$$

the converse of (3).

(5) $$\exists\, \phi_m \ni (\underset{m=\infty}{L}\, \phi_m = \infty \cdot K_{q_1q_2m}\, K_{q_2q_3m} \cdot \supset \cdot K_{q_1q_3\phi_m}),$$

i. e., there exists a function ϕ of m, with integral values ϕ_m, approaching infinity with m, and such that if $K_{q_1q_2m}$ and $K_{q_2q_3m}$ then $K_{q_1q_3\phi_m}$.†

As properties equivalent to (5) if the symmetry (2) is holding, we notice:

(6) $$\exists\, \phi_m \ni (\underset{m=\infty}{L}\, \phi_m = \infty \cdot K_{q_2q_1m}\, K_{q_2q_3m} \cdot \supset \cdot K_{q_1q_3\phi_m}),$$

which reads like (5). Further:

(7) $$\exists\, \phi_m \ni (\underset{m=\infty}{L}\, \phi_m = \infty \cdot K_{q_1q_2m}\, K_{q_3q_2m} \cdot \supset \cdot K_{q_1q_3\phi_m}),$$

(8) $$\exists\, \phi_m \ni (\underset{m=\infty}{L}\, \phi_m = \infty \cdot K_{q_2q_1m}\, K_{q_3q_2m} \cdot \supset \cdot K_{q_1q_3\phi_m}).$$

The functions ϕ of (5) to (8) need not be identical. We denote them by ϕ^5, ϕ^6, ϕ^7, ϕ^8, respectively. In addition to ϕ^{5678}, we shall need the following:

(a) $\phi^9_m = \phi^7_{m'}$, where m' is the lesser of ϕ^6_m and m,
(b) $\phi^{10}_m = \phi^8_{m'}$, where m' is the lesser of ϕ^6_m and m,
(c) $\phi^{11}_m = \phi^8_{m''}$, where m'' is the lesser of ϕ^7_m and m.

Evidently each of these three ϕ's approaches infinity with m.

We speak of K^1, K^{23}, K^{-34}, etc., as K-relations having the property (1), the properties (2) and (3), not the property (3) but the property (4), etc.

5. *Relations between the Properties of the K-relation.* Before taking up the relations between the above eight properties we prove the following lemmas:

(1) $$K^{167} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^9_m},$$

i. e., if the K-relation has the properties (1), (6), and (7), of § 4, then the K-relation also holds between q_2, q_1 and ϕ^9_m, where ϕ^9_m is defined in § 4, in terms of ϕ^6_m and ϕ^7_m, whose existence is assured by the presence of K^{67}.‡

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* Denoted by (m). Cf. Moore, *loc. cit.*, p. 27.

† In addition to the Fréchet property mentioned above compare also:

$$| a_1 - a_2 | \leq 1/c^m \cdot | a_2 - a_3 | \leq 1/c^m \cdot \supset \cdot | a_1 - a_3 | \leq 1/c\phi_m .$$

‡ This lemma and the succeeding ones aim at replacing the symmetry property (2) by a number of properties, collectively weaker, but for our purposes equally effective.

Suppose $K_{q_1q_2m}$. Then since K^6 we have: $K_{q_2q_3\phi^6_m}$. Let m' be the lesser of ϕ^6_m and m. Then since K^1 we have

$$K_{q_1q_2m'} \text{ and } K_{q_2q_3m'},$$

and so if we apply K^7, we obtain:

$$K_{q_2q_1\phi^7_{m'}}, \; i.\,e., \; K_{q_2q_1\phi^9_m}.$$

In a similar manner we show:

(2) $$K^{168} \cdot K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_1\phi^{10}_m}.$$

(3) $$K^{178} \cdot K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_1\phi^{11}_m}.$$

We have further:

(4) $$K^{46} \cdot K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_1\phi^6_m},$$

(5) $$K^{47} \cdot K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_1\phi^7_m},$$

(6) $$K^{48} \cdot K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_1\phi^8_m},$$

the proof of which is easily evident.

We then have the following propositions relating to the interrelations between the properties of K:

(7) $$K^2 :\supset: K^5 \cdot\backsim\cdot K^6 \cdot\backsim\cdot K^7 \cdot\backsim\cdot K^8,$$

i. e., if the K-relation has the property (2), then the properties (5), (6), (7), and (8) are equivalent. For we evidently can choose:

$$\phi^5_m = \phi^6_m = \phi^7_m = \phi^8_m.$$

(8) $$K^4 :\supset: K^6 \cdot\backsim\cdot K^7;$$

i. e., if the K-relation has the property (4), then properties (6) and (7) are equivalent. We show first that if (6) is present, then (7) is also. Suppose

$$K_{q_1q_2m} \text{ and } K_{q_3q_2m}.$$

Then by (4) above, we have:

$$K_{q_2q_1\phi^6_m} \text{ and } K_{q_2q_3\phi^6_m}.$$

Hence, applying K^6:

$$K_{q_1q_3\phi^6_{\phi^6_m}}; \; i.\,e., \; \phi^7_m = \phi^6_{\phi^6_m}.$$

Similarly if K^7 we show that

$$\phi^6_m = \phi^7_{\phi^7_m},$$

(9) $$K^4 :\supset: K^7 \cdot\supset\cdot K^8.$$

In a manner analogous to the preceding, it appears that:

$$\phi^8_m = \phi^7_{\phi^7_m},$$

(10) $$K^4 :\supset: K^8 \cdot\supset\cdot K^5.$$

Evidently

$$\phi^5_m = \phi^8_{\phi^8_m}.$$

(11) $$K^{14} :\supset: K^8 \cdot\supset\cdot K^7.$$

The proof is the result of an application of proposition (5), and the method of proposition (1).

The last four propositions may be gathered into a single one, and we thus get:

(12) $$K^{14} :\supset: K^6 \cdot\backsim\cdot K^7 \cdot\backsim\cdot K^8 \cdot\supset\cdot K^5;$$

i. e., under the hypothesis that K has the properties (1) and (4), the properties (6), (7), and (8) are equivalent, and any one implies (5).

(13) $$K^{167} \cdot\supset\cdot K^5.$$

This is a result of proposition (1), and the method of (1). It appears that

$$\phi^5_m = \phi^6_{m''},$$

where m'' is the lesser of ϕ^9_m and m.

In an entirely similar manner we have:

(14) $$K^{167} \cdot\supset\cdot K^8;\ K^{168} \cdot\supset\cdot K^{57};\ K^{178} \cdot\supset\cdot K^{56}.$$

We can write these as a series of continued implications and equivalences as follows:

(15) $$K^1 :\supset: K^{67} \cdot\backsim\cdot K^{78} \cdot\backsim\cdot K^{68} \cdot\supset\cdot K^5;$$

i. e., if K has the property (1), then the combinations (6) (7), (7) (8), (6) (8) are equivalent, and any combination implies property (5).

We shall show later, in § 9, that these constitute all of the relations which hold between the properties of K, the class $\mathfrak{Q}$ being general. It is too much of digression to take up this discussion at this point.

6. *The Relation between the Fréchet δ and the K-relation.* In the foot-note in § 4 we referred to the fact that Fréchet employs a distance function δ, having certain properties. It is possible to separate some of these properties,

and we thus obtain the following, of which the last seven are analogous to the properties of the K-relation above, with corresponding numbers:

(0)* $$\delta_{q_1q_2} \geqq 0 \text{ for every pair } q_1,\, q_2.$$

(2) $$\delta_{q_1q_2} = \delta_{q_2q_1}.$$

The δ is a symmetrical function of its arguments.

(3) $$\delta_{q_1q_2} = 0 \;\cdot\!\supset\!\cdot\; q_1 = q_2.$$

If for a pair of values $q_1 q_2$ the δ function has the value zero, the two members of the pair are identical.

(4) $$q_1 = q_2 \;\cdot\!\supset\!\cdot\; \delta_{q_1q_2} = 0.$$

(5)† $$\exists\ \phi_e \ni (\underset{e=0}{L}\, \phi_e = 0 \cdot \delta_{q_1q_2} \leqq e \cdot \delta_{q_2q_3} \leqq e \;\cdot\!\supset\!\cdot\; \delta_{q_1q_3} \leqq \phi_e).$$

(6) $$\exists\ \phi_e \ni (\underset{e=0}{L}\, \phi_e = 0 \cdot \delta_{q_2q_1} \leqq e \cdot \delta_{q_2q_3} \leqq e \;\cdot\!\supset\!\cdot\; \delta_{q_1q_3} \leqq \phi_e).$$

(7) $$\exists\ \phi_e \ni (\underset{e=0}{L}\, \phi_e = 0 \cdot \delta_{q_1q_2} \leqq e \cdot \delta_{q_3q_2} \leqq e \;\cdot\!\supset\!\cdot\; \delta_{q_1q_3} \leqq \phi_e).$$

(8) $$\exists\ \phi_e \ni (\underset{e=0}{L}\, \phi_e = 0 \cdot \delta_{q_2q_1} \leqq e \cdot \delta_{q_3q_2} \leqq e \;\cdot\!\supset\!\cdot\; \delta_{q_1q_3} \leqq \phi_e).$$

We denote by δ^2, δ^{34}, etc., a δ having the properties (2), the properties (3) and (4), etc.

In a system $(\mathfrak{Q}\,;\,\delta)$ we are able to define a K, and thus obtain a system $(\mathfrak{Q}\,;\,K)$ as follows:

$$K_{q_1q_2m} \equiv \delta_{q_1q_2} \leqq \frac{1}{2^m}\ ;$$

i. e., the K-relation holds between q_1, q_2 and m, if $\delta_{q_1q_2} = \frac{1}{2^m}$. Denote the K-relation thus defined by K_δ. Then it follows at once:

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* This property, not analogous to any properties of the K, plays a rôle only in that it avoids the persistent use of the absolute value of the δ. From any given δ we can obtain an equally effective δ having the property (0), by simply taking the absolute of the given δ. We shall suppose in the sequel that this has been done, and that we are operating with a δ having the property (0).

† *I. e.*, there exists a real-valued single-valued function ϕ_e which approaches zero as e approaches zero, such that if $\delta_{q_1q_2} \leqq e$ and $\delta_{q_2q_3} \leqq e$ then $\delta_{q_1q_3} \leqq \phi_e$. Moreover, we are able to say that there exists a function ϕ which is bounded for all finite values of e. For suppose it were not so. Then there would exist

(a) $\{e_n\}$, e_0, such that $e_n < e_0$;

(b) $\{q_{1n}\}$, $\{q_{2n}\}$, $\{q_{3n}\}$, such that $\delta_{q_{1n}q_{2n}} \leqq e_n$ and $\delta_{q_{2n}q_{3n}} \leqq e_n$ and $\delta_{q_{1n}q_{3n}} > n$.

But

$$\delta_{q_{1n}q_{2n}} \leqq e_0 \cdot \delta_{q_{2n}q_{3n}} \leqq e_0 \;\cdot\!\supset\!\cdot\; \delta_{q_{1n}q_{3n}} \leqq \phi_{e_0},$$

and we thus have a contradiction. We shall therefore suppose that the ϕ chosen is bounded for all finite values of e. Similarly for $\phi^6{}_e$, $\phi^7{}_e$, $\phi^8{}_e$ below.

(1) K_δ^1; *i. e*, the K-relation defined from the δ has the property (1) of § 4. We have further:

(2) If δ has any of the properties (2), (3), (4), (5), (6), (7), (8) above, the K_δ has the corresponding properties. In full:

$$\delta^2 \cdot\supset\cdot K_\delta^2;\ \delta^3 \cdot\supset\cdot K_\delta^3;\ \delta^4 \cdot\supset\cdot K_\delta^4;\ \delta^5 \cdot\supset\cdot K_\delta^5;\ \delta^6 \cdot\supset\cdot K_\delta^6;$$
$$\delta^7 \cdot\supset\cdot K_\delta^7;\ \delta^8 \cdot\supset\cdot K_\delta^8.$$

The first three of these cause no particular difficulty. As for $\delta^5 \cdot\supset\cdot K_\delta^5$, it is necessary to construct a ϕ^5. Let:

$$E_m = \overline{B}\left(\phi_e \,\Big|\, \frac{1}{2^{m-1}} < e \leqq \frac{1}{2^m}\right);$$

i. e., the least upper bound of the values of ϕ_e, while e lies between $\frac{1}{2^{m-1}}$ and $\frac{1}{2^m}$. In accordance with the foot-note on p. 247, this will evidently exist. Then if

$$\frac{1}{2^{m'-1}} < E_m \leqq \frac{1}{2^{m'}},$$

we set $\phi_m = m'$. In this way we define a function ϕ with integral values ϕ_m, and evidently

$$\underset{m=\infty}{L}\, \phi_m = \infty.$$

A similar construction holds for (6), (7), and (8).

We thus have the result that the Fréchet theory is a special case of a K-theory.

On the other hand, in a system $(\mathfrak{Q}; K)$ where the K-relation has the property (1), of § 4, we can define a δ, and hence obtain a system $(\mathfrak{Q}; \delta)$, in the following manner:

(a).* If $K_{q_1q_2m}(m)$, then $\delta_{q_1q_2} = 0$.

(b). If not (a), then set $\delta_{q_1q_2} = \frac{1}{2^{m'}}$, where†

$$m' = \overline{B}(m \ni K_{q_1q_2m}).$$

If we define a δ corresponding to a K-relation having the property (1), in this way we are able to state a proposition analogous to proposition (2) above.

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* If $K_{q_1q_2m}$ holds for all values of m, then $\delta_{q_1q_2} = 0$.

† m' is the least upper bound of the values of m for which the K-relation holds between q_1q_2m.

As a consequence, whatever theorems involving the properties of § 4 are holding in a system $(\mathfrak{Q}; K)$, are also holding in a system $(\mathfrak{Q}; \delta)$; and conversely, the theorems holding in a $(\mathfrak{Q}; \delta)$ will also hold in a $(\mathfrak{Q}; K)$, provided the K-relation in question has the property (1).

It seems then that the K-relation with the property (1) of § 4 is not more general than a δ. We can, however, regard the K-relation in the light of an operation which exhibits a δ having rational values only, corresponding to any given δ. If, moreover, the K-relation is sufficient for our theory, then we have shown, incidentally, that the essential part of the δ is a rational part. We shall therefore use the K-relation in preference to the δ.

7. *Limit defined in terms of the K-relation.* In terms of the K-relation we may define limit as follows:*

$$\underset{n}{L} q_n = q \;\therefore\equiv\therefore\; m :\supset: \exists\, n_m \ni n \geqq n_m \;.\supset.\; K_{q_n q m}.$$

Evidently such a limit is an L. We shall denote it by L_K. It follows then that any theory obtainable in a system $(\mathfrak{Q}; L)$ is also holding in a system $(\mathfrak{Q}; K)$ with $L = L_K$.

We have the following propositions relative to the properties of L_K:

(1) $$K \;.\supset.\; L_K^{2345},$$

where (2), (3), (4), (5) are the properties of L of § 3. This is at once evident from the definition of L_K.

(2) $$K^4 \;.\supset.\; L_K^6,$$

which is also an immediate consequence of the definition of L_K.

(3) $$K^{136} \;.\supset.\; L_K^1;$$

i. e., if K^{136}, then the L_K produces a unique limit. This may be shown as follows: suppose

$$\underset{n}{L} q_n = q', \text{ and } \underset{n}{L} q_n = q''.$$

---

* $\underset{n}{L} q_n = q$ is by definition the same as the statement: For every m there exists an integer n_m such that for every n greater than n_m we have $K_{q_n q m}$. Of course, this is not the only possible definition of limit in a K-situation; *e. g.*, we might define:

$$\underset{n}{L} q_n = q \;:\equiv:\; m :\supset: \exists\, n_m \ni n = n_m \;.\supset.\; K_{q_n q m}.$$

The above definition, however, is the analogue of the Fréchet definition:

$$\underset{n}{L} q_n = q \;.\equiv.\; \underset{n}{L} \delta_{q_n q} = 0.$$

Then by the definition of L_K we have:

$$m \cdot \supset \cdot \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot K_{q_n q'm} \text{ and } K_{q_n q''m},$$

the n_m being the greater of the values of n_m for q' and q''. Then K^6 gives us:

$$K_{q'q''\phi_m}(m).$$

Now since $\underset{m}{L}\phi_m = \infty$, and K has the property (1), we have:

$$K_{q'q''m}(m),$$

and so K^3 gives us

$$q' = q'':$$

Usually in a situation in which it is a question of arguing the fact that a sequence approaches an element as a limit, we do not obtain a statement which is identical with the above definition, but rather one of which the statement of the definition is a consequence. One of the cases of frequent occurrence is covered by the following lemma:*

$$(4) \qquad K^1 \cdot \{q_n\}, q :\supset: \exists\, \psi_n \ni (\underset{n}{L}\psi_n = \infty \cdot K_{q_n q\psi_n}) \cdot \supset \cdot \underset{n}{L} q_n = q.$$

For since $L\psi_n = \infty$, we have:

$$m :\supset: \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot \psi_n > m.$$

Since K^1, this n_m will also serve as the n_m of the definition of limit.

As a special case of this we might consider this proposition when for every n we have

$$q_n = q'.$$

Then we have:

$$(5) \quad K^1 \cdot q \cdot q_n = q'(n) :\supset: \exists\, \psi_n \ni (\underset{n}{L}\psi_n = \infty \cdot K_{q_n q\psi_n}) \cdot \supset \cdot K_{q'qm}(m).$$

This lemma was used in the proof of (3) above.

8. *On the Composition of Classes of Elements, and of K-relations.* Suppose two classes $\mathfrak{Q}' = [q']$, and $\mathfrak{Q}'' = [q'']$. We derive from these classes a product or composite class:

$$\mathfrak{Q}'\mathfrak{Q}'' = [(q', q'')];$$

i. e., a class $\mathfrak{Q} = [q]$ whose elements $q = (q', q'')$ are bipartite, q' and q'' independently ranging over the class $\mathfrak{Q}'$ and $\mathfrak{Q}''$ respectively. In practice, when there is no possibility of misinterpretation, we denote the element (q', q'') by $q'\,q''$.

---

* *I. e.*, if the K-relation has the property (1) of § 4, and the sequence $\{q_n\}$ and the element q are such that there exists a function ψ with integral values ψ_n, approaching infinity with n, and such that $K_{q_n q\psi_n}$, then the sequence has q as a limit.

Similarly, from a finite number of classes $\mathfrak{Q}'$, $\mathfrak{Q}''$, ..., $\mathfrak{Q}^n$ we obtain a composite class:

$$\mathfrak{Q}'\mathfrak{Q}''\ldots\mathfrak{Q}^n = [q'q''\ldots q^n].$$

Returning to the case $\mathfrak{Q}'\mathfrak{Q}''$, suppose there exist in $\mathfrak{Q}'$ and $\mathfrak{Q}''$ K-relations: K' and K'', respectively; *i. e.*, suppose we are dealing with the systems $(\mathfrak{Q}'; K')$ and $(\mathfrak{Q}''; K'')$. Then the existence of a system $(\mathfrak{Q}; K)$ is of interest. This depends upon the possibility of defining a K-relation from the K' and K''. We construct such a K-relation as follows:

Suppose $q_1 = q_1'q_1''$ and $q_2 = q_2'q_2''$. Then

$K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$, and m is the smaller of m' and m''.

Evidently such a K-relation will satisfy the condition that for every pair of elements q_1 and q_2, of the class $\mathfrak{Q}$, there exists an m such that $K_{q_1q_2m}$, in so far as such a condition is holding in the systems $(\mathfrak{Q}'; K')$ and $(\mathfrak{Q}''; K'')$.*

We consider how the presence of properties (1)–(8) of § 4 of the K' and K'' affects the presence of the corresponding properties of the K. We have:

$$K'^P \cdot K''^P \cdot\supset\cdot K^P,$$

where $P =$ (1), (2), (4), (5), (6), (7), (8), but

$$K'^{13} \cdot K''^{13} \cdot\supset\cdot K^3.$$

The proof of these propositions is very simple.

We can then state the result that if we have a theory in a system $(\mathfrak{Q}'; K')$, which concerns itself with properties (1)–(8) of § 4, a corresponding theory is holding in a composite class built up of two such systems.

A similar construction and result relative to the composite class and the composite K will evidently hold if we are dealing with any finite number of systems $(\mathfrak{Q}; K)$.

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* The above definition is not the only one possible. Others which suggest themselves are:

(a) $K_{q_1q_2m}$ if both $K'_{q'_1q'_2m}$ and $K''_{q''_1q''_2m}$.

(b) $K_{q_1q_2m}$ if either $K'_{q'_1q'_2m}$ or $K''_{q''_1q''_2m}$.

(c) $K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$ and m is the larger of m' and m''.

(d) $K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$ and $m = m' + m''$.

(e) $K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$ and $m = m' \times m''$.

All of these except the first have the defect that the theorem K'^3 and $K''^3 \cdot\supset\cdot K^3$ is not holding, and moreover it is not possible to find a simple condition on K' and K'' which will carry with it the presence of the property (3) in K. In the case of the first, we do not have a system $(\mathfrak{Q}; K)$ as defined in § 4, unless K' and K'' have the property (1) of § 4.

9. *On the Complete Existential Theory of the Properties of the K-relation.* We conclude this part by the construction of a complete existential theory of the properties (1)–(8) of the K-relation.

*Definition.** *The complete existential theory* of a set of properties P_1, $P_2, \ldots, P_n$ of systems Σ, consists of (a) the body of interrelations between the properties, and (b) a body of systems Σ, there being a system Σ for each of the conceivable combinations of properties $P_1, \ldots, P_n$ and their negatives $-P_1, \ldots, -P_n$, not excluded by the interrelations of (a).

In case a system Σ is obtainable for each of the 2^n combinations of $P_1, \ldots, P_n$ and their negatives $-P_1, \ldots, -P_n$, the set of properties $P_1, \ldots, P_n$ is said to be *completely independent and consistent.*

Evidently, if a set of n properties $P_1, \ldots, P_n$ are completely independent and consistent, they are also independent in the ordinary sense, *i.e.*, it is possible to find systems Σ having the properties:

$$-P_1, P_2, \ldots, P_n; \quad P_1, -P_2, P_3, \ldots, P_n; \quad \ldots; \quad P_1, P_2, \ldots, P_{n-1}, -P_n.$$

We propose to consider the eight properties of the K-relation given in § 4. In § 5 we have derived a set of relations between these properties. These relations exclude 176 of the $2^8 = 256$ combinations of the properties (1)–(8), and their negatives, leaving 80 to be discussed. To complete the existential theory it is necessary to obtain a K-relation for each of these eighty combinations of properties of K. We take up these eighty K-relations relative to classes $\mathfrak{O}$ of the following types:

(a) A class $\mathfrak{O}$ consisting of one element.

(b) A class $\mathfrak{O}$ consisting of two elements.

(c) A class $\mathfrak{O}$ consisting of three elements.

(d) A class $\mathfrak{O}$ consisting of four elements.

(e) A class $\mathfrak{O}$ consisting of a finite number of elements.

(f) A class $\mathfrak{O}$ consisting of a denumerable infinitude of elements.

(g) A class $\mathfrak{O}$ consisting of a non-denumerable infinitude of elements of the power of the interval $0 \ldots 1$. These are denoted by

$$\mathfrak{O}^{\mathrm{I}}, \mathfrak{O}^{\mathrm{II}_2}, \mathfrak{O}^{\mathrm{II}_3}, \mathfrak{O}^{\mathrm{II}_4}, \mathfrak{O}^{\mathrm{II}_n}, \mathfrak{O}^{\mathrm{III}}, \mathfrak{O}^{\mathrm{IV}},$$

respectively.

In the following table, which gives the eighty combinations of the presence and absence of the properties of K, let $+$ stand for the presence and $-$ for the absence of the property. The Roman numerals in the last column give the type of class $\mathfrak{O}$ of smallest dimension in which it is possible to determine a K-relation having the combination of properties in question.

* Cf. Moore, *loc. cit.*, p. 82.

TABLE I.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)			(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	
1	+	+	+	+	+	+	+	+	I, II_2	41	−	−	+	−	+	−	+	+	II_2
2	+	+	+	+	−	−	−	−	III	42	−	−	+	−	−	+	+	+	II_3
3	−	+	+	+	+	+	+	+	II_2	43	−	−	+	−	+	+	−	−	II_2
4	−	+	+	+	−	−	−	−	II_3	44	−	−	+	−	+	−	+	−	II_2
5	+	−	+	+	+	+	+	+	II_2	45	−	−	+	−	−	+	+	−	II_3
6	+	−	+	+	+	−	−	−	III	46	−	−	+	−	+	−	−	+	II_2
7	+	−	+	+	−	−	−	−	III	47	−	−	+	−	−	+	−	+	II_3
8	+	+	−	+	+	+	+	+	II_2	48	−	−	+	−	−	−	+	+	II_3
9	+	+	−	+	−	−	−	−	II_3	49	−	−	+	−	+	−	−	−	II_2
10	+	+	+	−	+	+	+	+	I, II_2	50	−	−	+	−	−	+	−	−	II_3
11	+	+	+	−	−	−	−	−	III	51	−	−	+	−	−	−	+	−	II_3
12	−	−	+	+	+	+	+	+	II_2	52	−	−	+	−	−	−	−	+	II_3
13	−	−	+	+	+	−	−	+	II_2	53	−	−	+	−	−	−	−	−	II_2
14	−	−	+	+	+	−	−	−	II_2	54	−	+	−	−	+	+	+	+	II_2
15	−	−	+	+	−	−	−	−	II_3	55	−	+	−	−	−	−	−	−	II_2
16	−	+	−	+	+	+	+	+	II_3	56	+	−	−	−	+	+	+	+	II_3
17	−	+	−	+	−	−	−	−	II_3	57	+	−	−	−	+	+	−	−	II_2
18	−	+	+	−	+	+	+	+	I, II_2	58	+	−	−	−	+	−	+	−	II_2
19	−	+	+	−	−	−	−	−	II_2	59	+	−	−	−	+	−	−	+	II_2
20	+	−	−	+	+	+	+	+	II_3	60	+	−	−	−	+	−	−	−	II_3
21	+	−	−	+	+	−	−	−	II_2	61	+	−	−	−	−	+	−	−	II_3
22	+	−	−	+	−	−	−	−	II_3	62	+	−	−	−	−	−	+	−	II_3
23	+	−	+	−	+	+	+	+	II_2	63	+	−	−	−	−	−	−	+	II_3
24	+	−	+	−	+	+	−	−	III	64	+	−	−	−	−	−	−	−	II_3
25	+	−	+	−	+	−	+	−	III	65	−	−	−	−	+	+	+	+	II_2
26	+	−	+	−	+	−	−	+	III	66	−	−	−	−	+	+	+	−	II_4
27	+	−	+	−	+	−	−	−	III	67	−	−	−	−	+	+	−	+	II_3
28	+	−	+	−	−	+	−	−	III	68	−	−	−	−	+	−	+	+	II_3
29	+	−	+	−	−	−	+	−	III	69	−	−	−	−	−	+	+	+	II_3
30	+	−	+	−	−	−	−	+	III	70	−	−	−	−	+	+	−	−	II_2
31	+	−	+	−	−	−	−	−	III	71	−	−	−	−	+	−	+	−	II_2
32	+	+	−	−	+	+	+	+	II_3	72	−	−	−	−	−	+	+	−	II_3
33	+	+	−	−	−	−	−	−	II_2	73	−	−	−	−	+	−	−	+	II_2
34	−	−	−	+	+	+	+	+	II_2	74	−	−	−	−	−	+	−	+	II_3
35	−	−	−	+	+	−	−	+	II_3	75	−	−	−	−	−	−	+	+	II_3
36	−	−	−	+	+	−	−	−	II_2	76	−	−	−	−	+	−	−	−	II_2
37	−	−	−	+	−	−	−	−	II_3	77	−	−	−	−	−	+	−	−	II_3
38	−	−	+	−	+	+	+	+	II_2	78	−	−	−	−	−	−	+	−	II_3
39	−	−	+	−	+	+	+	−	II_3	79	−	−	−	−	−	−	−	+	II_3
40	−	−	+	−	+	+	−	+	II_2	80	−	−	−	−	−	−	−	−	II_2

(a) The class $\mathfrak{Q}$ consisting of a single element. Let the element be q. Then to define the K-relation it is necessary to state the values of m for which K_{qqm} holds. Then we have:

$$(1) \qquad (\mathfrak{Q}^{\mathrm{I}}; K) \cdot\supset\cdot K^{235678}.$$

There is no difficulty about seeing that K^{25678}. As for K^3 we either have

$$K_{qqm}(m) \quad \text{or} \quad \exists\, m \ni - K_{qqm}.$$

In the first case K^3, and in the second K^3 *vacuously*.* We have further:

$$(2) \qquad (\mathfrak{Q}^{\mathrm{I}}; K) :\supset: K^4 \cdot\supset\cdot K^1.$$

For if K^4 we must have $K_{qqm}(m)$, and so K^1.

These two propositions exclude all but the following three combinations:

1. $+++++++++$; 10. $+++-++++$; 18. $-++-++++$.

For 1, let K_{qqm} hold for every m; for 10, let K_{qqm} hold for $m \leqq m_1$; and for 18, let K_{qqm} hold for $m \geqq m_1$.

The case in which the class $\mathfrak{Q}$ consists of only one element not being of frequent occurrence, we give K-relations satisfying the combinations 1, 10, 18 under the next head also.

(b) The class $\mathfrak{Q}$ consisting of two elements, q_1 and q_2. In this case we must assign values for which

$$K_{q_1q_1m}, \quad K_{q_1q_2m}, \quad K_{q_2q_1m}, \quad \text{and} \quad K_{q_2q_2m}$$

hold. We arrange the combinations possible in II_2 in tabular form, giving first the combination of properties, then the values of m for which the K-relations hold, and finally the functions ϕ^5, ϕ^6, ϕ^7, and ϕ^8, in case we have K^5, K^6, K^7 or K^8.

We thus obtain 34 systems in $\mathfrak{Q}^{\mathrm{II}_2}$. We are not able to obtain more on account of certain relations which exist between these properties in case the class $\mathfrak{Q}$ consists of two elements only. These relations, suggesting themselves in the attempt to obtain a K-relation satisfying combinations of properties which they exclude, are:

$$(1) \qquad K^{13} \cdot\supset\cdot K^{5678}.$$

* This term and concept, introduced by Moore (*Transactions of the American Mathematical Society*, III, p. 489, foot-note), is of considerable importance in this type of discussion. We shall make use of it frequently, especially as far as the property (3) is concerned.

TABLE II.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	$K_{q_1q_1m}$	$K_{q_1q_2m}$	$K_{q_2q_1m}$	$K_{q_2q_2m}$	ϕ
1	+	+	+	+	+	+	+	+	(m)	$m \leqq m_1$	$m \leqq m_1$	(m)	$\phi_m = m$
3	—	+	+	+	+	+	+	+	(m)	$m \geqq m_1$	$m \geqq m_1$	(m)	$\phi_m = m$
5	+	—	+	+	+	+	+	+	(m)	$m \leqq m_1$	$m \leqq m_2 < m_1$	(m)	$\phi_m = m$
8	+	+	—	+	+	+	+	+	(m)	(m)	(m)	(m)	$\phi_m = m$
10	+	+	+	—	+	+	+	+	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\phi_m = m$
12	—	—	+	+	+	+	+	+	(m)	$m \geqq m_1$	$m \geqq m_2 > m_1$	(m)	$\phi_m = m$
13	—	—	+	+	+	—	—	+	(m)	m odd	m even	(m)	$\phi^5_m = m$; $\phi^8_m = m + 1$
14	—	—	+	+	+	—	—	—	(m)	$m \geqq m_1$	$m < m_1$	(m)	$\phi^5_m = m$
18	—	+	+	—	+	+	+	+	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	$\phi_m = m$
19	—	+	+	—	—	—	—	—	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	$m < m_1$	
21	+	—	—	+	+	—	—	—	(m)	(m)	$m \leqq m_1$	(m)	$\phi^5_m = m$
23	+	—	+	—	+	+	+	+	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_2 < m_1$	$m \leqq m_1$	$\begin{cases} \phi_m = m : & m > m_1 \\ \phi_m = m_2 : & m < m_1 \end{cases}$
33	+	+	—	—	—	—	—	—	$m \leqq m_1$	(m)	(m)	$m \leqq m_1$	
34	—	—	—	+	+	+	+	+	(m)	(m)	$m \geqq m_1$	(m)	$\begin{cases} \phi_m = m : & m \geqq m_1 \\ \phi_m = m_1 : & m < m_1 \end{cases}$
36	—	—	—	+	+	—	—	—	(m)	(m)	m_1	(m)	$\phi^5_m = m$
38	—	—	+	—	+	+	+	+	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_2 > m_1$	$m \geqq m_1$	$\begin{cases} \phi_m = m : & m \geqq m_2 \\ \phi_m = m_2 : & m < m_2 \end{cases}$
40	—	—	+	—	+	+	—	+	m odd, m_1	m_1	m even, m_1	m_1	$\phi_{m_1} = m_1$; $\phi^{58}_m = m$; $\phi^6_m = m + 1$
41	—	—	+	—	+	—	+	+	m odd, m_1	m even, m_1	m_1	m_1	$\phi_{m_1} = m_1$; $\phi^{58}_m = m$; $\phi^7_m = m + 1$
43	—	—	+	—	+	+	—	—	$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	$m < m_1$	$\phi^{56}_m = m$
44	—	—	+	—	+	—	+	—	$m < m_1$	$m < m_1$	$m \geqq m_1$	$m \geqq m_1$	$\phi^{57}_m = m$
46	—	—	+	—	+	—	—	+	$m < m_1$	$m \geqq m_2 > m_1$	$m_1 < m < m_2$	$m < m_1$	$\phi^{58}_m = m$
49	—	—	+	—	+	—	—	—	$m \geqq m_1$	$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	$\phi^5_m = m$
53	—	—	+	—	—	—	—	—	$m < m_1$	$m \geqq m_1$	$m \geqq m_1$	$m < m_1$	
54	—	+	—	—	+	+	+	+	$m \geqq m_1$	(m)	(m)	$m \geqq m_1$	$\begin{cases} \phi_m = m : & m \geqq m_1 \\ \phi_m = m_1 : & m < m_1 \end{cases}$
55	—	+	—	—	—	—	—	—	$m < m_1$	(m)	(m)	$m \geqq m_1$	
57	+	—	—	—	+	+	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	(m)	$\phi^{56}_m = m$
58	+	—	—	—	+	—	+	—	(m)	(m)	$m \leqq m_1$	$m \leqq m_1$	$\phi^{57}_m = m$
59	+	—	—	—	+	—	—	+	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$\phi^{58}_m = m$
65	—	—	—	—	+	+	+	+	$m \geqq m_1$	(m)	$m \geqq m_1$	$m \geqq m_1$	$\begin{cases} \phi_m = m : & m \geqq m_1 \\ \phi_m = m_1 : & m < m_1 \end{cases}$
70	—	—	—	—	+	+	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \geqq m_1$	ϕ^{56}; cf. 65
71	—	—	—	—	+	—	+	—	$m \geqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	ϕ^{57}; cf. 65
73	—	—	—	—	+	—	—	+	$m \leqq m_1$	(m)	$m = m_1$	$m \leqq m_1$	ϕ^{58}; cf. 65
76	—	—	—	—	+	—	—	—	$m \geqq m_1$	(m)	$m < m_1$	$m \geqq m_1$	ϕ^5; cf. 65
80	—	—	—	—	—	—	—	—	$m < m_1$	(m)	$m \geqq m_1$	$m \geqq m_1$	

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This proposition holds also when $\mathfrak{Q}$ is any class of finite dimensions. We prove it by constructing the ϕ. Evidently *

$$\overline{m}' = \overline{B}\,(m \ni K_{q_i q_k m},\ i \neq k;\quad i,\ k = 1,\ 2,\ \ldots,\ n)$$

exists. Also, if K does not have the property (4), there will exist:

$$\overline{m}'' = \overline{B}\,(m \ni K_{q_i q_i m},\ i \neq j \ni K_{q_j q_j m}\,(m)).$$

Let $\overline{m}$ be the greater of these two, which, in case K^4, will evidently be $\overline{m}'$. Let further:

$$\underline{m}' = \underline{B}\,(\overline{B}\,(m \ni K_{q_i q_k m},\ i \neq k) \mid i,\ k = 1,\ 2,\ \ldots,\ n),$$
$$\underline{m}'' = \underline{B}\,(\overline{B}\,(m \ni K_{q_i q_i m},\ i \neq j \ni K_{q_j q_j m}\,(m)) \mid i)$$

and $\underline{m}$ the smaller of these. This will exist on account of the finiteness of the class $\mathfrak{Q}$. Then we construct:

$$\phi_m = \underline{m} \text{ when } m \leqq \overline{m}, \text{ and } \phi_m = m \text{ when } m > \overline{m}.$$

It is easy to see that this ϕ serves the desired purpose.

(2) $$K^{24-3} \cdot\supset\cdot K^{15678}.$$

Since K^4 we have

$$K_{q_1 q_1 m}\,(m), \text{ and } K_{q_2 q_2 m}\,(m).$$

Further, from K^{2-3},

$$K_{q_1 q_2 m}\,(m), \text{ and } K_{q_2 q_1 m}\,(m).$$

Hence K^{15678}.

(3) $$K^4 \cdot\supset\cdot K^5.$$

For if K^4 we can choose $\phi_m = m$.

(4) $$K^{1-2} \cdot\supset\cdot K^5.$$

We can construct a ϕ in a manner similar to that used in proposition (1).

(5) $$K^{167-3} \cdot\supset\cdot K^{2458}.$$

If K^{-3}, we have either $K_{q_1 q_2 m}\,(m)$ or $K_{q_2 q_1 m}\,(m)$. Suppose $K_{q_1 q_2 m}\,(m)$. Then

$$K^{16} \cdot\supset\cdot K_{q_2 q_2 m}\,(m),\ K^{17} \cdot\supset\cdot K_{q_2 q_1 m}\,(m), \text{ and } K^{16} \cdot\supset\cdot K_{q_1 q_1 m}\,(m).$$

And so we have K^{2458}.

(6) $$K^{48-3} \cdot\supset\cdot K^{67}.$$

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* $\overline{m}'$ is the least upper bound of the values of m for which $K_{q_i q_k m}$ holds, where $i \neq k$, and both i and k range over the values 1, 2, ..., n.

By K^4 we have $K_{q_1q_1m}(m)$, and $K_{q_2q_2m}(m)$. Also by K^{-3} we have either $K_{q_1q_2m}(m)$ or $K_{q_2q_1}(m)$. But in either case the presence of K^8 permits us to take

$$\phi^6_m = \phi^7_m = \phi^8_m.$$

(7) $$K^6 \cdot\supset\cdot K^5.$$

K^5 affects the following pairs of K-relations:

$$K_{q_1q_1m},\, K_{q_1q_1m}\,;\quad K_{q_1q_1m},\, K_{q_1q_2m}\,;\quad K_{q_1q_2m},\, K_{q_2q_1m}\,;\quad K_{q_1q_2m},\, K_{q_2q_2m}\,;$$
$$K_{q_2q_1m},\, K_{q_1q_1m}\,;\quad K_{q_2q_1m},\, K_{q_1q_2m}\,;\quad K_{q_2q_2m},\, K_{q_2q_1m}\,;\quad K_{q_2q_2m},\, K_{q_2q_2m}.$$

Of these K^6 relates to the following pairs:

$$K_{q_1q_1m},\, K_{q_1q_1m}\,;\quad K_{q_1q_1m},\, K_{q_1q_2m}\,;\quad K_{q_2q_2m},\, K_{q_2q_1m}\,;\quad K_{q_2q_2m},\, K_{q_2q_2m}.$$

Moreover, by K^6 we have:

$$K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_2\phi^6m}, \text{ and } K_{q_2q_1m} \cdot\supset\cdot K_{q_1q_1\phi^6m}.$$

So that

$$K_{q_1q_2m},\, K_{q_2q_1m} \cdot\supset\cdot K_{q_1q_1m\,\phi^6_m}$$

and

$$K_{q_2q_1m},\, K_{q_1q_2m} \cdot\supset\cdot K_{q_2q_2\phi^6_m}.$$

There thus remain to be treated only:

$$K_{q_1q_2m} \text{ with } K_{q_2q_2m} \text{ and } K_{q_2q_1m} \text{ with } K_{q_1q_1m}.$$

We build our ϕ^5 as follows: for m's common only to $K_{q_1q_2m}$ and $K_{q_2q_2m}$, and common only to $K_{q_2q_1m}$ and $K_{q_1q_1m}$, let $\phi^5_m = m$. For all other values of m let $\phi^5_m = \phi^6_m$. It is not difficult to show that this is an effective ϕ^5_m.

(8) $$K^7 \cdot\supset\cdot K^5.$$

The demonstration of this is analogous to that of (7).

(9) $$K^8 \cdot\supset\cdot K^5.$$

We show that it is possible to choose $\phi^5_m = \phi^8_{\phi^8_m}$, by taking up each individual pair to which K^5 applies.

(10) $$K^{67} \cdot\supset\cdot K^8.$$

The proof is similar to that of (7).

(11) $$K^{68-3} \cdot\supset\cdot K^7.$$

By writing down all pairs to which K^7 applies, and using the fact that since

K^{-3}, we have either $K_{q_1q_2m}(m)$ or $K_{q_2q_1m}(m)$, we find without much difficulty that $\phi^7_m = \phi^6_{\phi^8_{\phi^6_m}}$.

$$(12) \qquad K^{78-3} \ .\supset . \ K^6.$$

The proof of this is similar to that of (11). We find that $\phi^6_m = \phi^7_{\phi^8_{\phi^7_m}}$.

$$(13) \qquad K^{1-2-3} \ .\supset . \ K^4 \text{ or } K^6 \text{ or } K^7 \text{ or } K^8.$$

Since K^{-3}, we have either $K_{q_1q_2m}(m)$ or $K_{q_2q_1m}(m)$, but not both, since K^{-2}. Suppose $K_{q_1q_2m}(m)$. Then $K^{1-2} \ .\supset . \ \exists \ m_1 \ni K_{q_2q_1m}$ for $m \leqq m_1$ only. There are then four possibilities relative to $K_{q_1q_1m}$ and $K_{q_2q_2m}$:

(α) $K_{q_1q_1m}(m)$, and $K_{q_2q_2m}(m)$. Then K^4.

(β) $K_{q_1q_1m}(m)$, and $K_{q_2q_2m}$ $m \leqq m_2$. Then K^7.

(γ) $K_{q_1q_1m}$ for $m \leqq m_2$ and $K_{q_2q_2m}(m)$. Then K^6.

(δ) $K_{q_1q_1m}$ for $m \leqq m_2$ and $K_{q_2q_2m}$ for $m \leqq m_3$. Then K^8.

These relations make impossible the systems not included in the above enumeration of the systems possible in $\mathfrak{Q}^{II_2}$. In particular, (1) excludes 2, 6, 7, 11, 24, 25, 26, 27, 28, 29, 30, 31; (2) excludes further 9, 16, 17; (3) excludes 4, 5, 22, 37; (4) excludes 61, 62, 63, 64; (5) excludes 20, 32, 56; (6) excludes 35; (7) excludes 42, 45, 47, 50, 69, 72, 74, 77; (8) excludes 48, 51, 75, 78; (9) excludes 52, 79; (10) excludes 39, 66; (11) excludes 67; (12) excludes 68; (13) excludes 60. This totals 46, the number of systems which have not been obtained thus far.

(c) The class $\mathfrak{Q}$ consisting of three elements; *i. e.*, $\mathfrak{Q}^{II_3}$. We pass now to the case in which the class $\mathfrak{Q}$ consists of three elements. We do not take up the combinations of properties of the K-relation found to exist in a $(\mathfrak{Q}^{II_2}; K)$. As a matter of fact, we shall show later how to modify the systems obtained in $\mathfrak{Q}^{II_2}$, so as to procure systems in $\mathfrak{Q}^{II_3}$ having the same combinations of properties.

As regards the relations which hold between the properties of the K-relation when $\mathfrak{Q}$ consists of three elements, we have already seen that the relation

$$(1) \qquad K^{13} \ .\supset . \ K^{5678}$$

holds when $\mathfrak{Q}$ is of finite dimension, and hence when $\mathfrak{Q}$ has only three elements. There is but one other proposition holding in $\mathfrak{Q}^{II_3}$:

$$(2) \qquad K^{567-3} \ .\supset . \ K^8.$$

This is proved by considering all of the pairs of K-relations to which K^8 applies.

If we assume that $K_{q_1q_2m}(m)$, one relation of this type holding on account of K^{-8}, it is easily apparent that we have $\phi^8 = \phi^6_{\phi^7_{\phi^5}}$, excepting for the following four pairs of K-relations:

$K_{q_1q_3m} \cdot K_{q_1q_1m}$; $K_{q_1q_3m} \cdot K_{q_2q_1m}$; $K_{q_2q_3m} \cdot K_{q_1q_2m}$; $K_{q_2q_3m} \cdot K_{q_2q_2m}$, for which we have as a possible ϕ^8: $\phi^8 = \phi^7_{\phi^6_{\phi^5}}$. We then build a ϕ^8 as follows: Take $\phi^8 = \phi^6_{\phi^7_{\phi^5}}$ excepting for values of m for which both $K_{q_1q_3m}$ and $K_{q_1q_1m}$ or both $K_{q_2q_3m}$ and $K_{q_2q_2m}$, and not $K_{q_3q_1m}$ or $K_{q_3q_3m}$, and for which both $K_{q_1q_3m}$ and $K_{q_2q_1m}$ or both $K_{q_2q_3m}$ and $K_{q_1q_2m}$, and not $K_{q_3q_1m}$ or $K_{q_3q_2m}$ or $K_{q_3q_3m}$, hold. For these values of m we take $\phi^8 = \phi^7_{\phi^6_{\phi^5}}$. It is not difficult to show that this ϕ is uniquely determined for all values of m and that $\underset{m}{L}\phi_m = \infty$.

This proposition excludes but one further possibility: 66. There thus remain thirty-three combinations of properties of K which it is possible to obtain in a class $\mathfrak{Q}^{II_3}$, in addition to those already obtained in $\mathfrak{Q}^{II_2}$. These systems are given in Table III.

(d) The class $\mathfrak{Q}$ consisting of four elements. Passing to the case in which the class $\mathfrak{Q}$ consists of four elements, we obtain only one additional system, viz., one satisfying the combination:

66. $- - - - + + + -$.

The following K-relation will satisfy this combination of properties:

$K_{q_1q_1m}$: $m_2, m_3, m \geqq m_1$;	$K_{q_1q_2m}$: (m);	$K_{q_1q_3m}$: m_2, m_6;	$K_{q_1q_4m}$: m_3;
$K_{q_2q_1m}$: $m_2, m_3, m \geqq m_1$;	$K_{q_2q_2m}$: $m_2, m_3, m \geqq m_1$;	$K_{q_2q_3m}$: m_2, m_4;	$K_{q_2q_4m}$: m_3, m_6;
$K_{q_3q_1m}$: m_2;	$K_{q_3q_2m}$: m_2;	$K_{q_3q_3m}$: m_2;	$K_{q_3q_4m}$: m_4;
$K_{q_4q_1m}$: m_3;	$K_{q_4q_2m}$: m_3, m_5;	$K_{q_4q_3m}$: m_7;	$K_{q_4q_4m}$: m_2, m_3.

We suppose the $m_2, \ldots, m_7$ to be distinct, and less than m_1. The values of the ϕ-functions are as follows:

	m_2	m_3	m_4	m_6	$m \geqq m_1$	$m < m_1$
ϕ^5	m_2	m_3	m_6	m_3	m	m
ϕ^6	m_2	m_3	m_2	m_2	m	m_1
ϕ^7	m_2	m_3	m_2	m_3	m	m_1

In this situation K^{-8}, because the ϕ^8_m is not uniquely defined for the value m_4, a result of considering $K_{q_1q_2m_4}$ with $K_{q_2q_3m_4}$ and $K_{q_2q_3m_4}$ with $K_{q_3q_4m_4}$.

TABLE III.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	$K_{q_1q_1m}$	$K_{q_1q_2m}$	$K_{q_1q_3m}$	$K_{q_2q_1m}$	$K_{q_2q_2m}$
4	−	+	+	+	−	−	−	−	(m)	$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	(m)
9	+	+	−	+	−	−	−	−	(m)	(m)	$m \leqq m_1$	(m)	(m)
15	−	−	+	+	−	−	−	−	(m)	$m \geqq m_1$	$m < m_1$	$m \geqq m_2$	(m)
16	−	+	−	+	+	+	+	+	(m)	(m)	$m \geqq m_1$	(m)	(m)
17	−	+	−	+	−	−	−	−	(m)	(m)	$m < m_1$	(m)	(m)
20	+	−	−	+	+	+	+	+	(m)	(m)	$m \leqq m_1$	(m)	(m)
22	+	−	−	+	−	−	−	−	(m)	(m)	$m \leqq m_1$	(m)	(m)
32	+	+	−	−	+	+	+	+	(m)	(m)	$m \leqq m_1$	(m)	(m)
35	−	−	−	+	+	−	−	+	(m)	(m)	m_2	(m)	(m)
37	−	−	−	+	−	−	−	−	(m)	(m)	$m < m_1$	(m)	(m)
39	−	−	+	−	+	+	+	−	$m_1,\ m_2,\ m_3$	$m_1,\ m_2$	m_3	m_2	m_2
42	−	−	+	−	−	+	+	+	m_4	m_1	m_2	m_3	m_4
45	−	−	+	−	−	+	+	−	$m_1,\ m_2$	$m_1,\ m_2$	m_3	m_2	m_2
47	−	−	+	−	−	−	+	+	m_5	m_1	m_2	$m_3,\ m_4$	m_5
48	−	−	+	−	−	+	−	+	m_5	m_1	m_2	m_3	m_5
50	−	−	+	−	−	+	−	−	$m \leqq m_1$	$m > m_1$	$m \leqq m_1$	$m \leqq m_1$	$m > m_1$
51	−	−	+	−	−	−	+	−	$m \geqq m_1$	$m > m_1$	$m \leqq m_1$	$m > m_1$	$m > m_1$
52	−	−	+	−	−	−	−	+	$m \leqq m_1$	$m \geqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$
56	+	−	−	−	+	+	+	+	(m)	(m)	$m \leqq m_1$	(m)	(m)
60	+	−	−	−	+	−	−	−	$m \leqq m_1$	(m)	(m)	$m \leqq m_1$	$m \leqq m_1$
61	+	−	−	−	−	+	−	−	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	(m)
62	+	−	−	−	−	−	+	−	(m)	(m)	$m \leqq m_1$	(m)	(m)
63	+	−	−	−	−	−	−	+	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$
64	+	−	−	−	−	−	−	−	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$
67	−	−	−	−	+	+	−	+	$m = 2m',\ m_1$	(m)	m_1	$m = 2m',\ m_1$	$m = 2m',\ m_1$
68	−	−	−	−	+	−	+	+	$m = 2m',\ m_1$	(m)	$m = 2m' + 1,\ m_1$	$m = 2m',\ m_1$	$m = 2m',\ m_1$
69	−	−	−	−	−	+	+	+	$m = 2m',\ m_1$	(m)	m_1	$m = 2m',\ m_1$	$m = 2m',\ m_1$
72	−	−	−	−	−	+	+	−	$m = 2m',\ m_1$	(m)	m_1	$m = 2m',\ m_1$	$m = 2m',\ m_1$
74	−	−	−	−	−	+	−	+	$m = 3m',\ m_1$	(m)	m_1	$m = 3m',\ m_1$	$m = 3m',\ m_1$
75	−	−	−	−	−	−	+	+	$m = 3m',\ m_1$	(m)	$m = 3m' + 2,\ m_1$	$m = 3m',\ m_1$	$m = 3m',\ m_1$
77	−	−	−	−	−	+	−	−	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \geqq m_1$
78	−	−	−	−	−	−	+	−	$m \geqq m_1$	(m)	$m \leqq m_1$	(m)	(m)
79	−	−	−	−	−	−	−	+	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$

TABLE III—*Continued.*

$K_{q_2q_3m}$	$K_{q_3q_1m}$	$K_{q_3q_2m}$	$K_{q_3q_3m}$	Φ
$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	(m)	
(m)	$m \leqq m_1$	(m)	(m)	
$m \geqq m_1$	$m < m_1$	$m \geqq m_2$	(m)	
$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	(m)	$\Phi_m = m$
$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	(m)	
$m \leqq m_1$	$m \leqq m_2$	$m \leqq m_2$	(m)	$m_2 < m_1$; $\Phi_m = m_2 : m \leqq m_1$; $\Phi_m = m : m > m_1$
(m)	$m \leqq m_2$	(m)	(m)	
$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\Phi_m = m$
m_2	$m_3 < m_2$	m_3	(m)	$\Phi^{58}_m = m$; $\Phi^5_{m_2} = m_3$; $\Phi^8_{m_3} = m_2$
$m \geqq m_1$	$m < m_1$	$m \geqq m_2$	(m)	
m_4	m_1, m_3	m_1	m_3	$\Phi^5_m = m$; $\Phi^6_{m_1} = m_2$; $\Phi^6_{m_2} = m_2$; $\Phi^6_{m_3} = m_3$; $\Phi^6_{m_4} = m_3$; $\Phi^7_{m_1} = m_3$; $\Phi^7_{m_2} = m_2$; $\Phi^7_{m_3} = m_3$; $\Phi^7_{m_4} = m_2$
m_1	m_1	m_5	m_4	$\Phi^6_{m_1} = \Phi^6_{m_2} = m_4$; $\Phi^8_{m_1} = m_1$
m_1	m_4	m_5	m_2	$\Phi^{67}_{m_1} = \Phi^{67}_{m_2} = m_2$
m_1, m_4	m_1	m_6	m_5	$\Phi^8_{m_1} = m_1$; $\Phi^7_{m_1} = \Phi^7_{m_4} = m_5$
m_1, m_2	m_1	m_4	m_5	$\Phi^8_{m_1} = m_1$; $\Phi^6_{m_1} = \Phi^6_{m_2} = m_5$
$m > m_1$	$m \leqq m_1$	$m > m_1$	$m \geqq m_1$	$\Phi^6_m = m_1 : m \leqq m_1$; $\Phi^6_m = m : m < m_1$
$m > m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 50
$m \geqq m_1$	$m \geqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 50
$m \leqq m_1$	$m \leqq m_2 < m_1$	$m \leqq m_2 < m_1$	$m \leqq m_3 < m_2$	$\Phi_m = m_3 : m \leqq m_1$; $\Phi_m = m : m > m_1$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\Phi^5_m = m$
(m)	$m \leqq m_1$	(m)	(m)	$\Phi^6_m = m$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\Phi^7_m = m$
(m)	(m)	$m \leqq m_1$	$m \leqq m_1$	$\Phi^8_m = m$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	
m_1	m_1	$m = 2m' + 1, m_1$	$m = 2m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^{58}_m = m$; $\Phi^6_m = 2m$
m_1	m_1	m_1	$m = 2m', m_1$	Cf. 67
$m = 2m' + 1, m_1$	$m = 2m' + 1, m_1$	m_1	$m = 2m', m_1$	Cf. 67
$m = 2m' + 1, m_1$	m_1	m_1	$m = 2m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^{67}_m = 2m$
$m = 3m' + 1, m_1$	$m = 3m' + 1, m_1$	$m = 3m' + 2, m_1$	$m = 3m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^6_m = 3m$; $\Phi^8_m = m$
$m = 3m' + 1, m_1$	$m = 3m' + 1, m_1$	m_1	$m = 3m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^7_m = 3m$; $\Phi^8_m = m$
(m)	$m \leqq m_1$	$m \geqq m_1$	(m)	$\Phi_m = m_1 : m \leqq m_1$; $\Phi_m = m : m > m_1$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 77
(m)	$m \geqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 77

(e) The class $\mathfrak{Q}$ containing any finite number of elements. It is not possible to obtain any additional systems when we pass to the general finite case.

(f) The class $\mathfrak{Q}$ consisting of a denumerable infinitude of elements. We are thus forced to proceed to the case in which the class $\mathfrak{Q}$ is infinite. The simplest type of infinity is the denumerable type, and so we consider this first. It is possible to obtain systems $(\mathfrak{Q}^{\text{III}}; K)$ which have the twelve combinations of properties of the K-relation not yet considered. Suppose the elements to be q_i, $i = 1, 2, \ldots\ldots$ Then the K-relations may be defined as follows:

$$
2.\quad + + + + - - - - - : K_{q_iq_jm} : \begin{array}{lll} i \neq j \neq 1 \text{ or } 2 & : & m \leqq 2 \\ i = j & : & (m) \\ i \text{ or } j = 1 \text{ or } 2 & : & m \leqq i + j. \end{array}
$$

For from $K_{q_1q_j1+j}$ with $K_{q_jq_21+j}$ would follow $K_{q_1q_2\phi^6_{1+j}}$, which is contrary to $K_{q_1q_23}$. Since K^2 we have also K^{-6-7-8}.

$$
6.\quad + - + + + - - - : K_{q_iq_jm} \begin{array}{lll} : i = j & : & (m) \\ : i < j & : & m \overline{\overline{<}}\, 2 \\ : i > j & : & m \overline{\overline{<}}\, i. \end{array}
$$

(1) K^5. We show that $\phi^5_m = m$. Suppose $K_{q_iq_jm}$ and $K_{q_jq_km}$. Then if $i = j$ or $j = k$, we evidently have $\phi^5_m = m$. There remain to be considered thus:

(α) $i > j$, $j > k$. Then since $K_{q_iq_jm}$ holds for $m \leqq i$, $K_{q_jq_km}$ for $m \leqq j$, and $K_{q_iq_km}$ for $m \leqq i$, we evidently have $\phi^5_m = m$.

(β) $i < j$, $j > k$. Then since $K_{q_iq_jm}$ holds for $m \leqq 2$, and $K_{q_jq_km}$ for $m \leqq j$, while $K_{q_iq_km}$ at least for $m \leqq 2$, we also have $\phi^5_m = m$. Similarly it appears that this ϕ^5_m will work for the other two possibilities: $i > j$ and $j < k$, and $i < j$ and $j < k$.

(2) On the other hand we have K^{-6}. For suppose $i > 2$. Then if K^6 we should have from $K_{q_iq_1i}$ and $K_{q_iq_2i}$, $K_{q_1q_2\phi_i}$. But by hypothesis we have at most $K_{q_1q_22}$. Hence K^{-6}.

Since, further, K^6, K^7, K^8 are equivalent under K^{14} by § 5 (12), we have K^{-7} and K^{-8}.

$$
7.\quad + - + + - - - - - : K_{q_iq_jm} \quad \begin{array}{lll} i = j & : & (m) \\ i \neq j;\ i - j \neq 1 & : & m \leqq 2 \\ i - j = 1 & : & m \leqq 2i + 1. \end{array}
$$

For $K_{q_i q_{i+1} 2i+1}$ with $K_{q_{i+1} q_{i+2} 2i+1}$ would give us by K^5: $K_{q_i q_{i+2} \phi_{2i+1}}$, and we have at most $K_{q_i q_{i+2} 2}$. Hence K^{-5}. Again by K^6 we should have from $K_{q_i q_{i+1} 2i+1}$ with $K_{q_i q_i 2i+1}$: $K_{q_{i+1} q_i \phi_{2i+1}}$, and we have at most: $K_{q_{i+1} q_i 2}$. Hence K^{-6}, and also, by § 5 (12), K^{-7} and K^{-8}.

11. $+ + + - - - - - -$: $K_{q_i q_j m}$ $i \neq 1$ and $j \neq 1$: $m \leqq 2$,
$i = 1$ or $j = 1$: $m \leqq i + j$.

This is similar to 2 above.

24. $+ - + - + + - -$: $K_{q_i q_j m}$ $m \leqq j$.

It is not difficult to show that $\phi_m^{56} = m$. We see that K^{-7} by considering $K_{q_1 q_i m}$ with $K_{q_1 q_i m}$, and that K^{-8} by considering $K_{q_1 q_i m}$ with $K_{q_i q_i m}$.

25. $+ - + - + - + -$: $K_{q_i q_j m}$ $m \leqq i$. Cf. 24.

26. $+ - + - + - - +$: $K_{q_i q_j m}$ $m \leqq 2$, except when $i - j = 1$ and i is even, in which case $m \leqq 2i + 1$.

It is easily apparent that $\phi_m^{58} = m$. On the other hand, if K^7 and K^6 we should have from $K_{q_i q_{i+1} 2i+1}$: $K_{q_i q_i \phi^7_{2i+1}}$ and $K_{q_{i+1} q_{i+1} \phi^6_{2i+1}}$, which is contrary to $K_{q_i q_i 2}$ and $K_{q_{i+1} q_{i+1} 2}$.

27. $+ - + - + - - -$: $K_{q_i q_j m}$: $i \leqq j$ $m \leqq 1$
$i > j$ $m \leqq i$.

Cf. 6.

28. $+ - + - - + - -$: $K_{q_i q_j m}$: $m \leqq 2$ except $K_{q_i q_i m}$ for $i = 3n + 3$, and $K_{q_{i+1} q_i m}$, $K_{q_{i+2} q_i m}$, $K_{q_i q_{i+2} m}$ for $i = 3n + 1$, in which cases $m \overline{\overline{<}} n + 1$.

We show without much difficulty that $\phi_m^6 = m$. The negatives are easily obtained.

29. $+ - + - - - + -$: $K_{q_i q_j m}$: $m \leqq 2$ except $K_{q_i q_i m}$ for $i = 3n + 3$ or $3n + 1$, and $K_{q_i q_{i+1} m}$, $K_{q_i q_{i+2} m}$, $K_{q_{i+2} q_i m}$, for $i = 3n + 1$, in which cases $m \overline{\overline{<}} n + 1$.

Similar to 28.

30. $+-+-----+$: $K_{q_iq_jm}$: $m \leqq 2$ except $K_{q_iq_{i+1}m}$ for $i=3n+1$ or $3n+2$, in which case $m \leqq 2n+1$, and $K_{q_iq_{i-2}m}$ for $i=3n$, for which $m \leqq 2i-2$.

We have in this case $\phi^8_m = m-1$.

31. $+-+------$: $K_{q_iq_jm}$: $m \leqq i+j$.

We have thus obtained a situation for each of the above combinations of properties.

(g). However, the classes $\mathfrak{Q}$, to which we apply the K-relation and its properties, are generally of the denumerably infinite, or non-denumerably infinite type. In the above set of systems, however, there are considered only twelve which include a class $\mathfrak{Q}$ which is denumerably infinite, and no system in which the class $\mathfrak{Q}$ is non-denumerably infinite. The question naturally arises whether these situations are possible also in these two types of classes. We obtain the desired result by attempting to extend the given situations to classes of greater dimension, preserving the properties of the K-relation. The following scheme of extension will cover a large number of cases:

(α) Extension of $\mathfrak{Q}^{II_n}$ to $\mathfrak{Q}^{II_l}$, $l > n$.

Let the elements of $\mathfrak{Q}^{II_l}$ be $q_1, q_2, \cdots, q_l$. We put this set into correspondence with a set of n elements, $q'_1, q'_2, \cdots, q'_n$, by supposing q'_j to correspond to q_i, if $j \equiv (i-1) \bmod n$. This assigns to every member of the set $q_1, \cdots, q_l$, a definite correspondent of the set $q'_1, q'_2, \cdots, q'_n$. We suppose then that $K_{q_iq_jm}$ holds for values of m for which $K_{q'_{i'}q'_{j'}m}$ holds, where $q'_{i'}$ corresponds to q_i and $q'_{j'}$ to q_j. We then have

$$(1) \qquad K^P \text{ in } \mathfrak{Q}^{II_n} \;.\supset.\; K^P \text{ in } \mathfrak{Q}^{II_l},$$

where P is one of the properties: (1), (-1), (2), (-2), (-3), (4), (-4), (5), (-5), (6), (-6), (7), (-7), (8), (-8). There is no difficulty about seeing this. Hence it includes all of the properties and their negatives, excepting (3). If by K^{-4}, identically, we mean that for no i do we have $K_{q_iq_im}$, then we can say:

$$(2) \qquad K^{3-4} \text{ in } \mathfrak{Q}^{II_n} \;.\supset.\; K^{3-4} \text{ in } \mathfrak{Q}^{II_l}.$$

As a result we have then that if, in the combination K^{-34}, we have K^{-4} identically, all cases excepting those for which we have K^{34} may be extended as above, and the result will be a K in a $\mathfrak{Q}^{II_l}$ which will have the same combination of properties as the K in $\mathfrak{Q}^{II_n}$. A review of the above examples will reveal the fact that in all cases in which we have K^{3^-4}, we have K^{-4} identically. Hence this mode of extension will apply to all of the above systems excepting eleven, for which we have K^{34}.

(β) A similar scheme enables us to pass from $\mathfrak{Q}^{II_n}$ to $\mathfrak{Q}^{III}$. Suppose $\mathfrak{Q}^{III}$ to consist of the elements: $q_1, \dots, q_k, \dots$. We put q_k in correspondence with $q'_{k'}$ if $k \equiv (k'-1) \bmod n$. Then if we define our K-relation as in (α), we shall have the same results.

(γ) As for passing from $\mathfrak{Q}^{II_n}$ to $\mathfrak{Q}^{IV}$, where $\mathfrak{Q}^{IV}$ is a class of the non-denumerably infinite type, which is of the same power as the continuum, the interval $0 \leqq x \leqq 1$. Let us suppose that our elements are q_x, where $0 \leqq x \leqq 1$; *i.e.*,

$$\mathfrak{Q}^{IV} = [q_x;\ 0 \leqq x \leqq 1].$$

To set up a correspondence with $\mathfrak{Q}^{II_n}$, divide the interval $0 \leqq x \leqq 1$ into n equal parts and set

$$q_x \text{ into correspondence with } q'_i \text{ if } \frac{i-1}{n} < x \leqq \frac{i}{n}$$

and

$$q_x \text{ into correspondence with } q'_1 \text{ if } \quad 0 \leqq x \leqq \frac{1}{n}.$$

If we define our K-relation as in (α), the same results will also hold.

(δ) Finally, to extend $\mathfrak{Q}^{III}$ to $\mathfrak{Q}^{IV}$, set

$$q_x \text{ into correspondence with } q'_n \text{ if } \frac{1}{2^{n-1}} \leqq x \leqq \frac{1}{2^n},$$

and

$$q_0 \text{ into correspondence with } q'_1.$$

This produces results similar to the above.

We have thus shown that it is possible to obtain all but eleven of the above combinations of properties in $\mathfrak{Q}^{III}$ and $\mathfrak{Q}^{IV}$. It remains to consider these. We discuss each of these cases separately.

1. $+ + + + + + + + +$. In any class let $K_{q_1q_2m}$ hold for every m if $q_1 = q_2$, and for $m \leqq m_1$ if $q_1 \neq q_2$.

2. $+ + + + - - - - -$. This occurs for the first time in $\mathfrak{Q}^{III}$. So we need consider it only for $\mathfrak{Q}^{IV}$. Supposing the elements of $\mathfrak{Q}^{IV}$ to be q_x, we define our K as follows:

$$K_{q_xq_xm}(m), \quad K_{q_{x_1}q_{x_2}m} \text{ hold for } m \leqq 2,\ x_1,\ x_2 \neq 0 \text{ or } 1;$$

$$K_{q_xq_0m},\ K_{q_0q_xm},\ K_{q_1q_xm},\ K_{q_xq_1m}, \text{ for } m \leqq m_1, \text{ where } \frac{1}{2^{m_1}} < x.$$

3. $- + + + + + + +$. We suppose that in any class $K_{q_1 q_2 m}$ holds for every m if $q_1 = q_2$, and for $m \geqq m_1$ if $q_1 \neq q_2$.

4. $- + + + - - - - -$. In any class select three elements, and for these define the K-relations as given in II_3. For all other elements let $K_{q_1 q_2 m}$ hold for every m if $q_1 = q_2$, and for $m \geqq m_1$ if $q_1 \neq q_2$.

5. $+ - + + + + + +$. Let $K_{q_1 q_2 m}$ hold for every m if $q_1 = q_2$, and for $m \leqq m_1$ for $q_1 \neq q_2$, excepting q' and q'': $K_{q' q'' m}$ for $m \leqq m_2 < m_1$.

6. $+ - + + + - - -$. We need consider only $\mathfrak{L}^{IV}$. The method of constructing a K-relation from the one holding in $\mathfrak{L}^{III}$ is similar to that used in 2. This holds also of

7. $+ - + + - - - - -$.

12. $- - + + + + + +$. Proceed as in 5, substituting $>$ for $<$.

13. $- - + + + - - +$. Let $K_{q' q'' m}$ hold for every m if $q' = q''$. Let $K_{\underset{x_1}{q_i}\, \underset{x_2}{q_j}\, m}$ hold for m odd if $i > j$ or $x_1 > x_2$, and for m even if $i < j$ or $x_1 < x_2$. We have in this case $\phi^5_m = m$, and $\phi^8_m = m + 1$.

14. $- - + + + - - -$. Suppose $K_{q' q'' m}$ holds for every m if $q' = q''$, and $K_{\underset{x_1}{q_i}\, \underset{x_2}{q_j}\, m}$ holds for $m \geqq m_1$ if $i > j$ or $x_1 > x_2$, and for $m < m_1$ if $i < j$ and $x_1 < x_2$. We have $\phi^5_m = m$.

15. $- - + + - - - - -$. Proceed as in 4 above.

We have thus completed the study of the complete existential theory of our eight properties, and have shown that with the exception of the propositions of § 5, there exist no further relations between the properties in question.

While we do not have the entire set of properties completely independent, certain combinations of them are. For instance: (1), (2), (3), (4), (5) form a set of properties, which include all of the rest, and are completely independent.* A further set of completely independent properties is (1), (3), (6), (7). This set of properties is not quite as strong as the above. For if we have K^{12345}, we have also K^{678}, while if we have K^{1367}, we have only in addition K^{58}, and not necessarily K^2 or K^4.† There are a number of other interesting combinations of properties, but they are equivalent to these two, or to the combination K^{1346}.

* In so far as Fréchet assumes the properties (2), (3), (4), (5) as properties of the δ, and a system $(\mathfrak{L}\,;\,\delta^{2345})$ is a system $(\mathfrak{L}\,;\,K^{23451})$, we have shown incidentally that he has chosen a set of completely independent and consistent properties.

† In the next part we shall show that a K^{1367} is sufficient for the Fréchet δ-theory. We have thus a weaker set of properties than those assumed by Fréchet.

II.

Properties of Subclasses $\mathfrak{R}$ of the Class $\mathfrak{Q}$ of a System $(\mathfrak{Q}\,;\,K)$ and of Continuous Functions on $\mathfrak{R}$ to $\mathfrak{A}$.

We have been considering systems $(\mathfrak{Q}\,;\,L)$, and $(\mathfrak{Q}\,;\,K)$, and have in particular discussed properties of L and the K-relation. We now turn our attention to the properties of the class $\mathfrak{Q}$, considering in particular subclasses* $\mathfrak{R}$ of $\mathfrak{Q}$, and real-valued continuous functions on $\mathfrak{R}$.

We show in § 13 that it is not possible to obtain some theorems of the theory of point sets, relative to $\mathfrak{R}$ of systems $(\mathfrak{Q}\,;\,L)$, even though we suppose that L has all of the properties of § 3. Hence the treatment of subclasses $\mathfrak{R}$ of classes $\mathfrak{Q}$ is confined mostly to systems $(\mathfrak{Q}\,;\,K)$. However, a theory of sequentially continuous functions is obtainable in a system $(\mathfrak{Q}\,;\,L)$. A theory of difference continuous functions can be derived in a system $(\mathfrak{Q}\,;\,K)$. These are taken up in §§ 18 ff.

The theorems derived are in the main the theorems of Fréchet. However, instead of permanently conditioning the L and the K in the systems $(\mathfrak{Q}\,;\,L)$ and $(\mathfrak{Q}\,;\,K)$, we have preferred to indicate in each case the precise properties of L or K, sufficient to carry the argument. In this way it appears that it is not necessary to condition the L for the theorems on continuous functions. Further, that a K-relation having the properties (1), (3), (6), (7), of § 4, is sufficient for all the theorems, and in some cases even weaker conditions on the K will do. It will be noticed that the symmetry property (2) and the property (4) do not occur. The former is really a matter of convenience. It is avoided by the use of properties K^{167}, which combination we have seen is weaker than K^{125}, the combination it replaces. Property (4) serves to avoid the separate consideration of the limit of a sequence which consists of a finite number of elements only. Its presence as a condition restricts the generality of the theorems. We have therefore preferred to gain in generality at the expense of convenience, replacing the property (2) by a weaker combination, and taking up a more detailed discussion, if necessary, instead of using the property (4).

We take up first a consideration of the definitions of properties of the classes $\mathfrak{R}$ in a system $(\mathfrak{Q}\,;\,L)$, and the modification of these definitions in case we are operating in a system $(\mathfrak{Q}\,;\,K)$, passing thence to the consideration of sub-

* We shall suppose that $\mathfrak{R}$ denotes a subclass of $\mathfrak{Q}$ throughout this part.

classes $\mathfrak{R}$ of $\mathfrak{Q}$ in systems $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$, and the question of continuous functions of $\mathfrak{R}$ of $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$.

10. *Definitions.** Suppose $\mathfrak{S}$ a subclass of $\mathfrak{R}$. Then we have:

(1) q is a *limiting element* of $\mathfrak{R}$, if it is the limit of a sequence of distinct elements of the class $\mathfrak{R}$.† We denote limiting elements of $\mathfrak{R}$ by r'.

(2) The *derived class* $\mathfrak{R}'$ of $\mathfrak{R}$ is the class which consists of all of the limiting elements of the class $\mathfrak{R}$, *i. e.*, we have $\mathfrak{R}' = [\text{all } r']$.

(3) $\mathfrak{R}$ is *closed* if it contains its derived class $\mathfrak{R}'$, *i. e.*, if $\mathfrak{R}'^{\mathfrak{R}}$.

(4) $\mathfrak{R}$ is *dense in itself* if its derived class contains it, *i. e.*, if $\mathfrak{R}^{\mathfrak{R}'}$.

(5) $\mathfrak{R}$ is *perfect* if it is identical with its derived class, *i. e.*, if $\mathfrak{R} = \mathfrak{R}'$.

(6) $\mathfrak{R}$ is *compact* if every denumerable infinitude of elements of $\mathfrak{R}$ gives rise to at least one limiting element.‡

(7) $\mathfrak{R}$ is *extremal* if it is compact and closed.

(8) q is an *element of condensation* of $\mathfrak{R}$ if it is a limiting element of any subclass of $\mathfrak{R}$ obtained by removing a denumerable infinitude of elements from $\mathfrak{R}$. We denote an element of condensation by $r^{\wedge}$. Evidently a class $\mathfrak{R}$ has an element of condensation only if it is non-denumerable.

(9) $\mathfrak{R}$ is *condensed* if every non-denumerable set of its elements gives rise to an element of condensation.§

(10) An element q is *interior to* $\mathfrak{S}$ *relative to* $\mathfrak{R}$ ‖ if q is an element of $\mathfrak{S}$, and every sequence of $\mathfrak{R}$ which has q as a limit ultimately belongs to $\mathfrak{S}$.¶ In symbols:

$$q^{\text{interior } \mathfrak{S}(\mathfrak{R})} \equiv q^{\mathfrak{S}} \cdot \underset{n}{L} r_n = q \cdot \supset \cdot \exists\, n_1 \ni n \geqq n_1 \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

* Cf. Fréchet, *loc. cit.*, p. 6.

† Of course if the limit relation L is such that no sequence with all of its elements distinct has a limit, there will be no limiting elements. A similar statement holds relative to compact classes below.

‡ Accordingly every finite class is compact, for the definitional condition is satisfied vacuously, its hypothesis being incapable of fulfillment in a finite class. Cf. Moore, *Trans. Amer. Math. Soc.*, III, 489.

§ In accordance with the foot-note to (6) above, $\mathfrak{R}$ is condensed if it is denumerable; namely, in the vacuous sense.

‖ Denoted by $q^{\text{interior } \mathfrak{S}(\mathfrak{R})}$. Cf. relativity notation of Moore, *loc. cit.*, pp. 27 ff.

¶ Fréchet speaks of interiority in the strict sense. His definition reads:

$$q^{\text{interior } \mathfrak{S}(\mathfrak{R})} \equiv q^{\mathfrak{S}} \cdot \{ r_n \}^{\text{distinct}} \ni \underset{n}{L} r_n = q \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

This is evidently not what he means. Judging by the use made of the notion on p. 23, he assumes that he has defined interiority:

$$q^{\text{interior } \mathfrak{S}(\mathfrak{R})} \equiv q^{\mathfrak{S}} \cdot \{ r_n \}^{\text{distinct}} \ni \underset{n}{L} r_n = q \cdot \supset \cdot \exists\, n_1 \ni n \geqq n_1 \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

Our definition is, as is easily evident, equivalent to this one if we assume the L to have the properties used by Fréchet, viz., L^{126}.

11. *Two Propositions.* For an L_K defined relative to a K^1, we secure, as follows, a desirable transformation of the definition of interiority:*

(1) $$K^1 \cdot\!:\supset:\cdot\ s^{\text{interior } \mathfrak{S}(\mathfrak{R})} :\supset: \exists\ m_0 \ni (r \ni K_{rsm_0} \cdot\supset\cdot r^{\mathfrak{S}}).$$

As a converse of (1) we have:

(2) $$K \cdot s^{\mathfrak{S}} \cdot\!:\supset:\cdot\ \exists\ m_0 \ni (r \ni K_{rsm_0} \cdot\supset\cdot r^{\mathfrak{S}}) :\supset: s^{\text{interior } \mathfrak{S}(\mathfrak{R})}.$$

Joining these two propositions into a single equivalence we have:†

(3) $$K^1 \cdot s^{\mathfrak{S}} \cdot \mathfrak{S}^{\mathfrak{R}} \cdot\!:\supset:\cdot\ s^{\text{interior } \mathfrak{S}(\mathfrak{R})} :\backsim: \exists\ m_0 \ni (r \ni K_{rsm_0} \cdot\supset\cdot r^{\mathfrak{S}}).$$

We prove (1) first. If possible, suppose the proposition were not holding. Then we have:

$$m \cdot\supset\cdot \exists\ r_m \ni K_{r_m sm} \cdot r_m^{-\mathfrak{S}}.$$

Then evidently by § 7 (4): $\underset{m}{L}\, r_m = s$, and since $s^{\text{interior } \mathfrak{S}(\mathfrak{R})}$, we have:

$$\exists\ m_1 \ni m \geqq m_1 \cdot\supset\cdot r_m^{\mathfrak{S}}.$$

We have thus reached a contradiction, and our proposition therefore holds.

As for (2) suppose $\underset{n}{L}\, r_n = s$; *i. e.*,

$$m \cdot\supset\cdot \exists\ n_m \ni n \geqq n_m \cdot\supset\cdot K_{r_n sm}.$$

If, in particular, we take $m = m_0$, we shall have by the hypothesis of (2):

$$n \geqq n_{m_0} \cdot\supset\cdot r_n^{\mathfrak{S}}.$$

Hence s satisfies the conditions of interiority.

If we have K^1, we can define a concept analogous to that of the bounded point set, viz. *limited.* We define:‡

$$\mathfrak{R}^{\text{limited}} \equiv \mathfrak{R} \ni (\exists\ m \ni r_1, r_2 \cdot\supset\cdot K_{r_1 r_2 m}).$$

The property limited is related to the property compact as follows:§

(4) $$K^{17} \cdot \mathfrak{R} :\supset: \mathfrak{R}^{\text{compact}} \cdot\supset\cdot \mathfrak{R}^{\text{limited}}.$$

---

* The K-relation having the property (1), if an element s of a subclass $\mathfrak{S}$ of the subclass $\mathfrak{R}$ of $\mathfrak{Q}$ is interior to $\mathfrak{S}$ with respect to $\mathfrak{R}$, then there exists an m_0 such that every element r in the relation K_{rsm_0} belongs to $\mathfrak{S}$. The analogous proposition holds for the weaker interiority defined by Fréchet, if the K-relation has the property (3).

† A similar result holds in a system ($\mathfrak{Q}$; δ), δ unconditioned:

$$\delta \cdot \mathfrak{S}^{\mathfrak{R}} \cdot s^{\mathfrak{S}} \cdot\!:\supset:\cdot\ s^{\text{interior } \mathfrak{S}(\mathfrak{R})} :\backsim: \exists\ a \ni \delta_{rs} \leqq a \cdot\supset\cdot r^{\mathfrak{S}}.$$

In this form we have a conception of interiority which is analogous to the one of the linear point set theory.

‡ $\mathfrak{R}$ is limited, if there exists an m such that for every pair of elements r_1 and r_2 of the class $\mathfrak{R}$, we have $K_{r_1 r_2 m}$.

§ Cf. Fréchet, *loc. cit.*, p. 22.

If possible, suppose it were not so. Then

$$-m \cdot \supset \cdot \exists\, r_{1m} \cdot r_{2m} \ni -K_{r_{1m} r_{2m}(-m)}.$$

There are then four possibilities: (a) The sequences r_{1m} and r_{2m} each contain only a finite number of distinct elements. (b) The sequence r_{1m} contains only a finite number of distinct elements, while the sequence r_{2m} contains an infinitude of distinct elements. (c) The sequence r_{1m} contains an infinitude of distinct elements, while the sequence r_{2m} contains only a finite number of distinct elements. (d) Each sequence contains an infinitude of distinct elements. In case (a), there will be a certain pair of elements r_1, r_2 which will have like subscripts infinitely often. We argue a contradiction by considering that there exists an m_0 such that $K_{r_1 r_2 m_0}$. In case (b), so far as it is not covered by (a), we can show the existence of a set of integers m_n such that r_{2m_n} are all distinct and $\underset{n}{L}\, r_{2m_n} = q$, while $r_{1m_n} = r_1$ for every n. Then, since $(\mathfrak{Q}; K)$ is a system,

$$\exists\, m_0 \ni K_{r_1 q m_0}.$$

Further we have:

$$m_0 \cdot \supset \cdot \exists\, n_{m_0} \ni n \geqq n_{m_0} \cdot \supset \cdot K_{r_{2m_n} q m_0}.$$

Then, since K^7,

$$K_{r_1 r_{2m_n} \phi^7 m_0}, \quad i.\ e., \quad K_{r_{1m_n} r_{2m_n} \phi^7 m_0}.$$

By taking $n \geqq n_{m_0}$ and $-m_n < \phi^7_{m_0}$, we obtain a contradiction. Similarly for (c). As for case (d), so far as it is not covered by the preceding cases, we can obtain two sequences $\{r_{1m_n}\}$ and $\{r_{2m_n}\}$, each consisting of distinct elements, and further, since $\mathfrak{R}^{\text{compact}}$, in such a way that:

$$\exists\, (q_1, q_2) \ni \underset{n}{L}\, r_{1m_n} = q_1 \text{ and } \underset{n}{L}\, r_{2m_n} = q_2.$$

Since we have a system $(\mathfrak{Q}; K)$, we have:

$$\exists\, m_0 \ni K_{q_1 q_2 m_0}.$$

Also since $\underset{n}{L}\, r_{2m_n} = q_2$:

$$m_0 \cdot \supset \cdot \exists\, n''_{m_0} \ni n \geqq n''_{m_0} \cdot \supset \cdot K_{r_{2m_n} q_2 m_0}.$$

Applying K^7 to these two K-relations there results:

$$K_{r_{2m_n} q_1 \phi^7 m_0}.$$

From $\underset{n}{L}\, r_{1m_n} = q_1$ it follows:

$$\phi^7_{m_0} \;\cdot\!\supset\!\cdot\; \exists\, n'_{m_0} \ni n \geqq n'_{m_0} \;\cdot\!\supset\!\cdot\; K_{r_{1m_n}\, q_1\, \phi^7{}_{m_0}}.$$

And hence

$$K_{r_{1m_n}\, r_{2m_n}\, \phi^7 \phi^7{}_{m_0}}.$$

Hence, taking an integer n exceeding n'_{m_0} and n''_{m_0}, and also such that $-m_n < \phi^7_{\phi^7{}_{m_0}}$, we obtain the desired contradiction.

12. *Propositions on Compact Classes in* $(\mathfrak{Q}\,;L)$. We have as an immediate consequence of the definition of compact classes the propositions:*

(1) *The sum of a finite number of compact classes is compact.*

(2) *Any subclass of a compact class is compact.*

(3) *If every subclass of a class is compact, then the class is compact.* The last two propositions might be joined into one:

(4) $\mathfrak{R}^{\text{compact}} :\backsim: \mathfrak{S}^{\mathfrak{R}} \cdot\!\supset\!\cdot\, \mathfrak{R}^{\text{compact}}.$

(5)† $L^3 \cdot \mathfrak{R}^{\text{compact}} \cdot \mathfrak{R}_n \ni \mathfrak{R}_{n-1}{}^{\mathfrak{R}_n} \cdot \mathfrak{R}_n{}^{\text{closed, non vacuous}\,(n)} \cdot\!\supset\!\cdot\, \exists\, r \ni r^{\mathfrak{R}_n (n)}.$

The proof is identically the one given by Fréchet, *loc. cit.*, p. 7.

13. *On Derived Classes in a System* $(\mathfrak{Q}\,;L)$. It is not possible to obtain, even though we suppose that the limit relation has all of the properties of § 3, *i. e.*, L^{123456}, the theorem that the derived set of a given set is closed, or, in class terminology, that the derived class of every subclass of $\mathfrak{Q}$ is closed. This may be shown by the following example:‡

We suppose the class $\mathfrak{Q}$ to consist of the following elements, all of which are supposed distinct: $\{q_{ln}\}$, $\{q_l\}$, q_0 $(l, n = 1, 2, \ldots, \infty)$. The complete table for the limit L in this class is specified as follows:

(a) A sequence $\{q_{l_0 n}\}$, where l_0 is fixed, taken in any order, or any subsequence of such a sequence, or any sequence obtained by prefixing a finite number of elements of the class, shall have as limit q_{l_0}. (b) A sequence $\{q_l\}$, taken in any order, or any subsequence of such a sequence, or any sequence obtained by prefixing a finite number of elements, shall have as limit q_0. (c) Any identity sequence, or any sequence, which after a certain

*Cf. Fréchet, *loc. cit.*, p. 7. Notice, however, that there is no condition on L.

† If L has the property (3), and $\mathfrak{R}$ is compact, if, further, we have a sequence of subclasses of $\mathfrak{R}$, each of which is closed, contained in the preceding and containing at least one element, then there is an element common to all of these classes.

‡ The example given is a modification of the example given by Hahn, *Monatshefte* Vol. XIX, p. 248, so as to include the properties 4, 5, of L.

term consists of one element repeated, shall have the repeated element as limit. No other sequences shall have limits. Such a limit is evidently an L^{123456}. Consider now $\mathfrak{R} = [q_{ln}]$. Then $\mathfrak{R}' = [q_l]$, and $\mathfrak{R}'' = q_0$, so that $\mathfrak{R}''$ is not closed. This is due to the fact that the above properties are not sufficient to secure q_0 as the limit of some sequence built up of elements taken from $[q_{ln}]$ only. The connection between the class $\mathfrak{R}'$ and the class $\mathfrak{R}$ as given by these properties of limit is not sufficiently close. We therefore need a weaker system if we desire to have this as a theorem. We obtain this in systems $(\mathfrak{Q}; K)$, the K being suitably conditioned.

14. *Relative to Derived Classes in Systems* $(\mathfrak{Q}; K)$. We have the theorem:*

$$K^{15} : \supset : \mathfrak{R} \cdot \supset \cdot \mathfrak{R}'^{\text{closed}}.$$

Suppose

$$\underset{n}{L}\, r'_n = r'' \text{ and } \underset{l}{L}\, r_{nl} = r'_n,$$

where the r'_n are all distinct, and the r_{nl} are distinct for every n. Then we have by the definition of limit:

$$m \cdot \supset \cdot \exists\, n_m \ni K_{r'_{n_m} r''m}$$

and †

$$\exists\, l_m \ni \{r_{n_m l_m}\}^{\text{distinct}} \cdot K_{r_{n_m l_m} r'_{n_m} m}.$$

Then by the fact that K has the property (5) we have

$$K_{r_{n_m l_m} r'' \phi^5 m},$$

and so by § 7 (4): $\underset{m}{L}\, r_{n_m l_m} = r''$, *i. e.*, $r''^{\mathfrak{R}'}$ and therefore $\mathfrak{R}''^{\mathfrak{R}'}$.

15. *On Compact Classes and Their Derivatives in Systems* $(\mathfrak{Q}; K)$. We have: ‡

(1) $$K^{16} : \supset : \mathfrak{R}^{\text{compact}} \cdot \supset \cdot \mathfrak{R}'^{\text{compact}}.$$

We need consider only the case in which $\mathfrak{R}'$ contains an infinitude of elements.

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* Cf. Fréchet, *loc. cit.*, p. 18.

† There exists a sequence of l, l_m, such that the elements of the sequence $r_{n_m l_m}$ are all distinct, and $K_{r_{n_m l_m} r'_{n_m} m}$. The distinctness of the elements $r_{n_m l_m}$ can be argued by a step-by-step process from the fact that the elements of the sequences r_{nl} are distinct for every n.

‡ Cf. Fréchet, Rend. di Pal., XXX, p. 4. Also for (6) and (7) below.

Suppose then that $[r'_n]$ is a denumerable infinitude of distinct elements of the class $\mathfrak{R}'$. Then on account of the definition of $\mathfrak{R}'$ we have:

$$n \cdot\supset\cdot \exists \{r_{nl}\}^{\text{distinct}} \ni \underset{l}{L}\, r_{nl} = r'_n .$$

Then by the definition of limit,

$$\exists\, l_n \ni \{r_{nl_n}\}^{\text{distinct}} \text{ and } K_{r_{nl_n} r'_n n} .$$

Since $\mathfrak{R}$ is compact, there will be a subsequence of $\{r_{nl_n}\}$, $\{r_{n_k}\}$, and an element r' such that

$$\underset{k}{L}\, r_{n_k} = r' .$$

Then

$$m \cdot\supset\cdot \exists\, k_m \ni k \geqq k_m \cdot\supset\cdot K_{r_{n_k} r' m} .$$

But $K_{r_{n_k} r'_{n_k} n_k}$ and so if K^1:

$$m \cdot\supset\cdot \exists\, k'_m \ni k \geqq k'_m \cdot\supset\cdot K_{r_{n_k} r'_{n_k} m} .$$

Applying the fact that K^6, there results:

$$k \geqq k_m \text{ and } k \geqq k'_m \cdot\supset\cdot K_{r'_{n_k} r' \phi^6_m} .$$

Hence by § 7 (4) we have:

$$\underset{k}{L}\, r'_{n_k} = r' .$$

This shows that there exists a limiting element for $[r'_n]$, *i. e.*, $\mathfrak{R}'$ is compact.

We have as immediate corollaries:

(2) $$K^{156} :\supset: \mathfrak{R}^{\text{compact}} \cdot\supset\cdot \mathfrak{R}'^{\text{extremal}} .$$

(3) $$K^{156} :\supset: \mathfrak{R}^{\text{compact}} \cdot\supset\cdot (\mathfrak{R} + \mathfrak{R}')^{\text{extremal}} .$$

A theorem relating to the class $\mathfrak{R}$ itself is the following:*

(4) $$K^{1367} :\supset: \mathfrak{R}^{\text{compact}} \cdot\supset\cdot [s \ni s^{\mathfrak{R}, -\mathfrak{R}'}]^{\text{denumerable}} .$$

Suppose $s^{\mathfrak{R}, -\mathfrak{R}'}$. Then †

$$m_s = \overline{B}(m \ni K_{r'sm} \,|\, r'^{\mathfrak{R}'})$$

exists and is finite. For if it were not finite, we should have either: (a) By K^1

* If K has the properties (1), (3), (6), and (7), and $\mathfrak{R}$ is compact, then the class of elements consisting of elements of $\mathfrak{R}$ which are not limiting elements of $\mathfrak{R}$ is denumerable. Cf. Fréchet, *loc. cit.*, p. 20.

† m_s is the least upper bound of the values of m for which $K_{r'sm}$ holds, r' ranging over the class $\mathfrak{R}'$.

there would exist an r' such that $K_{r'sm}(m)$, in which case $s = r'$, if K^3, which evidently is contradictory to the hypothesis relative to s; or (b)

$$\exists \{r'_m\}^{\text{distinct}} \ni K_{r'_m sm}.$$

But then by § 7 (4) we should have $\underset{m}{L}\, r'_m = s$. If then K^{15}, we have by § 15 $s^{\mathfrak{R}'}$, which again is contradictory to the hypothesis.

We have further:

$$m \cdot \supset \cdot \exists \text{ finite number of } s \text{ such that } m_s = m.$$

If not, let there be an infinitude for m_0. Since $\mathfrak{R}$ is compact, there exists a limiting element for a sequence of these values. Let $\{s_n\}$ be the sequence, and r'_0 the limiting element. Then

$$m \cdot \supset \cdot \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot K_{s_n r'_0 m}.$$

Now by § 5 (1), if K^{167}, we have: $K_{r'_0 s_n \phi^9_m}$. To obtain a contradiction, choose m so that $\phi^9_m > m_0$. Since, then, there exist only a finite number of s such that $m_s = m$, the class $[s]$ is evidently denumerable.

As an immediate corollary we have:

(5) $$K^{1367} : \supset : \mathfrak{R}^{\text{compact}} \cdot \mathfrak{R}'^{\text{denumerable}} \cdot \supset \cdot \mathfrak{R}^{\text{denumerable}}.$$

(6) $$K^{17} : \supset : \mathfrak{R}^{\text{compact}} \cdot \mathfrak{R}_m = [r \ni (r_1, r_2)^{\mathfrak{R}_m} \cdot \supset \cdot -K_{r_1 r_2 m}] \cdot \supset \cdot \mathfrak{R}_m^{\text{finite}}.$$

Suppose if possible $\mathfrak{R}_{m_1}$ not finite. Then

$$\exists \{r_n\}^{\mathfrak{R}_{m_1}} \ni \underset{n}{L}\, r_n = q.$$

Then

$$m : \supset : \exists\, n_m \ni {}^{n_1}_{n_2} \geqq n_m \cdot \supset \cdot K_{r_{n_1} qm} \cdot K_{r_{n_2} qm}.$$

Then

$$K^7 \cdot \supset \cdot K_{r_{n_1} r_{n_2} \phi_m}.$$

If we choose m so that $\phi_m > m_1$, we have a contradiction. Hence

$$m \cdot \supset \cdot \mathfrak{R}_m^{\text{finite}}.$$

(7) $$K^{17} : \supset : \mathfrak{R}^{\text{compact}} \cdot \supset \cdot \exists\, \mathfrak{S}^{\text{denumerable}} \ni \mathfrak{R}^{\mathfrak{S}+\mathfrak{S}'}.$$

The proof of this is as in Fréchet: *Rendiconti di Palermo,* XXX, § 5, pp. 3 and 4.

16. *Classes of Elements of Condensation.* Denote the class of elements of condensation of $\mathfrak{R}$, by $\mathfrak{R}^{\llcorner}$, *i. e.*, $\mathfrak{R}^{\llcorner} = [r^{\llcorner}]$. Then we have:

(1) $$K^{15} \cdot \mathfrak{R} \cdot \supset \cdot \mathfrak{R}^{\llcorner\text{closed}}.$$

This is an immediate consequence of § 15.*

(2) $$K^{1367} :\supset: \mathfrak{R}^{\text{condensed}} \cdot\supset\cdot \mathfrak{R}^{\llcorner\text{dense in itself}}.$$

Let $r^{\llcorner}$ be any element of $\mathfrak{R}^{\llcorner}$. Then let

$$\mathfrak{R}_m = [\text{all } r \ni K_{rr^{\llcorner}m} \cdot -K_{rr^{\llcorner}m+1}].$$

Every element of the class $\mathfrak{R}$ excepting $r^{\llcorner}$ will be in a uniquely determined class $\mathfrak{R}_m$. For otherwise we should have $K_{rr^{\llcorner}m}(m)$, and then by K^3 $r = r^{\llcorner}$. There will be an infinitude of non-denumerable classes in the set $\mathfrak{R}$. If not, then:

$$\exists\, m_1 \ni m \geqq m_1 \cdot\supset\cdot \mathfrak{R}_m^{\text{denumerable}}.$$

But, by removing the denumerable class of elements consisting of the elements in the classes $\mathfrak{R}_m$ for $m \geqq m_1$, we obtain a class of which $r^{\llcorner}$ is not a limiting element, which would be contrary to the hypothesis, that $r^{\llcorner}$ is an element of condensation of the class. Denote by $\mathfrak{R}_{m_n}$ the classes of $[\mathfrak{R}_m]$ which are non-denumerable. Then:

$$\mathfrak{R}^{\text{condensed}} \cdot\supset\cdot \exists\, r^{\llcorner}_{m_n} \text{ elements of condensation of } \mathfrak{R}_{m_n}.$$

There are two possibilities: (a) the sequence $\{r^{\llcorner}_{m_n}\}$ contains only a finite number of distinct elements, and (b) the sequence $\{r^{\llcorner}_{m_n}\}$ contains an infinitude of distinct elements. Suppose if possible (a), and let $r^{\llcorner}_0$ be an element infinitely repeated, *i. e.*,

$$r^{\llcorner}_0 = r^{\llcorner}_{m_{n_k}}\ (k).$$

We shall have:

$$\exists\, \{r_{m_{n_k}l}\}^{\mathfrak{R}_{m_{n_k}}} \ni \underset{l}{L} r_{m_{n_k}l} = r^{\llcorner}_{m_{n_k}} = r^{\llcorner}_0.$$

Then

$$m \cdot\supset\cdot \exists l_m \ni l \geqq l_m \cdot\supset\cdot K_{r_{m_{n_k}l}r^{\llcorner}_0 m},$$

which holds in particular for $m = m_{n_k}\ (k)$. Also by hypothesis:

$$K_{r_{m_{n_k}l}r^{\llcorner}m_{n_k}}.$$

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* Cf. Fréchet, *loc. cit.*, p. 19. It may be remarked that Fréchet really proves: If $(\mathfrak{Q}; K)$ is such that the derived class of every subclass of $\mathfrak{Q}$ is closed, then the class of elements of condensation of any subclass of $\mathfrak{Q}$ is also closed.

Hence by K^6 we have:

$$K_{r^{\backprime}_0 r^{\backprime\phi}_{m_{n_k}}}.$$

This will be holding for every k, and hence by § 7 (5) and K^3 we have

$$r^{\backprime}_0 = r^{\backprime}.$$

But by the above we then have:

$$l \geqq l_{(m_{n_k}+1)} \cdot \supset \cdot K_{r_{m_{n_k}l} r^{\backprime}(m_{n_k}+1)},$$

which is contradictory to the definition of $\mathfrak{R}_{m_{n_k}}$. Hence case (a) can not occur. We are thus led to the result that we have (b), *i. e.*, that there is an infinitude of distinct elements in the sequence $\{r^{\backprime}_{m_n}\}$. Let $r^{\backprime}_{m_{n_k}}$ be this infinitude. Then for every k

$$\underset{l}{L}\, r_{m_{n_k}l} = r^{\backprime}_{m_{n_k}},$$

where $r_{m_{n_k}l}$ is composed of distinct elements of $\mathfrak{R}_{m_{n_k}}$. Then

$$m \cdot \supset \cdot \exists\, l_m \ni l \geqq l_m \cdot \supset \cdot K_{r_{m_{n_k}l} r^{\backprime}_{m_{n_k}}}{}^{m}.$$

This holds in particular for $m = m_{n_k}$ (k). Moreover, for every k:

$$K_{r_{m_{n_k}l} r^{\backprime}_{m_{n_k}}}.$$

Hence by K^7:

$$K_{r^{\backprime}_{m_{n_k}} r^{\backprime\phi^7}_{m_{n_k}}},$$

and by § 7 (4):

$$\underset{k}{L}\, r^{\backprime}_{m_{n_k}} = r^{\backprime}.$$

Since $r^{\backprime}$ was any element of $\mathfrak{R}^{\backprime}$, this proves that $\mathfrak{R}^{\backprime}$ is dense in itself.

We have the following corollary:

(3) $$K^{1367} :\supset: \mathfrak{R}^{\text{condensed}} \cdot \supset \cdot \mathfrak{R}^{\backprime\,\text{perfect}}.$$

By a method similar to § 15 (4) it can be shown that:

(4) $$K^{1367} :\supset: \mathfrak{R}^{\text{condensed}} \cdot \supset \cdot [s \ni s^{\mathfrak{R}, -\mathfrak{R}^{\backprime}}]^{\text{denumerable}}.$$

17. *Heine Borel Theorems.*

17a. *Heine Borel Property.* Suppose a class of classes $\mathfrak{S}$, in notation $[\mathfrak{S}]$. We consider a unipartite property P of such classes $[\mathfrak{S}]$, the notation $[\mathfrak{S}]^P$, denoting that the class $[\mathfrak{S}]$ has the property.

Every property P determines a derived property, likewise of classes $[\mathfrak{S}]$, the *Heine Borel Property* with respect to the property P, in notation $H\text{-}B(P)$. A class $[\mathfrak{S}]$ has the property $H\text{-}B(P)$ in case there exists a finite subclass $[\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_n]$ of $[\mathfrak{S}]$, which has the property P, in symbols:

$$[\mathfrak{S}]^{H\text{-}B(P)} \equiv [\mathfrak{S}] \ni ([\mathfrak{S}]^P \cdot \exists\, [\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_n]^{[\mathfrak{S}]\cdot P}).$$

17b. This Heine Borel Property occurs in the theory of linear point sets in the case where the class of classes consists of a set of closed intervals, and where the property P is that every point of a given interval lies within one of the intervals of the set. We shall be concerned with the Heine Borel Property relative to classes $\mathfrak{S}$ of elements s, where the property P is that every element of a given class $\mathfrak{R}$ is interior to some class $\mathfrak{S}$ of the class of classes $[\mathfrak{S}]$ relative to $\mathfrak{R}$. We shall denote this property by $I(\mathfrak{R})$, in symbols:

$$\mathfrak{R} \cdot [\mathfrak{S}]^{I(\mathfrak{R})} \equiv r^{\mathfrak{R}} \cdot\supset\cdot \exists\, \mathfrak{S}^{[\mathfrak{S}]} \ni r^{\text{interior } \mathfrak{S}(\mathfrak{R})}.$$

We then have the following theorem:*

$$(1) \qquad K^{15} \cdot \mathfrak{R}^{\text{extremal}} :\supset: [\mathfrak{S}]^{I(\mathfrak{R})\cdot\text{denumerable}} = [\mathfrak{S}_n] \cdot\supset\cdot [\mathfrak{S}_n]^{H\text{-}B(I)}.$$

Suppose that the theorem does not hold. Then there will be an r_1, which will not be interior to $\mathfrak{S}_1$ relative to $\mathfrak{R}$. Let $\mathfrak{S}_{n_1}$ be the first class of $[\mathfrak{S}_n]$ to which r_1 is interior. Then there will exist an r_2 not interior to $\mathfrak{S}_1, \ldots, \mathfrak{S}_{n_1}$, but interior to $\mathfrak{S}_{n_2}$. Proceeding in this manner, we obtain in the general case an element r_l not interior to $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{(n_l-1)}$, but interior to $\mathfrak{S}_{n_l}$. By way of contradiction we shall show that there exists a subsequence $\{r_{l_k}\}$ of $\{r_l\}$, which ultimately consists of elements which are interior to a single class of the set $[\mathfrak{S}_n]$. Since $\mathfrak{R}$ is extremal, we have:

$$\exists\, \{r_{l_k}\} \text{ of } \{r_l\} \ni \underset{k}{L}\, r_{l_k} = r_0.$$

Now r_0 will be interior to some class $\mathfrak{S}$ of the class $[\mathfrak{S}_n]$, let us say $\mathfrak{S}_i$. We then have by § 11 (1):

$$K^1 \cdot\supset\cdot \exists\, m_0 \ni K_{rr_0m_0} \cdot\supset\cdot r^{\mathfrak{S}_i}.$$

Now since $\underset{k}{L}\, r_{l_k} = r_0$:

$$m \cdot\supset\cdot \exists\, k_m \ni k \geqq k_m \cdot\supset\cdot K_{r_{l_k} r_0 m}.$$

* Cf. Fréchet, *loc. cit.*, p. 22. E. R. Hedrick (*Trans. Amer. Math. Soc.*, XII, 285) has recently shown that it is sufficient to replace the hypothesis K^{15} by the hypothesis that the derived class of any subclass of $\mathfrak{Q}$ is closed. He supposes a $(\mathfrak{Q}; L)$, the L being the one of Fréchet, an L^{126}. It is easily shown that an L^8 is sufficient. To be sure, the hypothesis that the derived class of any subclass of $\mathfrak{Q}$ be closed probably acts in a restrictive way on the L. What this restriction is does not seem to have been as yet determined.

If in particular we take $m = m_0$, we have:

$$k \geqq k_{m_0} \cdot \supset \cdot r_{l_k}^{\mathfrak{S}_i}.$$

To obtain r_{l_k} interior to $\mathfrak{S}_i(\mathfrak{R})$, suppose

$$\{r_{nk}\} \ni \underset{n}{L}\, r_{nk} = r_{l_k}.$$

Then

$$m \cdot \supset \cdot \exists\, n_{mk} \ni n \geqq n_{mk} \cdot \supset \cdot K_{r_{nk} r_{l_k} m},$$

and so if K^5:

$$k \geqq k_m \text{ and } n \geqq n_{mk} \cdot \supset \cdot K_{r_{nk} r_0 \phi_m}.$$

If then we take m_1 so that $\phi_{m_1} \geqq m_0$, we shall actually have:

$$k \geqq k_{m_1} \text{ and } n \geqq n_{m_1 k} \cdot \supset \cdot r_{nk}^{\mathfrak{S}_i},$$

that is, if k exceeds k_{m_0} and k_{m_1}, r_{l_k} will be interior to $\mathfrak{S}_i$ relative to $\mathfrak{R}$, which is the desired contradiction.

17c. *Separability.** $\mathfrak{R}$ is *separable relative to* $\mathfrak{Q}$ if there exists a denumerable subclass of $\mathfrak{Q}$ which together with its derivative contains the class $\mathfrak{R}$, symbolically:

$$\mathfrak{R}^{\text{separable}(\mathfrak{Q})} \equiv \exists\, \mathfrak{D}^{\text{denumerable}} \ni \mathfrak{R}^{\mathfrak{D}+\mathfrak{D}'}.$$

We have a special case of this when the class $\mathfrak{R}$ is the class $\mathfrak{Q}$ itself. In so far as the derived class of a subclass of $\mathfrak{Q}$ can not contain any elements not belonging to $\mathfrak{Q}$, we must have, in this case,

$$\mathfrak{Q}^{\text{separable}(\mathfrak{Q})} \equiv \exists\, \mathfrak{D}^{\text{denumerable}} \ni \mathfrak{Q} = \mathfrak{D} + \mathfrak{D}'.$$

The property separable is thus a bipartite property relating to the classes $\mathfrak{R}$ and $\mathfrak{Q}$. From this bipartite property we derive the property of separability of $\mathfrak{Q}$, *i. e.*, we have:

$$\mathfrak{Q}^{\text{separable}} \equiv \mathfrak{Q}^{\text{separable}(\mathfrak{Q})}.$$

We have the following propositions:

(1) $\mathfrak{Q}^{\text{separable}} :\supset: \mathfrak{R}^{\mathfrak{Q}} \cdot\supset\cdot \mathfrak{R}^{\text{separable}(\mathfrak{Q})}$.

(2) $\mathfrak{R}^{\mathfrak{Q}} \cdot\supset\cdot \mathfrak{R}^{\text{separable}(\mathfrak{Q})} :\supset: \mathfrak{Q}^{\text{separable}}$.

(3) $K^{15} :\supset: \mathfrak{R}^{\text{separable}(\mathfrak{Q})} \cdot\supset\cdot \mathfrak{R}'^{\text{separable}(\mathfrak{Q})}$.

(4) $K^{15} :\supset: \mathfrak{R}^{\text{separable}(\mathfrak{Q})} \cdot\supset\cdot (\mathfrak{R} + \mathfrak{R}')^{\text{separable}(\mathfrak{Q})}$.

(5) $\mathfrak{R}^{\text{separable}(\mathfrak{R})} \cdot\supset\cdot \mathfrak{R}^{\text{separable}(\mathfrak{Q})}$.

(6) $K^{17} :\supset: \mathfrak{R}^{\text{compact}} \cdot\supset\cdot \mathfrak{R}^{\text{separable}(\mathfrak{R})} \cdot\supset\cdot \mathfrak{R}^{\text{separable}(\mathfrak{Q})}$. Cf. § 15 (7).

* Cf. Fréchet, *loc. cit.*, p. 23. The definition of separability given by Fréchet is:

$$\mathfrak{Q}^{\text{separable}} \equiv \exists\, \mathfrak{D}^{\text{denumerable}} \ni \mathfrak{Q} = \mathfrak{D}'.$$

It is easily seen that our definition is somewhat weaker.

17d. *Generalization of the Theorem of Cauchy.** The class $\mathfrak{R}$ has a *generalization of the Theorem of Cauchy relative to the class* $\mathfrak{Q}$ if, for every sequence r_n of $\mathfrak{R}$, which is such that for every n there exists an n_m such that, if n_1 and n_2 exceed n_m, we have $K_{r_{n_1} r_{n_2} m}$, then there exists an element q, which is the limit of the sequence r_n. In notation:

$$\mathfrak{R}^{GC(\mathfrak{Q})} \equiv \{r_n\} \ni m \cdot\supset\cdot \exists\, n_m \ni {}^{n_1}_{n_2} \geqq n_m :\supset: K_{r_{n_1} r_{n_2} m} \cdot:\supset:\cdot \exists\, q \ni L r_n = q,$$

where $\mathfrak{R}^{GC(\mathfrak{Q})}$ denotes the fact that $\mathfrak{R}$ has a generalization of the theorem of Cauchy relative to $\mathfrak{Q}$. It is possible to replace the class $\mathfrak{R}$ by the class $\mathfrak{Q}$, and thus obtain $\mathfrak{Q}^{GC(\mathfrak{Q})}$, or simply $\mathfrak{Q}^{GC}$. We have at once:

(1) $$\mathfrak{Q}^{GC} \cdot\supset\cdot \mathfrak{R}^{GC(\mathfrak{Q})}.$$

(2) $$\mathfrak{R}^{\text{compact}} \cdot\supset\cdot \mathfrak{R}^{GC(\mathfrak{Q})}.$$

17e. We define finally a property D_m† of a class of classes $[\mathfrak{T}]$, relative to $\mathfrak{R}$. *The classes* $[\mathfrak{T}]$ *have the property* D_m *relative to* $\mathfrak{R}$, if every element r of the class $\mathfrak{R}$ belongs to at least one class of the set $[\mathfrak{T}]$, and if r_1 and r_2 belong to the same $\mathfrak{T}$, then $K_{r_1 r_2 m}$. Symbolically:

$$[\mathfrak{T}]^{D_m(\mathfrak{R})} \equiv \text{(a) } r \cdot\supset\cdot \exists\, \mathfrak{T}_r^{[\mathfrak{T}]} \ni r^{\mathfrak{T}_r}. \quad \text{(b) } (r_1, r_2)^{\mathfrak{T}} \cdot\supset\cdot K_{r_1 r_2 m}.$$

We have the following lemmas relative to the property D_m:

(1) $$K^{167} :\supset: \mathfrak{R} \cdot m \cdot\supset\cdot \exists\, [\mathfrak{T}_m]^{D_m(\mathfrak{R})}.$$

Suppose m_0 such that $m_0 \geqq m$, $\phi^7_{m_0} \geqq m$, and $\phi^9_{m_0} \geqq m$. Let

$$\mathfrak{T}_{m r_0} = [r \ni K_{r r_0 m_0} \cdot r_0].$$

Evidently every element r will belong to at least one $\mathfrak{T}_{mr}$. Further, if of a pair of elements r_1 and r_2 belonging to the same $\mathfrak{T}_{m r_0}$, one of them is r_0, we shall evidently have $K_{r_1 r_0 m}$ and $K_{r_0 r_2 m}$, since $m_0 \geqq m$, and $\phi^9_{m_0} \geqq m$. In the general case we obtain from K^7 and from $K_{r_1 r_0 m_0}$ and $K_{r_2 r_0 m_0}$: $K_{r_1 r_2 m}$. Hence we have $[\mathfrak{T}_{mr}]^{D_m(\mathfrak{R})}$.

We are, however, interested more especially in the case in which for every m it is possible to find a class $[\mathfrak{T}]^{D_m(\mathfrak{R})}$ which is denumerable or even finite. We have:

(2) $$K^{167} \cdot \mathfrak{R}^{\text{separable}(\mathfrak{Q})} :\supset: m \cdot\supset\cdot \exists\, [\mathfrak{T}_{mn}]^{D_m(\mathfrak{R}) \cdot I(\mathfrak{R})}.$$

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* Cf. Fréchet, *loc. cit.*, p. 23.

† *I. e.*, Development, or division. Cf. Fréchet, p. 25, f.

On account of the separability of $\mathfrak{K}$ relative to $\mathfrak{Q}$, let $\mathfrak{T} = [q_n] \ni \mathfrak{K}^{\mathfrak{T}+\mathfrak{T}'}$. Suppose further

$$m_0 \ni m_0 \geqq m, \text{ and } \phi^7_{m_0} \geqq m.$$

Then set

$$\mathfrak{T}_{mn} = [r \ni K_{q_n r m_0} \cdot q_n \text{ if } q_n^{\mathfrak{K}}].$$

Evidently

$$r \cdot \supset \cdot \exists\, n_r \ni r^{\mathfrak{T}_{n_r}}.$$

Further, suppose $(r_1, r_2)^{\mathfrak{T}_{mn}}$. Then using K^7 we obtain, from $K_{q_n r_1 m_0}$ and $K_{q_n r_2 m_0}$, $K_{r_1 r_2 \phi^7_{m_0}}$ and so $K_{r_1 r_2 m}$. Hence $[\mathfrak{T}_{mn}]^{D_m(\mathfrak{K})}$.

As for the result $[\mathfrak{T}_{mn}]^{I(\mathfrak{K})}$, there evidently exists

$$m_0' \ni \phi^7_{m'_0} \geqq m_0 \text{ and } m_0' \geqq m_0.$$

If now $\underset{l}{L}\, r_l = r$, we have:

$$m_0' :\supset: \exists\, l_{m'_0} \ni l \geqq l_{m'_0} \cdot\supset\cdot K_{r_l r m'_0}.$$

Also, on account of the separability of $\mathfrak{K}$, we have either:

$$r \cdot\supset\cdot \exists\, q_{n_r} \ni K_{q_{n_r} r m'_0};$$

and hence, if we apply K^7, there result $K_{q_{n_r} r_l m_0}$ and $K_{q_{n_r} r m_0}$, *i.e.*, by the definition of $\mathfrak{T}_{mn}$ and the definition of interiority,

$$r^{\text{interior } \mathfrak{T}_{mn_r}(\mathfrak{K})};$$

or $r = q_n$, in which case the interiority is immediate. We therefore have:

$$[\mathfrak{T}_{mn}]^{I(\mathfrak{K})}.$$

As for $[\mathfrak{T}]^{D_m(\mathfrak{K})}$ and finite, we have:

$$(3) \qquad K^{167} \cdot \mathfrak{K}^{\text{compact}} :\supset: \exists\, [\mathfrak{T}_{m1}, \mathfrak{T}_{m2}, \dots, \mathfrak{T}_{mn}]^{D_m(\mathfrak{K}) \cdot I(\mathfrak{K})}.$$

Since the class $\mathfrak{K}$ is compact, we have by § 15 (3) the class $(\mathfrak{K} + \mathfrak{K}')$ extremal. Since further, by § 17c (6), from $(\mathfrak{K} + \mathfrak{K}')^{\text{compact}}$ we obtain $(\mathfrak{K} + \mathfrak{K}')^{\text{separable}(\mathfrak{Q})}$, we can determine by (2) above:

$$[\mathfrak{T}_{mn'}]^{D_m(\mathfrak{K}+\mathfrak{K}') \cdot I(\mathfrak{K}+\mathfrak{K}')}, \quad n' = 1, 2, 3, \dots.$$

But by § 17b, $(\mathfrak{K} + \mathfrak{K}')^{\text{extremal}} \cdot m :\supset: \exists\, [\mathfrak{T}_{mn'}]^{I(\mathfrak{K}+\mathfrak{K}')} \cdot\supset\cdot [\mathfrak{T}_{mn'}]^{H\text{-}B(I(\mathfrak{K}+\mathfrak{K}'))}$; *i. e.*, there exists a finite set of classes $[\mathfrak{T}_{mn'}]^{I(\mathfrak{K}+\mathfrak{K}')}$. Evidently this set is such that $[\mathfrak{T}_{mn}]^{D_m(\mathfrak{K}) \cdot I(\mathfrak{K})}$.

The following converse of this lemma holds:*

$$(4) \qquad K^1 \cdot \mathfrak{R}^{GC(\mathfrak{Q})} :\supset: m \cdot\supset\cdot \exists\, [\mathfrak{T}_{m1}, \ldots, \mathfrak{T}_{mn}]^{D_m(\mathfrak{R})} \cdot:\supset:\cdot \mathfrak{R}^{\text{compact}}.$$

17f. *Heine Borel Theorem in the Non-denumerable Case.* We are now in position to prove the Heine Borel Theorem in the case in which the given $[\mathfrak{S}]^{I(\mathfrak{R})}$ is non-denumerable. We have the theorem:†

$$(1) \qquad \mathfrak{R} \cdot K^{167} :\supset: \mathfrak{R}^{\text{extremal}} \cdot [\mathfrak{S}]^{I(\mathfrak{R})} \cdot\supset\cdot [\mathfrak{S}]^{H\text{-}B(I)}.$$

By § 17e (3) we have:

$$\mathfrak{R}^{\text{extremal}} \cdot m \cdot\supset\cdot \exists\, [\mathfrak{T}_m]^{\text{finite} \cdot D_m(\mathfrak{R})}.$$

If the theorem is not true, then for every m it is not true for one of the finite set of classes $[\mathfrak{T}_m]$, *i. e.*:

$$m \cdot\supset\cdot \exists\, \mathfrak{T}_{0m} \ni [\mathfrak{S}]^{-H\,B(I(\mathfrak{T}_{0m}))}.$$

We argue a contradiction by showing that there exists an $\mathfrak{S}_0$ of the class $[\mathfrak{S}]$, such that every element of each class of a subclass of $[\mathfrak{T}_{0m} \mid m] : [\mathfrak{T}_{0m_n} \mid n]$, is interior to $\mathfrak{S}_0(\mathfrak{R})$. If r_m be any element of $\mathfrak{T}_{0m}$, we obtain a sequence $\{r_m\}$. There are two possibilities: (a) there is one element repeated infinitely often in the sequence, and (b) no element is repeated infinitely often.

(a) If possible, suppose one element is repeated infinitely often; *i. e.*, let

$$r_0 = r_{m_n}(n).$$

Then

$$\exists\, \mathfrak{S}_0 \ni r_0^{\text{interior } \mathfrak{S}_0(\mathfrak{R})}.$$

Then by § 11 (1)

$$K^1 \cdot\supset\cdot \exists\, m_0 \ni (r \ni K_{r r_0 m_0}) \cdot\supset\cdot r^{\mathfrak{S}_0}.$$

Let $r^{(m_n)}$ be any element of $\mathfrak{T}_{m_n}$. Then:

$$K_{r^{(m_n)} r_{m_n} m_n}; \quad i.\, e., \quad K_{r^{(m_n)} r_0 m_n}.$$

Suppose, further, that $\underset{l}{L}\, r_l = r^{(m_n)}$. Then

$$m \cdot\supset\cdot \exists\, l_m \ni l \geqq l_m \cdot\supset\cdot K_{r_l r^{(m_n)} m}.$$

This will hold in particular for $m = m_n$, and so we have by K^5 for every n:

$$l \geqq l_{m_n} \cdot\supset\cdot K_{r_l r_0 \phi^5 m_n}.$$

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* For proof see Fréchet, *loc. cit.*, p. 25. Note the weaker hypothesis here.

† Cf. Fréchet, *loc. cit.*, p. 26.

If now we choose n_0 so that

$$n \geqq n_0 \cdot \supset \cdot \; m_n \geqq m_0 \cdot \phi^5_{m_n} \geqq m_0,$$

we shall have

$$r^{(m_n)\text{ interior to } \mathfrak{S}_0(\mathfrak{R})}.$$

Hence we have the desired contradiction in this case.

(b) No element of the sequence $\{r_m\}$ is repeated infinitely often. Then, on account of the extremality of the class $\mathfrak{R}$,

$$\exists \; \{r_{m_n}\}, \; r_0 \ni \underset{n}{L}\, r_{m_n} = r_0.$$

The argument then proceeds very much as in case (a).

We have the following converse of this important theorem:

(2) $$K^{136} \cdot\!:\supset:\cdot \; \mathfrak{R} \cdot [\mathfrak{S}]^{I(\mathfrak{R})} \cdot \supset \cdot [\mathfrak{S}]^{H\text{-}B(I)} :\supset: \mathfrak{R}^{\text{extremal}}.$$

(a) $\mathfrak{R}$ *is closed.* Suppose it were not. Then

$$\exists \; \{r_n\} \ni \underset{n}{L}\, r_n = q \cdot q^{-\mathfrak{R}}.$$

Let

$$\mathfrak{S}_m = [r \ni \overline{} K_{rqm}].$$

Then

$$r \cdot \supset \cdot \exists \; m \ni r^{\text{interior } \mathfrak{S}_m(\mathfrak{R})}.$$

For if not:

$$\exists \; \{r_l\} \ni (\underset{l}{L}\, r_l = r \cdot m \cdot \supset \cdot \exists \; l_m \ni l \geqq l_m \cdot \supset \cdot K_{r_l q m}).$$

But then by definition of limit:

$$\underset{l}{L}\, r_l = q.$$

If now K^{136}, then by § 7 (3), limit is unique, and since $q^{-\mathfrak{R}}$ we have a contradiction. Evidently a finite number of the classes $\mathfrak{S}$ will contain only a finite number of elements of the sequence r_n, and so

$$[\mathfrak{S}_m]^{\text{finite}} \cdot \supset \cdot [\mathfrak{S}_m]^{-I(\mathfrak{R})}.$$

(b) $\mathfrak{R}$ *is compact.* If not, then there exists a sequence of distinct elements $\{r_n\}$ without a limiting element. Then

$$r \neq r_n \cdot \supset \cdot \exists \; \mu_r > \overline{B}(m \ni K_{r_n r m} \mid n)$$

$$r \neq r_{n_0} \cdot \supset \cdot \exists \; \mu_{r_{n_0}} > \overline{B}(m \ni K_{r_n r_{n_0} m} \mid n \neq n_0).$$

If K^{13}, this μ will be finite for every r. Let

$$\mathfrak{S}_{r_0} = [r \ni K_{rr_0\mu_{r_0}} \cdot r_0].$$

Then evidently

$$r_0^{\text{interior } \mathfrak{S}_{r_0}(\mathfrak{R})}.$$

But no $\mathfrak{S}_{r_0}$ contains two elements of the sequence r_n, and so

$$[\mathfrak{S}_{r_0}]^{-H\text{-}B(I)}.$$

The proofs of these theorems are sufficient to indicate the method of attack in case we are operating with a K having the properties (1), (3), (6), (7). In the theorems of Fréchet not taken up here, there is little difficulty in showing that such a K is sufficient. Instead of following up the matter of the properties of subclasses $\mathfrak{R}$ of $\mathfrak{Q}$ of systems $(\mathfrak{Q}\,; K)$ further, we turn our attention briefly to continuous functions on subclasses $\mathfrak{R}$ of systems $(\mathfrak{Q}\,; L)$ and $(\mathfrak{Q}\,; K)$.

Continuous Functions on Subclasses $\mathfrak{R}$ of Systems $(\mathfrak{Q}\,; L)$. §§ 18–20.

18. *Functions of Subclasses $\mathfrak{R}$ of $\mathfrak{Q}$ to the Class of real Numbers $\mathfrak{A}$. Sequential Continuity.* By a function on a subclass $\mathfrak{R}$ of $\mathfrak{Q}$ to the class of real numbers $\mathfrak{A}$, we mean a correspondence between elements of the class $\mathfrak{R}$ and real numbers $\mathfrak{A}$ such that to every element of the class $\mathfrak{R}$ there corresponds at least one real number. If this function or correspondence be denoted by μ, our definition might be stated symbolically:

$$r^{\mathfrak{R}} \cdot \supset \cdot \exists\, a^{\mathfrak{A}} \ni \mu_r = a.$$

We shall suppose, in particular, that we are dealing only with single-valued functions.

If μ is a function on the class $\mathfrak{R}$ to $\mathfrak{A}$, we say that μ *is continuous at the element* r in case

$$\underset{n}{L}\, r_n = r \cdot \supset \cdot \underset{n}{L}\, \mu_{r_n} = \mu_r,$$

where the first limit is an L,* the second a real-number limit.

μ is said to be *continuous on $\mathfrak{R}$ to $\mathfrak{A}$*, if it is continuous for every element r of $\mathfrak{R}$.

* If L is not unique, *i. e.*, $\underset{n}{L}\, r_n = r$ and $\underset{n}{L}\, r_n = r'$, and μ is continuous, then we must of course have $\mu_r = \mu_{r'}$.

19. *Bounded Properties of Continuous Functions.** We have the following theorem: †

(1) $$\mu^{\text{continuous on } \mathfrak{R}^{\text{extremal}} \text{ to } \mathfrak{A}} \cdot \supset \cdot \exists\, (b,\, B) \ni b \leqq \mu_r \leqq B\ (r^{\mathfrak{R}}).$$

We show that there exists an upper bound B, the lower bound being shown to exist in a similar manner. Suppose, if possible, that no upper bound exists. Then

$$n \cdot \supset \cdot \exists\, r_n \ni \mu_{r_n} > n.$$

Since μ is defined for every r, the sequence $\{r_n\}$ cannot contain an element infinitely repeated. Then since $\mathfrak{R}$ is extremal we have:

$$\exists\, \{r_{n_l}\}^{\text{distinct}} \cdot r \ni \underset{l}{L}\, r_{n_l} = r.$$

Now since μ is continuous on $\mathfrak{R}$, it will be continuous at r, and so we have:

$$\underset{l}{L}\, \mu_{r_{n_l}} = \mu_r.$$

But by the hypothesis on r_{n_l} we would have:

$$\underset{l}{L}\, \mu_{r_{n_l}} = \infty.$$

We have thus reached a contradiction.

If we denote by b and B the greatest lower and least upper bounds, respectively, of μ on $\mathfrak{R}$, we have: ‡

(2) $$\mu^{\text{continuous on } \mathfrak{R}^{\text{extremal}} \text{ to } \mathfrak{A}} \cdot \supset \cdot \exists\, (r_1, \text{ and } r_2) \ni \mu_{r_1} = b,\ \mu_{r_2} = B.$$

We show that

$$\exists\, r_2 \ni \mu_{r_2} = B.$$

Suppose it were not so. Then

$$n \cdot \supset \cdot \exists\, r_n \ni B > \mu_{r_n} > B - 1/n.$$

The sequence $\{r_n\}$ will again involve an infinitude of distinct elements, and thus using the condition that the class $\mathfrak{R}$ is extremal we obtain:

$$\exists\, \{r_{n_l}\}^{\text{distinct}} \cdot r \ni \underset{l}{L}\, r_{n_l} = r.$$

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* We consider here a number of the more important theorems of Fréchet (*loc. cit.*, pp. 8–15), to show the method of reasoning when we suppose L unconditioned. The theorems not taken up are also holding in the general ($\mathfrak{Q}$; L) situation; as a matter of fact, the proofs of Fréchet, when used with some care, will give the desired results.

† If μ is continuous on an extremal subclass $\mathfrak{R}$, then it is bounded on $\mathfrak{R}$. Cf. Fréchet, *loc. cit.*, p. 8.

‡ Cf. Fréchet, *loc. cit.*, p. 8.

Then on account of the continuity of μ:

$$\underset{l}{L}\,\mu_{r_{n_l}} = \mu_r.$$

But by the construction of $\{r_n\}$:

$$\underset{l}{L}\,\mu_{r_{n_l}} = B.$$

Hence $\mu_r = B$. Similarly we show that:

$$\exists\ r_1 \ni \mu_{r_1} = b.$$

20. *Sequences of Continuous Functions.* We define uniformity and quasi-uniformity of convergence of sequences first. Suppose a sequence of functions $\{\mu_n\}$ each on $\mathfrak{R}$ to $\mathfrak{A}$, converging to a function μ on $\mathfrak{R}$ to $\mathfrak{A}$, *i. e.*,

$$\underset{n}{L}\,\mu_{nr} = \mu_r,$$

or

$$r \cdot e \cdot\supset\cdot \exists\ n_{er} \ni n \geqq n_{er} \cdot\supset\cdot |\mu_{nr} - \mu_r| \leqq e.$$

We say that the convergence is *uniform* on $\mathfrak{R}$ if

$$e \cdot\supset\cdot \exists\ n_e \ni n \geqq n_e \cdot\supset\cdot |\mu_{nr} - \mu_r| \leqq e,$$

for every $r^{\mathfrak{R}}$. It evidently differs from the simple convergence in that the n_e does not depend upon the elements r. The convergence is said to be *quasi-uniform*, if

$$e \cdot l :\supset: \exists\ l_{el} \ni r \cdot\supset\cdot \exists\ n_{erl} \ni (l \leqq n_{erl} \leqq l_{el} \cdot |\mu_{n_{erl}r} - \mu_r| \leqq e).$$

Quasi-uniformity of convergence does not insure the convergence of a sequence. The convergence must be assumed separately if desired.

The subject of uniformity and quasi-uniformity is of importance in the convergence of a sequence of continuous functions to a continuous function. We have the following well-known theorem:

(1) *A uniformly convergent sequence of continuous functions converges to a continuous function.*

The hypothesis of the following theorem is, however, less exacting, and hence it covers the former also:*

(2) *A quasi-uniformly convergent sequence of continuous functions converges to a continuous function.*

* Cf. Fréchet, *loc. cit.*, p. 10.

Suppose a sequence $\{\mu_{nr}\}$ of continuous functions on $\mathfrak{R}$ to $\mathfrak{A}$, which converges to μ_r, *i. e.*,

$$\underset{n}{L}\,\mu_{nr} = \mu_r,$$

the convergence being supposed quasi-uniform. We wish to show that μ_r is continuous, *i. e.*,

$$\underset{i}{L}\, r_i = r_0 \;\cdot\!\supset\!\cdot\; \underset{i}{L}\,\mu_{r_i} = \mu_{r_0}.$$

Since the sequence converges at r_0, we have:

$$e \;\cdot\!\supset\!\cdot\; \exists\, n_{er_0} \ni n \geqq n_{er_0} \;\cdot\!\supset\!\cdot\; |\mu_{nr_0} - \mu_{r_0}| \leqq e.$$

On account of the quasi-uniformity of convergence of our sequence we have:

$$e \cdot n_{er_0} \;\cdot\!\supset\!\cdot\; \exists\, l_{en_{er_0}} \ni r_i \;\cdot\!\supset\!\cdot\; \exists\, n_{er_i n_{er}} = n_{ei} \ni n_{er_0} \leqq n_{ei} \leqq l_{en_{er_0}} \text{ and}$$
$$|\mu_{n_{ei} r_i} - \mu_{r_i}| \leqq e \qquad (1)$$

The continuity of the functions $\mu_{n_{ei}}$ gives:

$$e \;\cdot\!\supset\!\cdot\; \exists\, i_e \ni i \geqq i_e \;\cdot\!\supset\!\cdot\; |\mu_{n_{ei} r_i} - \mu_{n_{ei} r_0}| \leqq e, \qquad (2)$$

where i_e is the maximum of the i_e corresponding to the values of n between n_{er_0} and $l_{en_{er_0}}$. Finally, since n_{ei} exceeds n_{er_0}, we have:

$$|\mu_{n_{ei} r_0} - \mu_{r_0}| \leqq e. \qquad (3)$$

Adding the inequalities of (1), (2) and (3), we have:

$$|\mu_{r_i} - \mu_{r_0}| \leqq 3e,$$

subject to the condition $i \geqq i_e$, the n being only a subsidiary.

The following converse of this theorem holds also in this general situation:

(3) *If the class $\mathfrak{R}$ is extremal, and a sequence of continuous functions converges to a continuous function, then the convergence is quasi-uniform.*

If the sequence of functions in question be $\{\mu_{nr}\}$ and the limit function μ_r, it is necessary to show:

$$e \cdot l \;\cdot\!\supset\!\cdot\; \exists\, l_{el} \ni r \;\cdot\!\supset\!\cdot\; \exists\, n_{erl} \ni (l \leqq n_{erl} \leqq l_{el} \cdot |\mu_{n_{erl} r} - \mu_r| \leqq e).$$

Choose e and l arbitrarily. Let n_{erl} be the minimum of the numbers exceeding l, such that:

$$|\mu_{n_{erl} r} - \mu_r| \leqq e.$$

We wish to show that n_{erl} has a finite upper bound. If this is not the case we have:

$$i \geqq l \;\cdot\!\supset\!\cdot\; \exists\, r_i \ni n_{er_i l} \geqq i.$$

Since the sequence $\{\mu_{nr}\}$ is convergent for every r, no r_i can occur infinitely often, and we obtain from the extremality of $\mathfrak{R}$:

$$\exists \; \{r_{i_k}\}^{\text{distinct}} \cdot r \ni \underset{k}{L}\, r_{i_k} = r.$$

Since now μ_r is continuous,

$$e \cdot \supset \cdot \; \exists \; k_e \ni k \geqq k_e \cdot \supset \cdot \; |\mu_{r_{i_k}} - \mu_r| \leqq e.$$

On account of the continuity of μ_{nr},

$$e \cdot \supset \cdot \; \exists \; k_{en} \ni k \geqq k_{en} \cdot \supset \cdot \; |\mu_{nr_{i_k}} - \mu_{nr}| \leqq e.$$

Since further $\underset{n}{L}\, \mu_{nr} = \mu_r$,

$$e \cdot \supset \cdot \; \exists \; n_{er} \ni n \geqq n_{er} \cdot \supset \cdot \; |\mu_{nr} - \mu_r| \leqq e.$$

Hence,

$$3e \cdot \supset \cdot \; \exists \; n_{er} \cdot \exists \; k_{en} \ni n \geqq n_{er} \cdot k \geqq k_{en} \cdot \supset \cdot \; |\mu_{nr_{i_k}} - \mu_{r_{i_k}}| \leqq 3e.$$

Let n_0 be the greater of n_{er} and l. Then this last inequality will hold for $k \geqq k_{en_0}$. On account of the convergency of the sequence we have:

$$3e \cdot \supset \cdot \; \exists \; n_{ek} \ni n \geqq n_{ek} \cdot \supset \cdot \; |\mu_{nr_{i_k}} - \mu_{r_{i_k}}| \leqq 3e.$$

There being only a finite number of k less than k_{en_0}, we shall have a finite number of corresponding n_{ek}. Of these and of n_0 we choose the largest. This will serve as an upper bound for the $n_{er_{i_k}l}$. We have thus reached a contradiction in so far as we have demonstrated the existence of a finite bound for $n_{er_{i_k}l}$. Hence the theorem.

Continuous Functions on Subclasses $\mathfrak{R}$ of Systems $(\mathfrak{Q}; K)$.

21. *Difference Continuity.* In case we are operating in a system $(\mathfrak{Q}; K)$, it is possible to define a type of continuity which is analogous to the difference continuity of a function on a real interval. The K-relation serves to replace the absolute value of the difference. In order to distinguish this type from the continuity employed above, we call the former difference continuity, and the latter sequential continuity. We define:

$$\mu^{\text{difference continuous at } r_0} \equiv e \cdot \supset \cdot \; \exists \; m_{er_0} \ni K_{rr_0m_{er_0}} \cdot \supset \cdot \; |\mu_r - \mu_{r_0}| \leqq e.$$

We say that μ *is difference continuous on the class* $\mathfrak{R}$ if it is continuous at every element of $\mathfrak{R}$.

In order to make the theorems on sequentially continuous functions available here, we must establish some connection. This is contained in the theorem *

$$K^1 \cdot \mathfrak{R} \cdot \mu^{\text{on } \mathfrak{R} \text{ to } \mathfrak{A}} : \supset : \mu^{\text{sequentially continuous on } \mathfrak{R}} \cdot \backsim \cdot \mu^{\text{difference continuous on } \mathfrak{R}}.$$

(a) $\mu^{\text{sequentially continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{difference continuous on } \mathfrak{R}}$. We wish then to show that:

$$r_0^{\mathfrak{R}} \cdot e \cdot \supset \cdot \exists\, m_{er_0} \ni K_{rr_0m_{er_0}} \cdot \supset \cdot |\mu_r - \mu_{r_0}| \leqq e.$$

Suppose this were not so. Then:

$$\exists\, r_0 \cdot \exists\, e_0 \ni m \cdot \supset \cdot \exists\, r_m \ni K_{r_m r_0 m} \cdot |\mu_{r_m} - \mu_{r_0}| > e_0.$$

From § 7 (5) and K^1, it follows that:

$$\underset{m}{L}\, r_m = r_0.$$

Since now μ is sequentially continuous, we have:

$$\underset{m}{L}\, \mu_{r_m} = \mu_{r_0}, \quad i.\ e., \quad e \cdot \supset \cdot \exists\, m_e \ni m \geqq m_e \cdot \supset \cdot |\mu_{r_m} - \mu_{r_0}| \leqq e.$$

Since this will be holding for e_0 also, we have reached a contradiction.

(b) $\mu^{\text{difference continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{sequentially continuous on } \mathfrak{R}}$. Since μ is difference continuous on $\mathfrak{R}$, we have:

$$r_0 \cdot e \cdot \supset \cdot \exists\, m_e \ni K_{rr_0m_e} \cdot \supset \cdot |\mu_r - \mu_{r_0}| \leqq e.$$

Suppose $\{r_n\}$ is any sequence having r_0 as a limit, *i. e.*:

$$m \cdot \supset \cdot \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot K_{r_n r_0 m}.$$

This will hold in particular when $m = m_e$, and so we have:

$$e \cdot \supset \cdot \exists\, n_e = n_{m_e} \ni n \geqq n_e \cdot \supset \cdot |\mu_{r_n} - \mu_{r_0}| \leqq e,$$

which is the continuity as desired.

22. *Uniform Continuity.* We note that relative to sequential continuity it does not seem possible to define a uniform continuity. However, in the case of difference continuity such a possibility exists. We define:

$$\mu^{\text{uniformly continuous on } \mathfrak{R}} \equiv e \cdot \supset \cdot \exists\, m_e \ni K_{r_1 r_2 m_e} \cdot \supset \cdot |\mu_{r_1} - \mu_{r_2}| \leqq e,$$

the uniformity feature entering in that the m_e is independent of the r. If μ is uniformly continuous it is also continuous. On the other hand we have: *

$$K^{1367} \cdot \mathfrak{R}^{\text{extremal}} : \supset : \mu^{\text{continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{uniformly continuous on } \mathfrak{R}}.$$

* Cf. Fréchet, *loc. cit.*, p. 28.

If possible suppose that μ is not uniformly continuous. Then:

$$\exists\ e_0 \ni m \cdot\supset\cdot \exists\ (r_{1m} \cdot r_{2m}) \ni K_{r_{1m}r_{2m}m} \cdot |\mu_{r_{1m}} - \mu_{r_{2m}}| > e_0.$$

We thus obtain two sequences. There are four possibilities: (a) The sequences $\{r_{1m}\}$ and $\{r_{2m}\}$ each contain only a finite number of distinct elements; (b) The sequence $\{r_{1m}\}$ contains only a finite number of distinct elements, while the sequence $\{r_{2m}\}$ contains an infinitude of distinct elements; (c) The sequence $\{r_{1m}\}$ contains an infinitude of distinct elements, while the sequence $\{r_{2m}\}$ contains only a finite number of distinct elements; and (d) Each sequence $\{r_{1m}\}$ and $\{r_{2m}\}$ contain an infinitude of distinct elements.

In case (a) there will be one pair of elements which will have like subscripts infinitely often. These are shown to be equal by K^{18} and § 7 (5). We thus obtain a contradiction at once.

In case (b), so far as it is not covered by (a), it will be possible to select from the sequence $\{r_{2m}\}$ the sequence $\{r_{2m_n}\}$ of distinct elements, and from the sequence $\{r_{1m}\}$ the set $\{r_{1m_n}\}$ consisting of one element repeated, *i. e.*:

$$r_{1m_n} = r_1\,(n).$$

Then by our hypothesis:

$$K_{r_{1m_n}r_{2m_n}m_n}, \quad i.\ e., \quad K_{r_1 r_{2m_n}m_n}.$$

From K^{167} by § 5 (1) it follows that:

$$K_{r_{2m_n}r_1\phi^9_{m_n}}.$$

Since this holds for every n and $\underset{m}{L}\,\phi^9_m = \infty$, we have by § 7 (4):

$$\underset{n}{L}\, r_{2m_n} = r_1.$$

But μ is continuous. Hence:

$$e_0 \cdot\supset\cdot \exists\ n_{e_0} \ni n \geqq n_{e_0} \cdot\supset\cdot |\mu_{r_1} - \mu_{r_{2m_n}}| \leqq e_0 \text{ or } |\mu_{r_{1m_n}} - \mu_{r_{2m_n}}| \leqq e_0,$$

which is contrary to the definition of r_{1m} and r_{2m}. Hence we do not have (b).

We show similarly that case (c) can not occur.

In case (d), in so far as it is not covered by the preceding, it will be possible to select the sequences $\{r_{1m_n}\}$ and $\{r_{2m_n}\}$ each of distinct elements, and on account of the extremality of $\mathfrak{R}$ in such a way that

$$\underset{n}{L}\, r_{1m_n} = r_1.$$

Then $$m \cdot\supset\cdot \exists\ n'_m \ni n \geqq n'_m \cdot\supset\cdot K_{r_{1m_n} r_1 m}.$$

Also by K^1 and the hypothesis on r_{1m} and r_{2m}:

$$m \cdot \supset \cdot \exists\, n''_m \ni n \geqq n''_m \cdot \supset \cdot K_{r_{1m_n} r_{2m_n} m}.$$

Then using K^6 we have:

$$n \geqq n'_m \cdot n \geqq n''_m \cdot \supset \cdot K_{r_{2m_n} r_1 \phi^6 m}.$$

Hence, by K^1 and § 7 (4):

$$\underset{n}{L}\, r_{2m_n} = r_1.$$

But on account of the continuity of μ we shall have:

$$e/2 \cdot \supset \cdot \exists\, n_e \ni n \geqq n_e \cdot \supset \cdot |\mu_{r_{1m_n}} - \mu_{r_1}| \leqq e/2 \cdot |\mu_{r_{2m_n}} - \mu_{r_1}| \leqq e/2,$$

i. e.,

$$n \geqq n_e \cdot \supset \cdot |\mu_{r_{1m_n}} - \mu_{r_{2m_n}}| \leqq e.$$

But this is contradictory to our hypothesis on r_{1m_n} and r_{2m_n} if $e = e_0$.

We have thus shown that none of the cases (a), (b), (c), (d) can occur; that is, we have established the uniform continuity of μ.

We could proceed to consider finally the theorem of Fréchet, *loc. cit.*, p. 31, as extended by Hahn,* and show that this is also holding in case K^{1367}, *i. e.*, in case δ^{0367}. With some care in the use of the proof given by Hahn, there is little difficulty in proving the existence of the non-constant continuous function in a δ^{0367}, and deriving the theorem in question, *i. e*:

$$K^{1367} \cdot \mathfrak{R} :\supset: \mu^{\text{continuous on } \mathfrak{R} \text{ and bounded on } \mathfrak{R}} \cdot \supset \cdot \mathfrak{R}^{\text{extremal}}.$$

* *Monatshefte für Mathematik und Physik*, XIX, 251 ff. We might remark in this connection that, on account of what seems to be an oversight, the limit L, defined by Hahn on p. 250 of this note in the construction of a system $(\mathfrak{Q}; L)$ on which every continuous function is constant, is not unique. The uniqueness is secured if the series of inequalities in l. 8: $i_1 < i_2 < \ldots . < i_n$ is replaced by $k_1 < k_2 < \ldots . < k_n$.

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