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On the Galois Groups of Some Special Algebraic Equations.

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ON THE RECIPROCAL QUARTIC EQUATION.

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In this paper it is proposed to determine the Galois group of the reciprocal quartic equation,

$$x^4-ax^3+bx^2-ax+1=0, \tag{1}$$

for the domain of rationality $R(1)$. We shall establish the conditions for which the group of the equation is transitive, and hence the condition that the equation is irreducible; and consider the possible intransitive groups when these conditions are not fulfilled.

Calling the roots of (1) α_0, α_1, β_0, β_1; α_0 the reciprocal of β_0, and α_1 the reciprocal of β_1,

$$\alpha_0\beta_0=\alpha_1\beta_1=1. \tag{2}$$

If we assume that the roots of (1) are distinct, the Galois group of (1) is a subgroup of the group,

$$\begin{aligned}G_8\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\alpha_1)(\beta_0\beta_1);\\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)].\end{aligned}$$

For, any substitution not in G_8 leaves the relation (2) unaltered only if the equation (1) has a pair of equal roots. The group G_8 has only two transitive subgroups, viz:

$$\begin{aligned}C_4&\equiv[1;\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)],\\ G_4&\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].\end{aligned}$$

Hence, if we impose such conditions upon the coefficients of (1) that its group is either G_8, G_4, or C_4 we have the necessary and sufficient conditions that the equation be irreducible. For this purpose it is necessary to compute the values in terms of the coefficients of functions belonging to each of the following subgroups of G_8:

$$\begin{aligned}&G_4;\ C_4;\ H_4\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].\\ &G'_2\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1)];\ G''_2\equiv[1;\ (\alpha_0\beta_1)(\alpha_1\beta_0)].\end{aligned}$$

From the equation (1):

$$\alpha_0+\beta_0+\alpha_1+\beta_1=a, \tag{3}$$

$$(\alpha_0+\beta_0)(\alpha_1+\beta_1)=b-2. \tag{4}$$

By means of (3) and (4) we easily find:

$$\phi \equiv (\alpha_0+\beta_0)-(\alpha_1+\beta_1)=\sqrt{[2-b+\tfrac{1}{4}a^2]}, \text{ belonging to } H_4, \tag{5}$$

$$\psi \equiv (\alpha_0-\beta_0)(\alpha_1-\beta_1)=\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}, \text{ belonging to } G_4, \tag{6}$$

$$\phi.\psi \text{ belonging to } C_4. \tag{7}$$

With the aid of (5) and (6);

$$\phi' \equiv \alpha_0+\alpha_1=\sqrt{\{\tfrac{1}{4}a^2-(1+\tfrac{1}{2}b)-\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}\}}, \text{ belonging to } G'_2, \tag{8}$$

$$\phi'' \equiv \alpha_0+\beta_1=\sqrt{\{\tfrac{1}{4}a^2-(1+\tfrac{1}{2}b)+\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}\}}, \text{ belonging to } G''_2, \tag{9}$$

$$\phi''' \equiv (\alpha_0-\beta_1)^2=\sqrt{(b^2-4)} \text{ (when } a=0\text{), belonging to } G''_2. \tag{10}$$

The values for ϕ and ψ are given up to a rational factor; those for ϕ', ϕ'', ϕ''' up to a rational term.

In the study of the cases where the group is intransitive we shall make use of the two additional functions:

$$\chi_0 \equiv \alpha_0-\beta_0=+\sqrt{\{\tfrac{1}{2}a^2-(b+2)+a\sqrt{[\tfrac{1}{4}a^2+2-b]}\}},$$
$$\text{belonging to } H_2 \equiv [1;\ (\alpha_1\beta_1)], \tag{11}$$

$$\chi_1 \equiv \alpha_1-\beta_1=+\sqrt{\{\tfrac{1}{2}a^2-(b+2)-a\sqrt{[\tfrac{1}{4}a^2+2-b]}\}},$$
$$\text{belonging to } H'_2 \equiv [1;\ (\alpha_0\beta_0)]. \tag{12}$$

From the definition of the Galois group of an equation for a domain of rationality R it follows that if a rational function of the roots of the equation belonging to a group H has a value not in R the group of the equation is not contained in H. Hence we need to consider subgroups of a group H only when a function belonging to H has its value in R.

The group of (1) will depend on the character of the functions (5)-(12) and we have in the following three cases the conditions for which the group G is transitive:

1) ϕ irrational, ψ irrational, and $\phi.\psi$ irrational. $\therefore G=G_8$.

2) ϕ irrational, ψ irrational, and $\psi.\phi$ rational.

$\therefore G=C_4$ or a subgroup of C_4. But its subgroups are excluded as they are subgroups of H_4 to which ϕ belongs. $\therefore G=C_4$.

3) ϕ irrational, ψ rational. Then $G=G_4$, G'_2 or G''_2.

We distinguish two cases:

I) $a\neq 0$. If ϕ' and ϕ'' are irrational, *i. e.*, if the two values $+\sqrt{\{\tfrac{1}{4}a^2-(1+\tfrac{1}{2}b)\pm\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}\}}$ are irrational, G'_2 and G''_2 are excluded, and $G=G_4$.

II) $a=0$. In this case $\phi''=0$ and we use $\phi'''=\sqrt{(b^2-4)}\neq 0$ to exclude G'_2. $\phi'''\neq 0$, since $b=2$ makes ϕ rational and $b=-2$ makes the equation 1) have a pair of equal roots. If then ϕ' is irrational, *i. e.*, in this case, $\sqrt{(-2-b)}$ is irrational and if $\sqrt{(b^2-4)}\equiv\phi'''$ is irrational, $G=G_4$.

These are the only cases in which the group G is transitive. Therefore we conclude that the necessary and sufficient conditions for the irreducibility of (1) are, if $a \neq 0$,

I) $\sqrt{(2-b+\frac{1}{4}a^2)}$ irrational, and $\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$ irrational, or,

II) $\sqrt{(2-b+\frac{1}{4}a^2)}$ irrational, $\sqrt{[(1+\frac{1}{2}b)^2-a^2}$ rational, and $\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)\pm\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$ irrational.

III) When $a=0$, $\sqrt{(-2-b)}$, $\sqrt{(2-b)}$, $\sqrt{(b^2-4)}$ all irrational.

When these conditions are not fulfilled the equation is reducible and its group may be any of the intransitive subgroups of G_8. By examining the values of the set of functions (5)-(12) for any particular case the group G can be determined. The following cases arise:

1) ϕ rational, ψ irrational. Hence, $G=H_4$, H_2, or H'_2, dependent upon rational or irrational nature of χ_0, χ_1. Since ψ is irrational, χ_0 and χ_1 are not both rational. Therefore $G \neq G_1$.

2) ϕ rational, ψ rational. Hence, $G=G_2 \equiv [1; (\alpha_0\beta_0)(\alpha_1\beta_1)]$, the greatest common subgroup of G_4, $\check{H}_4$ and C_4, or if all remaining functions are rational, then $G=G_1$.

3) ϕ irrational, ψ rational. Hence, $G=G'_2$ or G''_2, according as ϕ' or ϕ'' is rational. The case ϕ', ϕ'' both irrational has been included in the previous discussion.

ON DeMOIVRE'S QUINTIC.

By DR. R. L. BORGER, University of Illinois.

§1. For the domain of rational numbers, DeMoivre's quintic

$$(1) \qquad x^5+px^3+\tfrac{1}{5}p^2x+r=0,$$

for values of p and r making the discriminant

$$\Delta \equiv \left(\frac{r}{2}\right)^2+\left(\frac{p}{5}\right)^5 \neq 0,$$

will be shown to have as its Galois group *either the metacyclic group* G_{20} *or a cyclic group* C_4. We may then readily deduce the following properties:

DeMoivre's quintic is solvable by radicals.

Either all the roots are real or only one root is real.

Not more than one root is rational; if the equation is reducible in $R(1)$, *its left member is the product of a linear and an irreducible quartic factor.*

If the equation is irreducible in $R(1)$, *any root is a rational function of an arbitrary pair of roots.*

To determine the Galois group of (1), we make use on the one hand of Cayley's resolvent sextic for any quintic, and on the other hand of the following lemma:*

If we know a rational function of the roots of an algebraic equation $f(x)=0$ having the properties:

(i) That it is formally invariant under the substitutions of a group G' and under no others.

(ii) That it has a value in the domain of rationality.

(iii) That it is distinct from its conjugates under the substitutions of the symmetric group $G_{n!}$, then the Galois group G of $f(x)=0$ is a subgroup of G'.

§2. We exclude those values of p and r for which the discriminant $\Delta=0$. They give rise to equal roots and these may be removed by the process of highest common divisor. The function

$$\phi \equiv (x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1)-(x_1x_3+x_2x_4+x_3x_5+x_4x_1+x_5x_2)$$

belongs to the group G_{10} consisting of the substitutions

*Dickson, *Algebraic Equations*, p. 59, §65.

$$1;\quad (12345);\quad (13524);\quad (14253);\quad (15432);$$
$$(12)(35);\quad (25)(34);\quad (15)(24);\quad (14)(32);\quad (13)(45).$$

Under the substitutions of G_{60}, ϕ takes six values, which are the roots of a resolvent sextic. For the general quintic Cayley has computed* this resolvent sextic, which becomes for equation (1):

$$\phi^6 - 7p^2\phi^4 + 11p^4\phi^2 - \frac{32\,(5^5r^2 + 2^2p^5)}{5^2\sqrt{5}}\,\phi + 4000pr^2 + \tfrac{3}{25}p^6 = 0. \tag{2}$$

One root of (2) is $\phi = p\sqrt{5}$. By differentiating (2) with respect to ϕ we see that this root is simple unless

$$121p^5 + 5^5r^2 = 0. \tag{3}$$

We now divide the discussion into the two cases
I) p and r not satisfying (3).
II) p and r satisfying (3).

I) In this case, ϕ is distinct from its conjugates under G_{60}. Hence $\phi^2 = 5p^2$ belongs to G_{20}† and is distinct from its conjugates. Hence (§1), G_{20} contains the Galois group of (1). The Galois group G for the domain $R(1)$ may then be

$$G_{20},\ G_{10},\ C_5,\ C_4,\ G_2,\ \text{or}\ G_1 \equiv 1.$$

The groups G_{10}, G_5, G_2, G_1 may be at once excluded. By the definition of the Galois group of an equation, every rational function of the roots which remains invariant under the substitutions of G is rationally known. If G is G_{10} or a subgroup of it, then ϕ, belonging to G_{10}, would be rationally known. Since $\phi = p\sqrt{5}$ this is impossible unless $p=0$. Hence when $p \neq 0$, G is not contained in G_{10}.

If $p=0$ we know that (1) reduces to a binomial equation p and its group is metacyclic when r is not the fifth power of a rational number. If $r = k^5$ (k rational) the group G is then C_4. Hence when p and r do not satisfy (3), $G = C_4$ or G_{20}.

§3. Next, we consider the case in which p and r satisfy (3). By solving (3) we find

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*Cayley, *Collected Mathematical Papers*, Vol. IV, p. 319.

†The substitutions of G_{20} are given by $\binom{a}{ax+\beta}$, $\binom{a = 1\ 2\ 3\ 4}{\beta = 0\ 1\ 2\ 3\ 4}$.

$$r = \frac{11p^2}{5^2}\sqrt{\frac{p}{5}}. \tag{4}$$

Since r must be a rational number, $\sqrt{\frac{p}{5}} = a$ (a rational),

$$p = 5a^2, \quad r = 11a^5. \tag{5}$$

Substituting these values in (1), we get

$$x^5 + 5a^2x^3 + 5a^4x + 11a^5 = 0. \tag{6}$$

This equation has the root $x = -a$, and the depressed equation is

$$x^4 - ax^3 + 6a^2x^2 - 6a^3x + 11a^4 = 0. \tag{7}$$

Calling the roots of (7) x_1, x_2, x_3, x_5, and setting

$$\begin{aligned} y_1 &= x_1x_2 + x_3x_5, \\ y_2 &= x_1x_3 + x_2x_5, \\ y_3 &= x_1x_5 + x_2x_3, \end{aligned}$$

we obtain the cubic resolvent of (7),

$$y^3 - 6a^2y^2 - 38a^4y + 217a^6 = 0. \tag{8}$$

The roots of (8) are:

(9) $y_1 = 7a^2$; $y_2 = \frac{a^2}{2}(-1 + 5^2\sqrt{5})$; $y_3 = \frac{a^2}{2}(-1 - 5^2\sqrt{5})$;
$y_1 = x_1x_2 + x_3x_5$ belongs to the group
$G_8 \equiv [1, (12); (35); (1325); (1523); (12)(35); (13)(25); (15)(23)]$.

And since y_1 is distinct from its conjugates under G_{24} the Galois group of (7) is G_8 or a subgroup of G_8. As y_2 and y_3 are irrational the group for the domain $R(1)$ cannot be contained in

$$G_4 \equiv [1; (12)(35); (13)(25); (15)(23)].$$

The function $\varphi \equiv (x_1 + x_2) - (x_3 + x_5) = a\sqrt{5}$ belongs to the group

$H_4 \equiv [1; (12); (35); (12)(35)]$. Since the value of ψ is irrational* G is not contained in H_4.

The function $\chi \equiv (x_1 - x_2)(x_3 - x_5)[x_1 + x_2 - (x_3 + x_5)] \equiv a^3 5^5$ belongs to the group $C_4 \equiv [1; (1325), (1523); (12)(35)]$, since χ is rational and takes two values under the substitutions of G_8.

Therefore, $G = C_4$ or one of its subgroups. G cannot be a subgroup of C_4 as the subgroups of C_4 are contained in H_4. Therefore, $G = C_4$.

Hence when p and r satisfy (3), the group of (1) is C_4.

§4. We have now proved that *the Galois group of DeMoivre's quintic for the domain $R(1)$ is either the cyclic group C_4 or the metacyclic group G_{20}.* We therefore get the following results:

I. *DeMoivre's quintic is solvable by radicals.* The solution can be effected by the well known substitution $x = y - p/5y$.

Since the group may be C_4 the equation may be reducible. Hence

II. *If the equation is reducible it must reduce to the product of a linear factor and an irreducible quartic factor.*

As an equivalent form of II, we have,

III. *DeMoivre's quintic can never have more than one rational root.*

By means of a property of metacyclic equations† we may also conclude that

IV. *All the roots of DeMoivre's quintic are real or only one of them is real.*

If the group of the equation is G_{20} the equation is metacyclic‡ and,

V. *Each root is a rational function of an arbitrary pair of them.*

This problem was suggested to me by Prof. L. E. Dickson and I wish to thank him for criticisms and suggestions in connection with its solution.

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*a is not equal to 0 because of (3) and not both p and $r = 0$.

†Weber, *Algebra*, I, p. 620, VIII.

‡Weber, *Algebra*, I, p. 618, VI.