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Iterated Limits in General Analysis

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Iterated Limits in General Analysis.

By Ralph E. Root.

Introduction.

In a former note* we have briefly indicated a method for the investigation of iterated limits of functions on an abstract range. It is the purpose of the present paper to give a more comprehensive account of the method there proposed. The paper has its origin in the thought that in most of the definitions of limit that are employed in current mathematics a notion analogous to that of "neighborhood" or "vicinity" of an element is fundamental. In the domain of general analysis† various ways of determining a neighborhood of an element have been employed, notably the notion of *voisinage* used by M. Fréchet,‡ and the relations K_1 and K_2 used by E. H. Moore, either as undefined or as defined in terms of a "development" of the class of elements constituting the fundamental domain.§

A definite class of elements being assumed, the notion of "neighborhood" of an element is essentially that of a subclass having a special relation to the element. In taking this relation as undefined and at the basis of our system of postulates we occupy a position intermediate, as regards generality, between the extreme position of those who take the notion of "limit" itself as undefined,‖ and that of those who define "limit" by means of other relations which give rise to notions analogous to that of "neighborhood." The character and form of the postulates adopted are determined largely by two fundamental require-

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* *Bull. Am. Math. Soc.*, Vol. XVII (July, 1911), p. 538.

† The term "general analysis" is here used in a technical sense to indicate mathematical analysis pertaining to a class of elements whose character is not specified.

‡ "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII.

§ E. H. Moore, "Introduction to a Form of General Analysis," pp. 125 and 138.

‖ For example, Fréchet in the first chapter of the paper referred to above, and F. Riesz in his paper before the International Congress of Mathematicians at Rome, 1908 ("Stetigkeitsbegriff und abstracte Mengenlehre," *Atti*, Vol. II, pp. 18–24).

ments;* first, to provide for an adequate treatment of ideal limiting elements, and second, to insure the persistence of the specified conditions under composition of ranges.

In Chapter I, we consider a class $\mathfrak{P}$ of elements and an undefined relation R between subclasses of $\mathfrak{P}$, the system $(\mathfrak{P}; R)$ being subjected to a set of postulates that permit the definition of ideal elements in such fashion that the system, when once extended by the adjunction of ideal elements, is closed to further extension in this manner. It is shown also that from two or more systems a composite system may be derived, and that the composite system satisfies the postulates if and only if the postulates are satisfied by every component system.

A somewhat less restrictive body of postulates, considered in Chapter II, pertain to a system $(\mathfrak{P}; \mathfrak{U}; T)$, $\mathfrak{P}$ being a class of elements, $\mathfrak{U}$ a class of ideal elements, and T a relation between subclasses of $\mathfrak{P}$ and individual elements of $\mathfrak{P}$ or $\mathfrak{U}$. A subclass $\mathfrak{N}$ of $\mathfrak{P}$ having the relation T to an element p of $\mathfrak{P}$ or to an ideal element u of $\mathfrak{U}$ may be thought of as a generalized neighborhood of p or u. The postulates of Chapter I, with the definition of ideal elements for the system $(\mathfrak{P}; R)$, lead to a system satisfying the postulates of Chapter II. We obtain for our system a generalization of a portion of the theory of point-sets by establishing relations between our postulates and the more general conditions involved in the notion of "limit" as used by Fréchet, and those involved in the "Verdichtungstelle" of F. Riesz.

In the third chapter a system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is supposed to satisfy the postulates of Chapter II, and functions μ defined on the range $\mathfrak{P}$, a subclass of $\overline{\mathfrak{P}}$, are studied relative to limits and continuity. The treatment is not intended to be exhaustive, the theorems developed being such as are suggested by familiar theorems on multiple sequences and functions of real variables. Interesting features of the general theory associate themselves with the presence of ideal elements in the system, and with the study of a property of functions which has much the same force as uniform continuity, but which we have called *extensible* continuity.

The fourth chapter is given to applications of the general theory through direct specialization of the system and particular determination of other arbitrary features. Special systems $(\mathfrak{P}; \mathfrak{U}; T)$ are specified, by consideration of

* E. R. Hedrick (*Transactions*, Vol. XII (1911), p. 289) obtains by his "inclosable" property of the fundamental domain essentially a generalization of the notion "neighborhood," but his assumptions are made from a different point of view and, involving a certain uniformity, are more restrictive than the postulates of the present paper.

which the theorems of Chapter III pertain to: The theory of multiple sequences; functions of real variables; functions on a range for which there is defined a relation of the type of either of the relations K_1 and K_2 used by Professor Moore; functions on a range subject to the *voisinage* used by Fréchet; and functions on a range whose elements are real-valued functions on an arbitrary range. In some cases the system $(\mathfrak{P}; \mathfrak{U}; T)$ is reached by the mediation of a system $(\mathfrak{P}; R)$, and in some cases directly. In the applications, under certain restrictions on the class $\mathfrak{P}$, the property extensible continuity is found to be equivalent to uniform continuity in each case where the latter is defined.

We find it advantageous to draw largely upon the notation and terminology used by Professor Moore in his work on General Analysis. Convenience and economy of notation are conserved by the adoption of letters for elements, classes, etc., whose connotation renders frequent explanatory remarks unnecessary. Classes of elements are denoted by $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}$, etc., while their elements are denoted by p, q, r, etc., respectively. Classes of classes are, in general, denoted by u, v, w, etc.; properties and relations by P, Q, R, etc., or simply by the numerals attached to their definitions. Superscripts denote, in general, defining properties or conditions, the character of the superscript as well as of the base symbol serving to determine the nature of the limitation. Thus, $\mathfrak{R}^{\mathfrak{P}}$ states that $\mathfrak{R}$ is a subclass of $\mathfrak{P}$, $p^{\mathfrak{R}}$ that p is an element of $\mathfrak{R}$, $\mathfrak{P}^{P}$ that $\mathfrak{P}$ has the property P, etc. The symbol $\supset$ is a sign of implication, to be used in the statement of a proposition. That which precedes the sign of implication is hypothesis or given data, and that which follows is conclusion or a true statement concerning the given data. Thus, if A and B are propositions, $A . \supset . B$ is read "A implies B" or "if A then B," and if x represents a number in a certain interval and F a definite function on the interval, the proposition "for every two numbers x_1 and x_2 of the interval $F(x_1) - F(x_2) < k$" may be written, $x_1 . x_2 . \supset . F(x_1) - F(x_2) < k$. The reversed symbol $\subset$ denotes "is implied by" and $\backsim$ is the symbol of logical equivalence, "implies and is implied by." In a complex statement the symbols $\supset$, $\subset$ and $\backsim$ carry punctuation marks, ., :, .:, etc., the primary implication of the proposition being indicated by the greater number of dots. The mark $\exists$ is read "there exists," and the mark $\ni$ may be read "such that" or "where" as the sense of the proposition demands.

The independent use of the symbolical statement of propositions is confined largely to the proofs of theorems, where it is most useful in conserving precision and brevity, and where the technical symbols may be least objectionable to the general reader.

CHAPTER I.

THE SYSTEM $(\mathfrak{P}; R)$: EXTENSION AND COMPOSITION OF SYSTEMS.

§ 1. *Introductory: The System* $(\mathfrak{P}; R)$.

In this chapter we consider a system $(\mathfrak{P}; R)$ consisting of a class $\mathfrak{P}$ of elements p and a relation R on ordered pairs of subclasses of $\mathfrak{P}$. While the relation R is of the definite type indicated, it is not further specifically defined. We specify a system $(\mathfrak{P}; R)$ by specifying the class $\mathfrak{P}$ and the relation R, *i. e.*, a criterion which determines for every two subclasses $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathfrak{P}$ whether or not $\mathfrak{R}_1$ has the relation R to $\mathfrak{R}_2$.

For example, take for $\mathfrak{P}$ the class of all points of an ordinary Euclidean plane. Consider a circle as the class of all points within and on its circumference, then we may specify a relation R in terms of geometry as follows: Every circle whose radius is different from zero has the relation R to the point at its center considered as singular subclass, and every two concentric circles whose radii are different from zero have the relation R to each other. In no other case does the relation R hold.

In this example we have a definite system $(\mathfrak{P}; R)$. The pertinence of the relation R as specified to the study of limits of functions defined for a set of points in the plane is obvious. A study of the current theory of real-valued functions, in particular in connection with questions of continuity and iterated limits, leads to a determination of bodies of postulates on systems $(\mathfrak{P}; R)$ which serve to validate a theory of continuous functions and multiple and iterated limits associated with such systems $(\mathfrak{P}; R)$ in general.

Subclasses of $\mathfrak{P}$ are, in general, denoted by $\mathfrak{R}$, and the notation $\mathfrak{R}_1 R \mathfrak{R}_2$ indicates that $\mathfrak{R}_1$ has the relation R to $\mathfrak{R}_2$, while $\mathfrak{R}_1 {}^-R \mathfrak{R}_2$ indicates that $\mathfrak{R}_1$ does not have the relation R to $\mathfrak{R}_2$. In case it is desired to imply that a subclass consists of a single element, we may for simplicity, and for our purposes without confusion, use the notation for single elements. Thus $\mathfrak{R} R p$ indicates that the class $\mathfrak{R}$ has the relation R to the singular subclass whose element is p. The letter v denotes a class of subclasses $\mathfrak{R}$ of $\mathfrak{P}$, and, for a given element p, v_p is the class of all subclasses $\mathfrak{R}$ having the relation R to p, *i. e.*,

$$v_p = [\text{all } \mathfrak{R} \ni \mathfrak{R} R p].$$

Thus, in the example above, v_p is the family of concentric circles whose common center is at the point p, excluding the point circle of the family.

§ 2. *The Postulates and Certain Fundamental Definitions.*

Preliminary to the statement of postulates for a system $(\mathfrak{P}; R)$, we note that a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$ may have one or more of the following properties:

1. Every member $\mathfrak{R}$ of the class v contains at least one element p.

2. The relation R holds between every two classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ that are members of v.

3. There exists a sequence $\{\mathfrak{R}_n\}$ of members of the class v such that for every $\mathfrak{R}$ of v there is a number $n_{\mathfrak{R}}$ such that for $n > n_{\mathfrak{R}}$ the class $\mathfrak{R}_n$ is contained in $\mathfrak{R}$.

4. For every $\mathfrak{R}$ of v there exists an $\mathfrak{R}_1$ of v such that for every p in $\mathfrak{R}_1$ there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ having the relation R to the singular class p.

5. If v_1 is a class containing v and having properties 1, 2, 3 and 4, then $v = v_1$.

6. If v_1 is a class having properties 1, 2, 3, 4 and 5, and not containing v, then there exists a member $\mathfrak{R}_1$ of v_1 and a member $\mathfrak{R}_2$ of v such that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have no common elements.

7. For every element p of $\mathfrak{P}$ there is an $\mathfrak{R}$ of v which does not contain p.

These definitions may be more concisely stated in symbols as follows:

1. $\mathfrak{R}^v . \supset . \exists\, p^{\mathfrak{R}}$.

2. $\mathfrak{R}_1^v . \mathfrak{R}_2^v . \supset . \mathfrak{R}_1 R \mathfrak{R}_2$.

3. $\exists \{\mathfrak{R}_n\} \ni [(n . \supset . \mathfrak{R}_n^v) . (\mathfrak{R}^v :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} . \supset . \mathfrak{R}_n^{\mathfrak{R}})]$.

4. $\mathfrak{R}^v :\supset: \exists\, \mathfrak{R}_1^v \ni p^{\mathfrak{R}_1} . \supset . \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R p$.

5. $v^{v_1} . v^{1.2.3.4} . \supset . v = v_1$.

6.* $v_1^{1.2.3.4.5} . v^{-v_1} . \supset . \exists\, (\mathfrak{R}_1^{v_1} . \mathfrak{R}_2^v) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} . p^{\mathfrak{R}_2}$.

7. $p . \supset . \exists\, \mathfrak{R}^v \ni p^{-\mathfrak{R}}$.

These properties, 1–7, may be called propositional properties.† It is not here asserted that any of the defining propositions are true with respect to any class v, but it is clear that the question whether or not a given one of these propositions is true with respect to a given class v is a question of the presence or absence of a definite property for the class.

The desired postulates might now be stated in the following form:

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* The minus sign here signifies negation. Thus $^{-}\exists$ is read "there does not exist," and $p^{-\mathfrak{R}}$ indicates that p is not an element of the subclass $\mathfrak{R}$. v^{-v_1} indicates that v is not a subclass of v_1.

† See E. H. Moore, *loc. cit.*, p. 20.

(A) *For every element p the class v_p has properties* 1–6, *i. e.*,

$$p\,.\supset.\,v_p^{1.2.3.4.5.6}$$

(B) *For every element p it is true that every $\mathfrak{R}$ of v_p contains p, while if p_1 is distinct from p there is an $\mathfrak{R}$ of v_p not containing p_1.* In symbols:

$$p:\supset:(\mathfrak{R}Rp\,.\supset.\,p^{\mathfrak{R}})\,.\,(p_1\neq p\,.\supset.\,\exists\,\mathfrak{R}\ni\mathfrak{R}Rp\,.\,p_1^{-}\ ).$$

But for convenience of reference, as well as to provide for discussion of the independence of the conditions on the system, we separate these assumptions into simpler components, which we state explicitly in the following seven postulates:

I. $\mathfrak{R}Rp\,.\supset.\,p^{\mathfrak{R}}.$

II. $\mathfrak{R}_1Rp\,.\,\mathfrak{R}_2Rp\,.\supset.\,\mathfrak{R}_1R\mathfrak{R}_2.$

III. $p\,.:\supset:.\,\exists\,\{\mathfrak{R}_n\}\ni[(n\,.\supset.\,\mathfrak{R}_nRp)\,.\,(\mathfrak{R}Rp:\supset:\exists\,n_{\mathfrak{R}}\ni n>n_{\mathfrak{R}}\,.\supset.\,\mathfrak{R}_n^{\mathfrak{R}})].$

IV. $\mathfrak{R}Rp:\supset:\exists\,\mathfrak{R}_1\ni[\mathfrak{R}_1Rp\,.\,(p_1^{\mathfrak{R}_1}\,.\supset.\,\exists\,\mathfrak{R}_2^{\mathfrak{R}}\ni\mathfrak{R}_2Rp_1)].$

V. $v^{1.2.3.4}\,.\,(\mathfrak{R}Rp\,.\supset.\,\mathfrak{R}^{v}):\supset:v=v_p.$

VI. $v^{1.2.3.4.5}\,.\,v_p^{-v}\,.\supset.\,\exists\,(\mathfrak{R}_1^{v}\,.\,\mathfrak{R}_2^{v_p})\ni{}^{-}\exists\,p_1\ni p_1^{\mathfrak{R}_1}\,.\,p_1^{\mathfrak{R}_2}.$

VII. $p_1\neq p\,.\supset.\,\exists\,\mathfrak{R}\,\varepsilon\,\mathfrak{R}Rp\,.\,p_1^{-\mathfrak{R}}.$

Postulates I and VII are together equivalent to the statement (B). A corollary of postulate I is that for every p the class v_p has property 1, while postulates II–VI state that for every p the class v_p has the respective properties 2–6.

The following examples are pertinent to the question of consistency and independence of the postulates. Example 0 is an instance of a system satisfying the seven postulates, and the remaining examples each violate one postulate and satisfy all the others, the examples being numbered in the order of the postulates violated.

Ex. 0. The class $\mathfrak{P}$ is the class of all complex numbers. The notation $\mathfrak{R}_{dp}$, where d is a positive real number and p is an element of $\mathfrak{P}$, stands for the sub class of $\mathfrak{P}$ consisting of all elements p_1 of $\mathfrak{P}$ such that $|p_1-p|\leqq d$, that is,

$$\mathfrak{R}_{dp}\equiv[\text{all } p_1\ni|p_1-p|\leqq d].$$

The relation R is specified as follows: For every p and every d the relation $\mathfrak{R}_{dp}Rp$ holds, and for every p and every d_1 and d_2 the relation $\mathfrak{R}_{d_1p}R\,\mathfrak{R}_{d_2p}$ holds. The relation holds in no other case.

Ex. 1. The system $(\mathfrak{P};R)$ is specified as in example 0, except that $\mathfrak{R}_{dp}$ does not contain the element p, hence

$$R_{dp}\equiv[\text{all } p_1\neq p\ni|p_1-p|\leqq d].$$

Ex. 2. The class $\mathfrak{P}$ is the class of all complex numbers, and the notation $\mathfrak{R}_{dp}$ has the same significance as in example 0. For every p and for every d the relation $\mathfrak{R}_{dp} R p$ holds, but in no other case does the relation R hold.

In this example postulates V and VI are satisfied vacuously, *i. e.*, their hypotheses are incapable of fulfilment, there being no class v which has properties 2 and 3.

Ex. 3. The class $\mathfrak{P}$ is the class of all points of a given Euclidean plane. The designation "line" is used to indicate a subclass $\mathfrak{R}$ constituting the class of all points of a line. Every "line" has the relation R to every one of its points, and every two intersecting "lines" have the relation R to each other. In no other case does the relation R hold.

Here "intersecting" is interpreted as "having a point in common," so that a "line" has the relation R to itself. Postulates V and VI are again satisfied vacuously.

Ex. 4. The system $(\mathfrak{P}; R)$ is as specified in example 0, except that in the particular case $p = 0$ the classes R_{dp} consist only of real elements p_1, *i. e.*,

$$\mathfrak{R}_{d0} \equiv [\text{all real } p_1 \ni |p_1| \leqq d].$$

Ex. 5. Again, the class $\mathfrak{P}$ is the class of all complex numbers, and the notation $\mathfrak{R}_{dp}$ has the same significance as in example 0. The relation R is as specified in example 0 except that for the particular element $p_0 = 0$ the relation $\mathfrak{R}_{dp_0} R p_0$ holds only in case d is less than or equal to unity.

Ex. 6. The class $\mathfrak{P}$ consists of two elements, p_1 and p_2.* The cases in which the relation R holds are listed as follows:

$$p_1 R p_1, \quad p_2 R p_2, \quad \mathfrak{P} R \mathfrak{P}.$$

Ex. 7. Again, $\mathfrak{P}$ is a class consisting of two elements, p_1 and p_2. Following is the list of cases in which the relation R holds:

$$\mathfrak{P} R p_1, \quad \mathfrak{P} R p_2, \quad \mathfrak{P} R \mathfrak{P}.$$

In this instance postulate VI is satisfied vacuously, since any class v that has properties 1–5 possesses the single member $\mathfrak{P}$, and is therefore coincident with both v_{p_1} and v_{p_2}.

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* It should be remembered that elements do not enter in the relation R. The notation for elements is substituted for class notation as a matter of convenience. The class v_{p_1} consists of one member, the class $\mathfrak{R}$ having the single element p_1, and has no member in common with that class v whose only member is $\mathfrak{P}$.

§ 3. *Extension of the System by the Adjunction of Ideal Elements*

Making use of properties 1–7 defined in § 2, properties that may be possessed by a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$, we proceed to the definition of ideal elements for the system $(\mathfrak{P}; R)$.

Def. 1. An *ideal element* of the system $(\mathfrak{P}; R)$ is a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$ having properties 1–7.

The letter u invariably stands for an ideal element.

THEOREM I. *If v is a class having properties* 1–6, *then v is an ideal element u, or there is an element p such that $v = v_p$.*

Proof: If v has property 7, it is a u by definition; if it has not property 7, then there is an element p common to all classes $\mathfrak{R}$ of v. Since v_p has properties 1–6, we clearly have $v = v_p$.

Let $\mathfrak{U}$ denote the class of all ideal elements of the system $(\mathfrak{P}; R)$, and let $\mathfrak{Q}$ be a class consisting of the elements of $\mathfrak{P}$, together with all ideal elements, *i. e.*, $\mathfrak{Q} \equiv \mathfrak{P} + \mathfrak{U}$. We denote elements of $\mathfrak{Q}$, in general, by q, subclasses of $\mathfrak{Q}$ by $\mathfrak{S}$, and classes of subclasses by w. A certain technical form of correspondence between classes is of frequent occurrence, and it is therefore convenient to adopt a special symbol, $\|$, to be read *corresponds to*, which we define as follows:

Def. 2. $\mathfrak{S} \,\|\, \mathfrak{R}$ indicates that $\mathfrak{S}$ consists of the elements of $\mathfrak{R}$ together with every ideal element u such that there is a subclass $\mathfrak{R}_1$ of $\mathfrak{R}$ which belongs to the class* u. In symbols:

$$\mathfrak{S} \,\|\, \mathfrak{R} :\equiv: \mathfrak{S} = \mathfrak{R} + [\text{all } u \ni \exists\, \mathfrak{R}_1^{\mathfrak{R}} \ni \mathfrak{R}_1^{u}].$$

Def. 3. $w \,\|\, v$ indicates that w consists of all classes $\mathfrak{S}$ for which there exist classes $\mathfrak{R}$ in v such that $\mathfrak{S} \,\|\, \mathfrak{R}$. In symbols:

$$w \,\|\, v :\equiv: w = [\text{all } \mathfrak{S} \ni \exists\, \mathfrak{R}^{v} \ni \mathfrak{S} \,\|\, \mathfrak{R}].$$

It is obvious that for every $\mathfrak{R}$ there is a unique $\mathfrak{S}$ such that $\mathfrak{S} \,\|\, \mathfrak{R}$, and that for two distinct classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the corresponding classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are distinct. It follows that for every v there is a unique w, and that for two distinct classes v_1 and v_2 the corresponding w_1 and w_2 are distinct.

Let S be a relation of the same type as R defined as follows:

Def. 4. The relation $\mathfrak{S}_1 S \mathfrak{S}_2$ holds if and only if one of the following conditions is fulfilled:

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* No confusion need arise from the fact that the letter u denotes at the same time an element of $\mathfrak{Q}$ and a class of subclasses of $\mathfrak{Q}$, as well as of $\mathfrak{P}$. It was this double rôle that led to the adoption of small letters as notation for classes of subclasses in general.

$$(a) \quad \exists\, (\mathfrak{R}_1 \,.\, \mathfrak{R}_2) \ni \mathfrak{R}_1 R \mathfrak{R}_2 \,.\, \mathfrak{S}_1 \parallel \mathfrak{R}_1 \,.\, \mathfrak{S}_2 \parallel \mathfrak{R}_2 .$$
$$(b) \quad \exists\, (\mathfrak{R} \,.\, u) \ni \mathfrak{R}^u \,.\, \mathfrak{S}_1 \parallel \mathfrak{R} \,.\, (q^{\mathfrak{S}_2} \,.\, \supset .\, q = u).$$

In condition (b) the ideal element u considered as a class of subclasses contains $\mathfrak{R}$, and considered as an element of $\mathfrak{Q}$ constitutes the singular subclass $\mathfrak{S}_2$.

We now have a definite system $(\mathfrak{Q}; S)$, which we shall call the *extended system* derived from $(\mathfrak{P}; R)$. We investigate the character of this extended system with respect to the seven postulates. The properties 1–7 defined for a class v are defined also for a class w, if in the notation we replace v by w, p by q, $\mathfrak{R}$ by $\mathfrak{S}$ and R by S, and with similar changes of notation we have the seven postulates stated for the system $(\mathfrak{Q}; S)$.

THEOREM II. *The seven postulates are satisfied by the extended system* $(\mathfrak{Q}; S)$.

In proving this theorem it it convenient to establish first the following lemma:

LEMMA. *The necessary and sufficient condition that a class w shall have properties 1–4 or 1–5 or 1–6 is that there shall exist a class v having the corresponding properties such that $w \parallel v$.*

In considering the necessity of the condition we have available in each case the fact that w has property 2. This is sufficient to secure the existence of a v such that $w \parallel v$. It is sufficient, then, to assume a definite w corresponding to a definite v and prove the following propositions:

$$(a) \quad w^{1.2.3.4} \,.\, \supset .\, v^{1.2.3.4}, \qquad (b) \quad v^{1.2.3.4} \,.\, \supset .\, w^{1.2.3.4},$$
$$(c) \quad w^{1.2.3.4.5} \,.\, \supset .\, v^{5}, \qquad (d) \quad v^{1.2.3.4.5} \,.\, \supset .\, w^{5},$$
$$(e) \quad w^{1.2.3.4.5.6} \,.\, \supset .\, v^{6}, \qquad (f) \quad v^{1.2.3.4.5.6} \,.\, \supset .\, w^{6}.$$

(a): Since w has property 1, every $\mathfrak{S}$ of w contains a q, that is, either a p or a u. Every $\mathfrak{R}$ of v has a corresponding $\mathfrak{S}$ in w, hence it contains either this element p or an $\mathfrak{R}_1$ of this class u. Therefore v has property 1. That v has property 2 is evident from definition 4, and from property 3 of w there is a sequence $\{\mathfrak{S}_n\}$ such that the sequence $\{\mathfrak{R}_n\}$, where $\mathfrak{S}_n \parallel \mathfrak{R}_n$, is effective in establishing property 3 for v. Since w has property 4, we have

$$(1) \qquad \mathfrak{S}^w : \supset : \exists\, \mathfrak{S}_1^w \ni q^{\mathfrak{S}_1} \,.\, \supset .\, \exists\, \mathfrak{S}_2^{\mathfrak{S}} \ni \mathfrak{S}_2 S q,$$

and we wish to prove

$$(2) \qquad \mathfrak{R}^v : \supset : \exists\, \mathfrak{R}_1^v \ni p^{\mathfrak{R}_1} \,.\, \supset .\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R p.$$

Given an $\mathfrak{R}$ of v, take $\mathfrak{S}$ such that $\mathfrak{S} \parallel \mathfrak{R}$, then $\mathfrak{S}^w$ and (1) applies. Take $\mathfrak{R}_1$ so that $\mathfrak{S}_1 \parallel \mathfrak{R}_1$, then $\mathfrak{R}_1^v$ and every p of $\mathfrak{R}_1$ is in $\mathfrak{S}_1$. For every p of $\mathfrak{R}_1$, then, there is a subclass $\mathfrak{S}_2$ of $\mathfrak{S}$ such that $\mathfrak{S}_2 S p$, and there is an $\mathfrak{R}_2$ such that $\mathfrak{S}_2 \parallel \mathfrak{R}_2$, then clearly $\mathfrak{R}_2^v$ and $\mathfrak{R}_2 R p$. Thus v has property 4.

(b): The proof is obvious for properties 1, 2 and 3. As to property 4, we have condition (2) above and wish to prove (1). Given an $\mathfrak{S}$ of w, there is a corresponding $\mathfrak{R}$ in v, so that (2) applies to provide an $\mathfrak{R}_1$ fulfilling the conclusion of (2). Take $\mathfrak{S}_1$ to correspond to $\mathfrak{R}_1$, then $\mathfrak{S}_1^w$ and every p in $\mathfrak{S}_1$ is in $\mathfrak{R}_1$ and an $\mathfrak{S}_2$ corresponding to an $\mathfrak{R}_2$ furnished by (2) meets the requirements of (1). Further, every u in $\mathfrak{S}_1$ possesses a member $\mathfrak{R}_2$ which is a subclass of $\mathfrak{R}_1$, and since $\mathfrak{R}_1$ is necessarily a subclass of $\mathfrak{R}$, we see that the class $\mathfrak{S}_2$ corresponding to $\mathfrak{R}_2$ is a subclass of $\mathfrak{S}$; and clearly $\mathfrak{S}_2 S u$, therefore w has property 4.

Propositions (c) and (d) are easily verified by the use of (a) and (b).

(e): From property 6 of w we have

$$(3) \qquad w_1^{1.2.3.4.5} \cdot w^{-w_1} \cdot \supset \cdot \exists\, (\mathfrak{S}_1^{w_1} \cdot \mathfrak{S}_2^{w}) \ni {}^{-}\exists\, q \ni q^{\mathfrak{S}_1} \cdot q^{\mathfrak{S}_2},$$

and we wish to prove

$$(4) \qquad v_1^{1.2.3.4.5} \cdot v^{-v_1} \cdot \supset \cdot \exists\, (\mathfrak{R}_1^{v_1} \cdot \mathfrak{R}_2^{v}) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} \cdot p^{\mathfrak{R}_2}.$$

If v_1 has properties 1–5 and does not contain v, then there is a w_1 such that $w_1 \parallel v_1$ which does not contain w and which, by (b) and (d), has properties 1–5. Proposition (3) is now applicable, and the $\mathfrak{S}_1$ and $\mathfrak{S}_2$ thus available have corresponding classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ which obviously fulfil the conclusion of (4).

The proof of (f) is similar to that of (e).

The proof of the theorem is now easily completed. In analogy with previous notation, we denote by w_q the class of all classes $\mathfrak{S}$ such that $\mathfrak{S} S q$, and we observe that for every p we have $w_p \parallel v_p$, while for every u we have $w_u \parallel u$. Since every v_p and every u have properties 1–6, it follows from the lemma that the class w_q has properties 1–6. Postulate I being obviously fulfilled, it remains to consider postulate VII. We wish to show that

$$q_1 \neq q_2 \cdot \supset \cdot \exists\, \mathfrak{S} \ni (\mathfrak{S} S q_1 \cdot q_2^{-\mathfrak{S}}).$$

If both q_1 and q_2 are elements of $\mathfrak{P}$, postulate VII on the system $(\mathfrak{P}; R)$ assures us of a class $\mathfrak{R}$ such that the corresponding class $\mathfrak{S}$ is effective. If either q_1 or q_2 is a u, then, since by property 7 of u and postulate I the class u is not contained in any class v_p, the corresponding class w_u is not contained in any class w_p, and therefore property 6 of w_u is effective, the desired conclusion being an immediate consequence.

It is desirable now to show that the extended system $(\mathfrak{Q}; S)$ is closed to this process of extension; that is, if we repeat the process of extension the second extended system coincides with the first. We may state the required theorem in the form:

THEOREM III. *No ideal elements arise in the extended system* $(\mathfrak{Q}; S)$.

Proof: Suppose a class w to have properties 1–6, then there is a class v having properties 1–6 such that $w \parallel v$. By theorem I v is a u or a v_p, and in either case w is a w_q and therefore does not have property 7.

Following is an instance of a system which illustrates effectively the operation of the foregoing definition of extension:

Example. The class $\mathfrak{P}$ is the class of all rational numbers. If p_1 and p_2 are two distinct rational numbers, then, if p_1 is less than p_2, the class $\mathfrak{R}_{p_1 p_2}$ is the class of all rational numbers on the interval $p_1 p_2$. That is,

$$\mathfrak{R}_{p_1 p_2} \equiv [\text{all } p \ni p_1 \leqq p \leqq p_2].$$

The relation $\mathfrak{R}_{p_1 p_2} R\, \mathfrak{R}_{p_3 p_4}$ holds if and only if the intervals $p_1 p_2$ and $p_3 p_4$ have a common sub-interval, *i. e.*, if $p_1 < p_4$ and $p_3 < p_2$; and the relation $\mathfrak{R}_{p_1 p_2} R\, p$ holds if and only if $p_1 < p < p_2$. In no other case does the relation R hold.

It is not difficult to see that the system $(\mathfrak{P}; R)$ here specified satisfies the seven postulates. We proceed, therefore, to investigate the matter of ideal elements. Consider a class v having properties 1–6. By a little attention to the requirements of properties 3 and 4 we see that there exists a sequence $\{\mathfrak{R}_n\}$ of members of v such that for every n the class $\mathfrak{R}_n$ is of the form $\mathfrak{R}_{\underline{p}_n \bar{p}_n}$, where the sequence $\{\underline{p}_n\}$ is an increasing monotonic sequence of distinct elements and the sequence $\{\bar{p}_n\}$ is a decreasing monotonic sequence of distinct elements; and, further, such that every member $\mathfrak{R}$ of v contains a member $\mathfrak{R}_n$ of the sequence. In view of property 5, then, the two sequences $\{\underline{p}_n\}$ and $\{\bar{p}_n\}$ have a common limit. If this limit is a rational number p, then v coincides with v_p, while if the limit is an irrational number a, then v consists of all classes $\mathfrak{R}_{p_1 p_2}$ such that $p_1 < a < p_2$. In the latter case v has property 7 and is an ideal element of the system $(\mathfrak{P}; R)$. Since it is obvious that for every irrational number a the class v consisting of all classes $\mathfrak{R}_{p_1 p_2}$ such that $p_1 < a < p_2$ has properties 1–7, we see that the ideal elements u of the system $(\mathfrak{P}; R)$ are in reciprocal one-to-one correspondence with the irrational numbers in such fashion that, if u corresponds to a, then

$$u \equiv [\text{all } \mathfrak{R}_{p_1 p_2} \ni p_1 < a < p_2].$$

We may therefore consider our definition of ideal elements, in this instance, as a definition of irrational numbers.

The extended system $(\mathfrak{Q}; S)$ is seen to be as follows: $\mathfrak{Q}$ is the class of all real numbers; two intervals with rational limits that have a common interior element have the relation S to each other, and every interval with rational limits has the relation S to every one of its interior elements (considered as singular class), but in no other case does the relation S hold.

§ 4. *Composition of Systems.**

Two classes of elements, $\mathfrak{P}'$ and $\mathfrak{P}''$, determine a "product" or composite class, $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, consisting of all elements p of the form $p = (p', p'')$, where p' and p'' belong to the classes $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively. It should be borne in mind that these bi-partite elements, $p = (p', p'')$, which we shall denote simply by $p'p''$, are not in any sense products of the elements p' and p'', but rather that p is a notation for the pair $p'p''$. From $\mathfrak{R}'$ and $\mathfrak{R}''$, subclasses of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, we have $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, a subclass of $\mathfrak{P}$ consisting of all elements p of the form $p = p'p''$, where p' and p'' belong to $\mathfrak{R}'$ and $\mathfrak{R}''$ respectively. Similarly, if v' is a class of subclasses $\mathfrak{R}'$ of $\mathfrak{P}'$, and v'' is a class of subclasses $\mathfrak{R}''$ of $\mathfrak{P}''$, we have the composite class $v = v'v''$, the class of all $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, where $\mathfrak{R}'$ and $\mathfrak{R}''$ belong to the respective classes v' and v''.

If R' and R'' are relations of the type discussed in § 1, defined for $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, then from the two systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$ we derive what we shall call the composite of these two systems, the system $(\mathfrak{P}; R)$, where $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and R is a relation of the same type as R' and R'' defined as follows:

Def. 4. For a system $(\mathfrak{P}; R)$, composite of $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, the relation $\mathfrak{R}_1 R \mathfrak{R}_2$ holds if and only if there exist classes $\mathfrak{R}_1'$, $\mathfrak{R}_1''$, $\mathfrak{R}_2'$ and $\mathfrak{R}_2''$ such that $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$, and such that the relations $\mathfrak{R}_1' R' \mathfrak{R}_2'$ and $\mathfrak{R}_1'' R'' \mathfrak{R}_2''$ hold.

It may be observed that the effectiveness of the foregoing definition is independent of the conditions imposed upon the component systems. It is convenient throughout the present section to regard the systems involved as unconditioned, except as conditions are specified in the hypotheses of the several theorems.

THEOREM IV. *The seven postulates are satisfied by the system* $(\mathfrak{P}; R)$, *composite of* $(\mathfrak{P}'; R')$ *and* $(\mathfrak{P}''; R'')$, *if and only if they are satisfied by both component systems.*

* Compare T. H. Hildebrandt, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 250.

We consider first the following lemma:

LEMMA I. *If $v = v'v''$, then the following propositions hold:*

(a) $v^{1.2.3.4} . \supset . v'^{1.2.3.4} . v''^{1.2.3.4}$, (b) $v'^{1.2.3.4} . v''^{1.2.3.4} . \supset . v^{1.2.3.4}$,
(c) $v^{1.2.3.4.5} . \supset . v'^{5} . v''^{5}$, (d) $v'^{1.2.4.4.5} . v''^{1.2.3.4.5} . \supset . v^{5}$,
(e) $v^{1.2.3.4.5.6} . \supset . v'^{6} . v''^{6}$, (f) $v'^{1.2.3.4.5.6} . v''^{1.2.3.4.5.6} . \supset . v^{6}$.

(a) and (b) are sufficiently obvious if only we remember the significance of the relation $v = v'v''$.

(c): Suppose a class v_1' to contain v' and to have properties 1–4. The composite class $v_1 = v_1'v''$ then contains v and, by (a) and (b), has properties 1–4; by property 5 of v then $v_1 = v$, and therefore $v_1' = v'$. In similar manner it may be shown that v'' has property 5.

(d): Suppose v_1 to contain v and to have properties 1–4, and consider the two classes v_1' and v_1'' defined as follows:

$$v_1' = [\text{all } \mathfrak{R}' \ni \exists\, (\mathfrak{R}'' . \mathfrak{R}^{v_1}) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''],$$
$$v_1'' = [\text{all } \mathfrak{R}'' \ni \exists\, (\mathfrak{R}' . \mathfrak{R}^{v_1}) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''].$$

It is not difficult to see that these classes have properties 1–4, and also that v_1' contains v' and v_1'' contains v''; then by property 5 of v' and v'', we have $v_1' = v'$ and $v_1'' = v''$, and consequently $v_1 = v$.

(e): Suppose v_1' has properties 1–5 and $v'^{-v_1'}$, then if $v_1 = v_1'v''$ we have $v_1^{1.2.3.4.5}$ and v^{-v_1}; hence by property 6 of v,

$$\exists\, (\mathfrak{R}_1^{v_1} . \mathfrak{R}_2^{v}) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} . p^{\mathfrak{R}_2}.$$

The classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are of the forms $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$, where $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ are members of v_1' and v' respectively, and $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ are members of v''. By properties 1 and 3 of v'', $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ have common elements, therefore $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ can have no elements in common, and consequently v' has property 6. In like manner it may be shown that v'' has property 6.

(f): Suppose $v_1^{1.2.3.4.5}$ and v^{-v_1}; take v_1' and v_1'' defined as in the proof of (d) above, then as before they have properties 1–4, and by (b) the composite class $\bar{v}_1 = v_1'v_1''$ has these four properties. Since $\bar{v}_1$ clearly contains v_1, and v_1 has property 5, we have $\bar{v}_1 = v_1$; hence by (c) v_1' and v_1'' have property 5. Since v is not contained in v_1, either v' is not contained in v_1', or v'' is not contained in v_1''; suppose the former, then since v' has property 6, we have

$$\exists\, (\mathfrak{R}_1'^{v_1'} . \mathfrak{R}_2'^{v'}) \ni {}^{-}\exists\, p' \ni p'^{\mathfrak{R}_1'} . p'^{\mathfrak{R}_2'}.$$

Take $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ so that $\mathfrak{R}_1''$ is a member of v_1'' and $\mathfrak{R}_2''$ is a member of v'', and take $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$; then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are members of v_1 and v_2 respectively, and clearly they can have no common elements. Thus we show that v has property 6.

We record here for future reference certain results reached incidentally in the foregoing proofs:

LEMMA II. (a) *If v has properties* 1–4 *and*

$$v' = [\text{all } \mathfrak{R}' \ni \exists\, (\mathfrak{R}'' \,.\, \mathfrak{R}^v) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''],$$

then v' has properties 1–4.

(b) *If v has properties* 1–5, *then there exist classes v' and v'' such that $v = v'v''$.*

Taking up now the proof of the theorem, we assume that the composite system $(\mathfrak{P}; R)$ satisfies the seven postulates and show* that as a consequence both component systems satisfy them. For a given p' there exist elements p and p'' such that $p = p'p''$ and $v_p = v'_{p'}v''_{p''}$, and since every $\mathfrak{R}$ of v_p contains p, clearly every $\mathfrak{R}'$ of $v'_{p'}$ contains p', so that postulate I is satisfied by $(\mathfrak{P}'; R')$. It follows also from lemma I that postulates II–VI are satisfied by this system. If $p_1' \neq p_2'$, then taking $p_1 = p_1'p''$ and $p_2 = p_2'p''$, where p'' is any element of $\mathfrak{P}''$, we have $p_1 \neq p_2$; hence there is an $\mathfrak{R}$ not containing p_2 such that $\mathfrak{R}\,R\,p_1$. But such an $\mathfrak{R}$ is of the form $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, where $\mathfrak{R}'\,R'\,p_1'$ and $\mathfrak{R}''\,R''\,p''$; and since p_1 is in $\mathfrak{R}$, we see that p_1' is in $\mathfrak{R}'$ and p'' is in $\mathfrak{R}''$. Clearly then, p_2' is not in $\mathfrak{R}'$, so that the system $(\mathfrak{P}'; R')$ satisfies postulate VII. In like manner the system $(\mathfrak{P}''; R'')$ is shown to satisfy the seven postulates.

The remainder of the theorem, *i. e.*, that if the postulates are satisfied by both component systems then they are satisfied by the composite system, is sufficiently evident without detailed discussion.

From the system $(\mathfrak{P}; R)$, composite of the two systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, we may derive an extended system $(\mathfrak{Q}; S)$ by the process defined in § 3; or we may take first the extended systems $(\mathfrak{Q}'; S')$ and $(\mathfrak{Q}''; S'')$ and form their composite system, which we denote by $(\bar{\mathfrak{Q}}; \bar{S})$. It is desirable to compare the two systems $(\mathfrak{Q}; S)$ and $(\bar{\mathfrak{Q}}; \bar{S})$ thus derived. Let it be assumed that the systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$ satisfy the seven postulates; then the systems $(\mathfrak{Q}; S)$ and $(\bar{\mathfrak{Q}}; \bar{S})$ also satisfy them. The class $\bar{\mathfrak{Q}}$ consists of all elements of the four forms

$$(1)\ \bar{q} = p'p'', \qquad (2)\ \bar{q} = u'u'', \qquad (3)\ \bar{q} = p'u'', \qquad (4)\ \bar{q} = u'p''.$$

* It is necessary to exclude the trivial case in which the class $\mathfrak{P}$ has no elements.

Now all elements $p = p' p''$ belong to $\mathfrak{P}$ and are therefore in $\mathfrak{Q}$; also, since all classes u' and u'' have properties 1–7, every class $u = u' u''$ has properties 1–7 and is an ideal element of $(\mathfrak{P}; R)$ and consequently appears in $\mathfrak{Q}$. Further, it is clear that a class v of the form $v'_{p'} u''$ or $u' v''_{p''}$ has properties 1–7 and is therefore an element of $\mathfrak{Q}$.

Conversely, every element of $\mathfrak{Q}$ is either a p or a u; every p is of the form $p = p' p''$ and therefore is in $\mathfrak{Q}$; every u is a class having properties 1–7, hence, by lemma II above, is of the form $u = v' v''$; and by lemma I above v' and v'' have properties 1–6; therefore by theorem I v' is a $v'_{p'}$ or a u, and v'' is a $v''_{p''}$ or a u'', so that every u of $\mathfrak{Q}$ is of the form $u' u''$ or $v'_{p'} u''$ or $u' v''_{p''}$.

We arrive, then, at the following theorem:

THEOREM V. *If from two systems, $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, which satisfy the seven postulates we derive $(\mathfrak{Q}; S)$ by composition then extension, and $(\overline{\mathfrak{Q}}; \overline{S})$ by extension then composition, the two systems $(\mathfrak{Q}; S)$ and $(\overline{\mathfrak{Q}}; \overline{S})$ are related as follows:*

(a) *The elements q of $\mathfrak{Q}$ are in reciprocal one-to-one correspondence with the elements $\overline{q}$ of $\overline{\mathfrak{Q}}$ in such manner that, if q corresponds to $\overline{q}$, then $q = p' p''$ and $\overline{q} = p' p''$, or q is of the form $u = u' u''$, where $\overline{q} = u' u''$, or of the form $u = v'_{p'} u''$, where $\overline{q} = p' u''$, or of the form $u = u' v''_{p''}$, where $\overline{q} = u' p''$.*

(b) *If under the correspondence of* (a) *q corresponds to $\overline{q}$, then the classes $\mathfrak{S}$ such that $\mathfrak{S} S q$ are in reciprocal one-to-one correspondence with the classes $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}} \overline{S} \overline{q}$ in such manner that, if $\mathfrak{S}$ corresponds to $\overline{\mathfrak{S}}$, then under the correspondence of* (a) *the elements of $\mathfrak{S}$ correspond to elements of $\overline{\mathfrak{S}}$, and those elements of $\overline{\mathfrak{S}}$ which do not correspond to elements of $\mathfrak{S}$ are of the form $p' u''$ or $u' p''$.*

For a given $\overline{q}$ a class $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}} \overline{S} \overline{q}$ must be of the form $\overline{\mathfrak{S}} = \mathfrak{S}' \mathfrak{S}''$, and if $\mathfrak{S}'$ contains an element p' for which there is no subclass $\mathfrak{R}'$ of $\mathfrak{S}'$ such that $\mathfrak{R}' R' p'$, it is obvious that for every u'' in $\mathfrak{S}''$ the element $p' u''$ is in $\overline{\mathfrak{S}}$, while the corresponding element $v'_{p'} u''$ is not in the class $\mathfrak{S}$ that corresponds to $\overline{\mathfrak{S}}$ under the correspondence of (b).*

* An important fact to be noted here is that it is with respect to the relations S and $\overline{S}$ that this discrepancy appears. Our purpose in defining a relation S and an extended system $(\mathfrak{Q}; S)$ is to show more completely than would otherwise be possible the operation of our definition of ideal elements. We make no further use of relations S, but treat ideal elements u as associated with systems $(\mathfrak{P}; R)$. A considerable simplification is introduced in Chapter II. It is sufficient for our purpose that if $\mathfrak{U}'$ is the class of ideal elements arising in $(\mathfrak{P}'; R')$ and $\mathfrak{U}''$ is the class of ideal elements arising in $(\mathfrak{P}''; R'')$, then the class $\mathfrak{U}$ of ideal elements arising in the composite system $(\mathfrak{P}; R)$ may be regarded as identical with the sum of the composite classes $\mathfrak{U}'\mathfrak{U}''$, $\mathfrak{U}'\mathfrak{P}''$ and $\mathfrak{P}'\mathfrak{U}''$. And this is seen to be permissible, in the sense that $\mathfrak{U}$ consists of all classes u of the form $u' u''$ or $u' v''_{p''}$ or $v'_{p'} u''$.

Consider a set of r systems, $(\mathfrak{P}^1; R^1)$, $(\mathfrak{P}^2; R^2)$, ..., $(\mathfrak{P}^r; R^r)$. From the first two we may construct the composite system $(\mathfrak{P}^{1,2}; R^{1,2})$, and from this system and $(\mathfrak{P}^3; R^3)$ we obtain the composite system $(\mathfrak{P}^{1,2,3}; R^{1,2,3})$. Continuing in this way we arrive at what may be called the iterated composite system $(\mathfrak{P}^{1,\ldots,r}; R^{1,\ldots,r})$. From the definition of the composite of two given systems it is at once evident that this iterated composite system is as follows: $\mathfrak{P}^{1,\ldots,r}$ is the class of all elements of the form $p = p^1 p^2 \ldots . p^r$, where $p^1, p^2, \ldots,$ and p^r belong to the respective classes $\mathfrak{P}^1, \mathfrak{P}^2, \ldots, \mathfrak{P}^r$. That is, $\mathfrak{P}^{1,\ldots,r} = \mathfrak{P}^1 \mathfrak{P}^2 \ldots . \mathfrak{P}^r$. If $\mathfrak{R}_1^{1,\ldots,r}$ and $\mathfrak{R}_2^{1,\ldots,r}$ are subclasses of $\mathfrak{P}^{1,\ldots,r}$, then the relation $\mathfrak{R}_1^{1,\ldots,r} R^{1,\ldots,r} \mathfrak{R}_2^{1,\ldots,r}$ holds if and only if there exist classes $\mathfrak{R}_1^1, \mathfrak{R}_1^2, \ldots, \mathfrak{R}_1^r$ and $\mathfrak{R}_2^1, \mathfrak{R}_2^2, \ldots, \mathfrak{R}_2^r$ such that $\mathfrak{R}_1^{1,\ldots,r} = \mathfrak{R}_1^1 \mathfrak{R}_1^2 \ldots . \mathfrak{R}_1^r$ and $\mathfrak{R}_2^{1,\ldots,r} = \mathfrak{R}_2^1 \mathfrak{R}_2^2 \ldots . \mathfrak{R}_2^r$ and the relations $\mathfrak{R}_1^1 R^1 \mathfrak{R}_2^1$, $\mathfrak{R}_1^2 R^2 \mathfrak{R}_2^2$, ..., $\mathfrak{R}_1^r R^r \mathfrak{R}_2^r$ are all fulfilled.

It is clear that any other iterated composite system derived from this same set of r systems taken in different order can differ from the one considered in notation only. It will be observed, also, that if we form the composites of groups of systems into which this set of r systems may be divided, and then take the composite of these composites, we arrive at a composite system differing only in notation from the iterated composite system first considered. Thus we see that, aside from possible differences of notation, there is a unique composite system of any finite number of systems of the type $(\mathfrak{P}; R)$.

An obvious generalization of theorem IV is:

THEOREM VI. *The composite system of a finite number of systems satisfies the seven postulates if and only if every component system satisfies them.*

As a partial generalization of theorem V we have:

THEOREM VII. *If* $(\mathfrak{P}; R)$ *is the composite of the systems* $(\mathfrak{P}^1; R^1)$, $(\mathfrak{P}^2; R^2)$, ..., $(\mathfrak{P}^r; R^r)$, *then all elements of the form* $q = q^1 q^2 \ldots . q^r$, *where at least one* q^i *is an ideal element of the corresponding system, and the remainder are elements of the corresponding classes* $\mathfrak{P}^i$, *may be regarded as ideal elements of the system* $(\mathfrak{P}; R)$, *in the sense that the class* $u = v^1_{q^1} v^2_{q^2} \ldots . v^r_{q^r}$ *(in which, if* q^i *is a* u^i, $v^i_{u^i}$ *is understood to be identical with the class* u^i*) has the properties 1–7. And every ideal element of the composite system is a composite class of the type indicated.*

CHAPTER II.

PROPERTIES OF A SYSTEM FOR WHICH IDEAL ELEMENTS ARE DEFINED.

§ 5. *Postulates for a System* $(\mathfrak{P}; \mathfrak{U}; T)$, *an Instance of which is Associated with Every System* $(\mathfrak{P}; R)$.

We have seen in Chapter I that the postulates I–VII on the system $(\mathfrak{P}; R)$ permit of an effective definition of ideal elements for the system, and that the postulates persist under the process of composition, this process being suitably defined. A body of postulates more simple in form and more general in application, yet adequate for a development of a theory of multiple and iterated limits, may be stated for a system for which a definition of ideal elements is assumed to exist independently. Special instances of systems may arise, where ideal elements may be defined in terms of special features of the system in such manner that the enlarged system shall possess all the properties that are essential for the application of our general theory of functions, but where it may be very difficult or impossible to treat the system as a special instance of a system $(\mathfrak{P}; R)$ satisfying our postulates.* For this reason, as well as for the sake of simplicity, we specify here the conditions on which we rely for the development of the following theory.

The system consists of a class $\mathfrak{P}$ of elements p, a class $\mathfrak{U}$ of ideal elements u, and a relation T between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ and elements of the class $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$. The notation for this system is $(\mathfrak{P}; \mathfrak{U}; T)$, but the symbol T is largely suppressed in practice. That a given subclass $\mathfrak{R}$ has the relation T to a given element q is indicated by $\mathfrak{R}^q$, while $\mathfrak{R}^{-q}$ indicates that the relation T does not hold between $\mathfrak{R}$ and q. A relation T is said to be defined for the classes $\mathfrak{P}$ and $\mathfrak{U}$ when a criterion exists determining for every $\mathfrak{R}$ and every q whether $\mathfrak{R}^q$ or $\mathfrak{R}^{-q}$.

Following are the postulates: †

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* Instances of this sort are found in the systems $(\mathfrak{P}; K_2)$ of § 17 and $(\mathfrak{P}; V)$ of § 18.

† We discriminate notationally between the present postulates and those of § 2 by the use of parentheses with the numerals. The postulates may be read as follows:

(I) If $\mathfrak{R}$ has the relation T to p, then p is an element of $\mathfrak{R}$.

(II) Every $\mathfrak{R}$ having the relation T to an ideal element u contains at least one element p.

(III) For every q there exists a sequence $\{\mathfrak{R}_n\}$ of classes having the relation T to q such that for every $\mathfrak{R}$ having the relation T to q there exists a number $n_{\mathfrak{R}}$ such that for $n > n_{\mathfrak{R}}$ $\mathfrak{R}_n$ is a subclass of $\mathfrak{R}$.

(VI) For every $\mathfrak{R}$ having the relation T to an element q there exists another class $\mathfrak{R}_1$ having the relation T to q such that for every p in $\mathfrak{R}_1$ there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ having the relation T to p.

(V) If q_1 is distinct from q_2 there exist classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ having the relation T to q_1 and q_2 respectively such that there is no element common to $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

(I) $\mathfrak{R}^p \,.\supset.\, p^{\mathfrak{R}}.$

(II) $\mathfrak{R}^u \,.\supset.\, \exists\, p^{\mathfrak{R}}.$

(III) $q.:\supset:.\, \exists\, \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n^q) \,.\, (\mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})].$

(IV) $R^q :\supset: \exists\, \mathfrak{R}_1^q \ni (p^{\mathfrak{R}_1} \,.\supset.\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2^p).$

(V) $q_1 \neq q_2 \,.\supset.\, \exists\, (\mathfrak{R}_1^{q_1} \,.\, \mathfrak{R}_2^{q_2}) \ni \,^{-}\exists\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}.$

A system $(\mathfrak{P}; \mathfrak{U}; T)$ may be derived from a system $(\mathfrak{P}; R)$ as follows: Let $\mathfrak{P}$ and $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ be respectively the class $\mathfrak{P}$ of the system $(\mathfrak{P}; R)$ and the class of ideal elements arising in $(\mathfrak{P}; R)$; and let a subclass $\mathfrak{R}$ have the relation T to an element p if and only if $\mathfrak{R} R p$, and let $\mathfrak{R}$ have the relation T to an ideal element u if and only if $\mathfrak{R}$ is a member of the class u which constitutes the ideal element of the system $(\mathfrak{P}; R)$.

It may easily be seen that if the system $(\mathfrak{P}; R)$ satisfies the seven postulates of § 2 the resulting system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the five postulates stated above. By the mediation of the foregoing definition of T in terms of R, either example 0 of § 2, or the example of § 3, may serve to establish the consistency of the present postulates; and in this same way examples 1, 3, 4 and 7 of § 2 serve as instances of systems satisfying respectively all but (I), all but (III), all but (IV) and all but (V). To complete the proof of the independence of the five postulates we have only to show a system failing to satisfy (II) but satisfying the remaining four postulates. Such a system is the following: $\mathfrak{P}$ is the class of positive integers, and $\mathfrak{U}$ consists of a single ideal element u. The relation $\mathfrak{R}^p$ holds if and only if $\mathfrak{R}$ contains only the single element p, while $\mathfrak{R}^u$ holds if and only if $\mathfrak{R}$ is the null class.

From two systems, $(\mathfrak{P}'; \mathfrak{U}'; T')$ and $(\mathfrak{P}''; \mathfrak{U}''; T'')$, we derive a composite system $(\mathfrak{P}; \mathfrak{U}; T)$, where $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and $\mathfrak{U}$ consists of all elements of the form $u'u''$ or $u'p''$ or $p'u''$, *i. e.*, $\mathfrak{U} = \mathfrak{U}'\mathfrak{U}'' + \mathfrak{U}'\mathfrak{P}'' + \mathfrak{P}'\mathfrak{U}''$, so that if $\mathfrak{Q}' = \mathfrak{P}' + \mathfrak{U}'$ and $\mathfrak{Q}'' = \mathfrak{P}'' + \mathfrak{U}''$ and $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$, then $\mathfrak{Q}$ is the product or composite class $\mathfrak{Q}'\mathfrak{Q}''$. The relation $\mathfrak{R}^q$ holds if and only if there exist classes $\mathfrak{R}'$ and $\mathfrak{R}''$, subclasses of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, such that $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, and $\mathfrak{R}'^{q'}$ and $\mathfrak{R}''^{q''}$, where $q = q'q''$. Obviously, this definition of the composite system is consistent with the definition employed in § 4 and the above definition of T in terms of R.

Analogous to theorem IV of Chapter I is the following theorem, the proof of which should cause no difficulty:

THEOREM I. *The composite system* $(\mathfrak{P}; \mathfrak{U}; T)$ *of two systems* $(\mathfrak{P}'; \mathfrak{U}'; T')$ *and* $(\mathfrak{P}''; \mathfrak{U}''; T'')$ *satisfies the postulates* (I)–(V) *if and only if both component systems satisfy them.*

As in § 4, the definition of the composite of two systems leads to a unique composite system for any finite number of systems of the type considered. We clearly have here a theorem analogous to theorem VI of Chapter I.

§ 6. *Limiting Elements: Postulates of F. Riesz.*

The notion of limiting element of a subclass is of primary importance for the purpose in hand. From the point of view of the present investigation, it is sufficient to define limiting elements only for subclasses* $\mathfrak{R}$ of $\mathfrak{P}$.

Def. 1. An element q is a *limiting element* of the subclass $\mathfrak{R}_1$ of $\mathfrak{P}$ if every $\mathfrak{R}$ such that $\mathfrak{R}^q$ contains an element of $\mathfrak{R}_1$ distinct from q, *i. e.*, if

$$\mathfrak{R}^q \,.\supset .\, \exists\, p \neq q \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}}.$$

If q is a p, it is an *actual* limiting element, and if a u, it is an *ideal* limiting element; and if q is an element of $\mathfrak{R}_1$, it is a *proper* limiting element of $\mathfrak{R}_1$, and if not an element of $\mathfrak{R}_1$, it is an *improper* limiting element of $\mathfrak{R}_1$. Evidently, a proper limiting element is always actual, and an ideal limiting element is always improper, but a limiting element may be for the same $\mathfrak{R}_1$ both actual and improper.

In his paper† on "Stetigkeitsbegriff und abstracte Mengenlehre" before the International Congress of Mathematicians at Rome, 1908, F. Riesz proposed a set of postulates on which to build, for an abstract class, a generalization of the theory of point-sets. He first assumes an abstract system which we may denote by $(\mathfrak{P}; C)$, where $\mathfrak{P}$ is a class of elements p, and C is a relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ and individual elements p, in the sense that p is a limiting element or element of condensation (Verdichtungstelle) of the class $\mathfrak{R}$. It is of interest here to note that if the abstract class of Riesz be identified with our class $\mathfrak{P}$, and if the relation C be assumed to hold if and only if the element p is a limiting element of $\mathfrak{R}$ by definition 1, then the resulting system is found to satisfy the postulates of Riesz. We establish this fact by proving the following theorem, the four propositions of the theorem being somewhat more general than the four conditions necessary to secure the postulates.

THEOREM II. (a) *Every limiting element of $\mathfrak{R}$ is a limiting element of every class $\mathfrak{R}_1$ containing $\mathfrak{R}$.*

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* To define limiting elements for a general subclass of $\mathfrak{Q}$ it would be necessary to resort to a situ ation analogous to that of § 3; but since the extended system $(\mathfrak{Q}; S)$ is of the same character as $(\mathfrak{P}; R)$, we should thus revert essentially to the situation found in the special case when the class $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ contains no elements. This special case furnishes a close analogue to the usual method of procedure in the matter of limits.

† *Atti*, etc., pp. 18–24.

(b) *If* $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$, *then every limiting element of* $\mathfrak{R}$ *is a limiting element either of* $\mathfrak{R}_1$ *or of* $\mathfrak{R}_2$.

(c) *Only infinite classes have limiting elements.*

(d) *Each limiting element of* $\mathfrak{R}$ *is uniquely determined by the totality of all subclasses of* $\mathfrak{R}$ *of which it is a limiting element.*

Proof: Proposition (a) is immediately evident from definition 1.

(b): Let q be a limiting element of $\overline{\mathfrak{R}} = \overline{\mathfrak{R}}_1 + \overline{\mathfrak{R}}_2$. We are to show that q is a limiting element of $\overline{\mathfrak{R}}_1$ or of $\overline{\mathfrak{R}}_2$. By postulate (III) we have

$$(1) \qquad \exists \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n^q) \,.\, (\mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})];$$

then by definition 1 we see that

$$(2) \qquad n \,.\supset.\, \exists\, p_n \ni p_n \neq q \,.\, p_n^{\mathfrak{R}_n} \,.\, p_n^{\overline{\mathfrak{R}}}.$$

A sequence $\{p_n\}$ thus secured satisfies the condition

$$(3) \qquad \mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, p_n^{\mathfrak{R}}.$$

Since either $\overline{\mathfrak{R}}_1$ or $\overline{\mathfrak{R}}_2$ must contain an infinite subsequence of $\{p_n\}$, we may suppose that $\overline{\mathfrak{R}}_1$ contains the sequence $\{p_{n_m}\}$, where, if $m_1 \neq m_2$, then $n_{m_1} \neq n_{m_2}$. For a given $\mathfrak{R}$ only a finite number of terms of the sequence $\{p_{n_m}\}$ can precede the term $p_{n_{\mathfrak{R}}}$ in the sequence $\{p_n\}$; hence by (3) we have

$$\mathfrak{R}^q \,.\supset.\, \exists\, m \ni p_{n_m}^{\mathfrak{R}},$$

and since by (2) every p_{n_m} is distinct from q, we see that q is a limiting element of $\overline{\mathfrak{R}}_1$.

(c): Let q be a limiting element of $\overline{\mathfrak{R}}$. If possible, let $\overline{\mathfrak{R}}$ consist of a finite set of elements, $p_1, p_2, \ldots, p_n$. By postulate (V) we see that for every element p_i distinct from q there exists a class $\mathfrak{R}_i$ not containing p_i such that $\mathfrak{R}_i^q$. By an application of postulate (III) we secure a class $\mathfrak{R}$ such that $\mathfrak{R}^q$, which is a common subclass of all the classes $\mathfrak{R}_i$. This class $\mathfrak{R}$ clearly contains no element of $\overline{\mathfrak{R}}$ distinct from q; hence we reach a contradiction.

(d): We are to show that if q_1 and q_2 are distinct limiting elements of $\overline{\mathfrak{R}}$, then there is a subclass of $\overline{\mathfrak{R}}$ that has one of these as limiting element but not the other. By postulate (V)

$$\exists\, (\mathfrak{R}_1^{q_1} \,.\, \mathfrak{R}_2^{q_2}) \ni \,\bar{\exists}\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}.$$

Denote by $\overline{\mathfrak{R}}_1$ the greatest common subclass of $\overline{\mathfrak{R}}$ and $\mathfrak{R}_1$; then it is clear that q_2 is not a limiting element of $\overline{\mathfrak{R}}_1$. For a given $\mathfrak{R}$ such that $\mathfrak{R}^{q_1}$ there exists by postulate (III) a common subclass $\mathfrak{R}_3$ of $\mathfrak{R}$ and $\mathfrak{R}_1$ such that $\mathfrak{R}_3^{q_1}$. Since q_1 is a limiting element of $\overline{\mathfrak{R}}$, $\mathfrak{R}_3$ must contain an element p of $\overline{\mathfrak{R}}$, distinct from q_1, and this element p is obviously in $\overline{\mathfrak{R}}_1$. Thus every $\mathfrak{R}$ such that $\mathfrak{R}^{q_1}$ contains an element of $\overline{\mathfrak{R}}_1$ distinct from q_1; that is, q_1 is a limiting element of $\overline{\mathfrak{R}}_1$.

The four postulates of F. Riesz are equivalent to the four propositions of this theorem if we restrict limiting elements q to actual limiting elements p, and if in (b) $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have no common elements.

For the purpose of introducing ideal elements into the abstract class $\mathfrak{P}$, Riesz considers a system which we may denote by $(\mathfrak{P}; V)$, where V is a relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ of the same type as our relation R. He postulates for the system $(\mathfrak{P}; V)$ four properties as follows:

(1) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the relation V, and $\mathfrak{R}_3$ contains $\mathfrak{R}_1$ and $\mathfrak{R}_4$ contains $\mathfrak{R}_2$, then $\mathfrak{R}_3$ and $\mathfrak{R}_4$ have the relation V.

(2) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the relation V, and $\mathfrak{R}_1$ is divided into two classes, then at least one of these has the relation V to $\mathfrak{R}_2$.

(3) Two singular subclasses can not have the relation V.

(4) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ both have the relation V to a given singular class p, then they have the relation V to each other.

A definition of C in terms of V is given by Riesz, by which the relation C holds for a class $\mathfrak{R}$ and an element p if and only if the class $\mathfrak{R}$ and the singular class whose element is p have the relation V. A system $(\mathfrak{P}; C)$ thus obtained from a system $(\mathfrak{P}; V)$ which satisfies the first three conditions above has the first three properties postulated for the system $(\mathfrak{P}; C)$.

From a system $(\mathfrak{P}; \mathfrak{U}; T)$ we obtain a system $(\mathfrak{P}; V)$ as follows: The class $\mathfrak{P}$ of the system $(\mathfrak{P}; V)$ is identical with the class $\mathfrak{P}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$. The relation V holds between two subclasses of $\mathfrak{P}$ if and only if the two have a common limiting element (actual or ideal) or one subclass contains a limiting element of the other.*

It is easily seen that if the system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the five postulates of § 5, then the resulting system $(\mathfrak{P}; V)$ fulfils the four conditions prescribed by Riesz. In fact, the propositions (a), (b) and (c) of theorem II contain sufficient conditions on the system $(\mathfrak{P}; \mathfrak{U}; T)$ to secure this result.

Riesz defines an ideal element as a class v of subclasses $\mathfrak{R}$ which satisfies the following conditions:

(a) If $\mathfrak{R}$ belongs to v and $\mathfrak{R}_1$ contains $\mathfrak{R}$, then $\mathfrak{R}_1$ belongs to v.

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* Compare Riesz, *loc. cit.*, p. 23.

(b) If $\mathfrak{R}$ belongs to v and consists of two subclasses, $\mathfrak{R}_1$ and $\mathfrak{R}_2$, then either $\mathfrak{R}_1$ or $\mathfrak{R}_2$ belongs to v.

(c) Every two classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of v have the relation V.

(d) The class v is not contained in any different class v_1 having properties (a), (b) and (c).

(e) No element p is contained in every $\mathfrak{R}$ of v or has the relation V to every $\mathfrak{R}$ of v.

A special case of theorem II, (d), is the proposition: Every ideal element u is uniquely determined by the totality of all subclasses of $\mathfrak{P}$ of which it is a limiting element. It is not difficult to see that such a totality of classes for a given u constitutes a class v fulfilling the five conditions just given. In fact, condition (a) follows from proposition (a) of theorem II, condition (b) from (b) of theorem II, condition (c) from the definition of V in terms of T, and conditions (d) and (e) from (d) of theorem II. Thus every element of the class $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ corresponds uniquely to an ideal element of the system $(\mathfrak{P}; V)$.

It may be observed that by the mediation of the definition of a system $(\mathfrak{P}; \mathfrak{U}; T)$ in terms of a system $(\mathfrak{P}; R)$ there is associated with every system $(\mathfrak{P}; R)$ a definite system $(\mathfrak{P}; V)$, and that if the former satisfies the seven postulates of § 2, the latter must fulfil the conditions stated by Riesz. It is clear, also, that every ideal element arising in the system $(\mathfrak{P}; R)$ by our definition corresponds uniquely to an ideal element arising in the associated system $(\mathfrak{P}; V)$ by the definition given by Riesz.

§ 7. *The Fréchet Limit: Properties of Classes.*

In his thesis, "Sur quelques points du calcul fonctionnel," Paris, 1906, M. Fréchet* makes use of an undefined relation between sequences of elements and individual elements. By imposing certain conditions on this relation he is able to develop a theory, analogous to the theory of point-sets and of continuous functions, in which an element that has the undefined relation to a sequence plays the rôle of limit of the sequence. It is of advantage here to show that the notion of limit of a sequence of elements as defined below satisfies the conditions stated by Fréchet.

Def. 2. The sequence $\{p_n\}$ has the limit q if and only if for every $\mathfrak{R}$ such that $\mathfrak{R}^q$ there is a term of the sequence such that all following terms are in the class $\mathfrak{R}$. In symbols: †

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* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII.

† The notation $\lim_{n=\infty} p_n = q$ is here replaced by the more convenient but equally suggestive notation $L_n p = q$. Note that a sequence may have an ideal element as limit.

$$L\,p_n = q \;.:\; \equiv \;:.\; (\{p_n\} \,.\, q) \ni (\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\, \supset .\, p_n^{\mathfrak{R}}).$$

The conditions stated by Fréchet for the definition of the limit relation between sequence and element are implied by the following theorem:

THEOREM III. (a) *A sequence formed by repeating a single element has that element for limit.*

(b) *A sequence can not have two distinct limits.*

(c) *If a sequence* $\{p_n\}$ *has a limit* q, *then every subsequence* $\{p_{n_m}\}$ *such that* n_m *becomes infinite with* m *has the limit* q.

Proof: Proposition (a) is an immediate consequence of postulate (I).

(b): Suppose a sequence $\{p_n\}$ to have two distinct limits, q_1 and q_2. By postulate (V) we have

$$\exists\, (\mathfrak{R}_1^{q_1} \,.\, \mathfrak{R}_2^{q_2}) \ni -\exists\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}.$$

But by the supposition

$$\exists\, n_{\mathfrak{R}_1} \ni n > n_{\mathfrak{R}_1} \,.\, \supset .\, p_n^{\mathfrak{R}_1} \quad \text{and} \quad \exists\, n_{\mathfrak{R}_2} \ni n > n_{\mathfrak{R}_2} \,.\, \supset .\, p_n^{\mathfrak{R}_2}.$$

By considering p_n such that n is greater than both $n_{\mathfrak{R}_1}$ and $n_{\mathfrak{R}_2}$, we reach a contradiction.

(c): By hypothesis we have

$$\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\, \supset .\, p_n^{\mathfrak{R}},$$

and since n_m becomes infinite with m there is for every $n_{\mathfrak{R}}$ a number $m_{\mathfrak{R}}$ such that if m is greater than $m_{\mathfrak{R}}$, then n_m is greater than $n_{\mathfrak{R}}$; thus

$$\mathfrak{R}^q : \supset : \exists\, m_{\mathfrak{R}} \ni m > m_{\mathfrak{R}} \,.\, \supset .\, p_{n_m}^{\mathfrak{R}},$$

which is the required condition.

The three propositions of the theorem are equivalent to the properties of the Fréchet limit, except that he uses instead of (c) a less restrictive condition, obtained by adding to the hypothesis of (c) the restriction that the elements of the subsequence are taken in the same order as in the original sequence. In the proof just given we made use only of postulates (I) and (V), so that these two conditions on a system $(\mathfrak{P}; \mathfrak{U}; T)$ are sufficient for the development of a theory at least as extensive as that pertaining to the class (L) of Fréchet.*

* Fréchet denotes by (L) what we should represent by $(\mathfrak{P}; L)$, where L is a relation between sequences of elements of $\mathfrak{P}$ and individual elements of $\mathfrak{P}$. Observe that the limit relation which we have defined differs in type from that of Fréchet to the extent that we include ideal limiting elements. One sees, however, that the presence of ideal elements does not interfere with the application to the present situation of the theorems proved by Fréchet on the basis of his limit relation and without the aid of his *écart* or his *voisinage*.

THEOREM IV. *A necessary and sufficient condition that q shall be a limiting element of $\mathfrak{R}$ is that there exist a sequence $\}p_n\{$ of distinct elements of $\mathfrak{R}$ such that $L\,p_n = q$.*

It is necessary: By postulate (III) we have

$$\exists \,\}\mathfrak{R}_n\{ \ni [(n \,.\supset.\, \mathfrak{R}_n^q) \,.\, (\mathfrak{R}^q :\supset: \exists\, n_{\bar{\mathfrak{R}}} \ni n > n_{\bar{\mathfrak{R}}} \,.\supset.\, \mathfrak{R}_n^{\bar{\mathfrak{R}}})],$$

and, q being a limiting element of $\mathfrak{R}$, we have by definition 1,

$$n. \supset . \exists\, p_n \neq q \ni p_n^{\mathfrak{R}_n} \,.\, p_n^{\mathfrak{R}}.$$

The sequence $\}p_n\{$ thus secured is such that $L\,p_n = q$, and since the elements of the sequence constitute a class having the limiting element q, we see by theorem II, (c), that the number of distinct elements of the sequence is not finite. There exists, then, an infinite subsequence $\}p_{n_m}\{$ of distinct terms such that n_m becomes infinite with m, and which, by theorem III, (c), has the limit q.

It is sufficient: This proposition is a direct result of proposition (a) of theorem II.

Theorem IV shows that our definition of limiting element of a subclass is consistent with the definition employed by Fréchet. It may be noticed that in establishing these relations with the work of Fréchet, and the relations to the work of Riesz discussed in the previous section, no use has been made of postulate (IV); it is clear, therefore, that while we have sacrificed much in the matter of generality, we gain somewhat in the extent of the theory available for our system. We consider here certain properties of subclasses that are found useful in the next chapter.

Def. 3. The *derived class* of a subclass $\mathfrak{R}$ is the class of all limiting elements of $\mathfrak{R}$.

Def. 4. A subclass is *closed* if it contains its derived class.

Def. 5. A subclass $\mathfrak{R}$ is *compact* * if every infinite subclass of $\mathfrak{R}$ has at least one limiting element.

The propositions of the following theorem, which are given by Fréchet, are seen to be valid here, his proof of (d) being entirely applicable to the present situation, and the first three propositions being obvious deductions from the definition of compactness.

* See Fréchet, *loc. cit.*, p. 6. Here, again, attention must be called to the fact that we recognize ideal limiting elements.

THEOREM V. (a) *Every subclass of a compact class is compact.*

(b) *If every subclass of $\mathfrak{R}$ is compact, then $\mathfrak{R}$ is compact.*

(c) *A class formed of a finite number of compact classes is compact.*

(d) *If every member of a sequence $\{\mathfrak{R}_n\}$ of subclasses of a compact class $\mathfrak{R}$ is closed, contains the succeeding member, and contains at least one element, then there is an element common to all classes of the sequence.*

A proposition somewhat different in content from this last, but permitting of a very similar proof, is stated by Riesz,* and may be stated here as follows:

THEOREM VI. *If every member of a sequence $\{\mathfrak{R}_n\}$ of infinite subclasses of a compact class $\mathfrak{R}$ contains the succeeding member, then the members of the sequence possess at least one common limiting element.*

An important proposition in the theory of point-sets and in the analogous theories† in the domain of general analysis is the following: "The derived class of every subclass is closed." The following theorem, in the proof‡ of which we find the first use for postulate (IV), reduces to this proposition in case no ideal elements exist, *i. e.*, in case $\mathfrak{U}$ is the null class.

THEOREM VII. *If $\overline{\mathfrak{R}}_1$ is the class of all actual limiting elements of $\overline{\mathfrak{R}}$, then every limiting element of $\overline{\mathfrak{R}}_1$ is a limiting element of $\overline{\mathfrak{R}}$.*

Proof: By postulate (IV) we have for a given limiting element q of $\overline{\mathfrak{R}}_1$,

$$(1) \qquad \mathfrak{R}^q :\supset: \exists\, \mathfrak{R}_1^q \ni p^{\mathfrak{R}_1} .\supset. \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2^p .$$

Now such a class $\mathfrak{R}_1$ must contain an element p of $\overline{\mathfrak{R}}_1$ distinct from q. Then there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ such that $\mathfrak{R}_2^p$, and p being a limiting element of $\overline{\mathfrak{R}}$,

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* *Loc. cit.*, p. 20.

† E. R. Hedrick, "On Properties of a Domain for which Any Derived Set is Closed," *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 289.

‡ Fréchet shows (*loc. cit.*, p. 15) that this proposition does not follow from the hypotheses he has made on the class (L). He secures this theorem only after the introduction of the notion of *voisinage*. The following example shows that the theorem is not a consequence of postulates (I), (II), (III) and (V), which, as we have shown, are together as strong as the postulates on the class (L) combined with the postulates of Riesz. We specify a system $(\mathfrak{P};\ \mathfrak{U};\ T)$ as follows: $\mathfrak{P}$ consists of an element p, a sequence $\{p_n\}$ of elements, and a double sequence $\{p_{mn}\}$ of elements. Two elements having different notations are distinct. $\mathfrak{U}$ is the null class. The relation $\mathfrak{R}^p$ holds if and only if $\mathfrak{R}$ consists of the element p together with all, excepting a finite number, of the elements of the sequence $\{p_n\}$. For a given n, the relation $\mathfrak{R}^{p_n}$ holds if and only if $\mathfrak{R}$ consists of the element p_n together with all, excepting a finite number, of the elements of the simple sequence $\{p_{mn}\}$. For a given m and n, $\mathfrak{R}^{p_{mn}}$ holds if and only if $\mathfrak{R}$ consists of the single element p_{mn}. This system satisfies postulates (I), (II), (III) and (V). For a given n the element p_n is the limit of the sequence $\{p_{mn}\}$; and the sequence $\{p_n\}$ has the limit p. The subclass $\mathfrak{R}$ consisting of the elements of the double sequence $\{p_{mn}\}$ has for its derived class the class $\mathfrak{R}_1$, which consists of the elements of the sequence $\{p_n\}$. The only limiting element of $\mathfrak{R}_1$ is p, which is not in $\mathfrak{R}_1$. Thus the derived class $\mathfrak{R}_1$ is not closed.

there is by theorem IV a sequence of distinct elements of $\overline{\mathfrak{R}}$ in $\mathfrak{R}_2$, and therefore in $\mathfrak{R}$. Clearly then we have

$$\mathfrak{R}^q \cdot \supset \cdot \exists\, p \neq q \ni p^{\overline{\mathfrak{R}}} \cdot p^{\mathfrak{R}};$$

that is, q is a limiting element of $\overline{\mathfrak{R}}$.

The proofs of the following propositions relative to a composite system should cause no difficulty.

THEOREM VIII. *If* $(\mathfrak{P}; \mathfrak{U}; T)$ *is the composite system of the systems* $(\mathfrak{P}^1; \mathfrak{U}^1; T^1)$, $(\mathfrak{P}^2; \mathfrak{U}^2; T^2)$, ..., $(\mathfrak{P}^r; \mathfrak{U}^r; T^r)$, *and if* $\mathfrak{R} = \mathfrak{R}^1 \mathfrak{R}^2 \ldots\ldots \mathfrak{R}^r$, *and* $q = q^1 q^2 \ldots\ldots q^r$, *the following propositions hold:*

(a) *If, for every* n, $p_n = p_n^1 p_n^2 \ldots\ldots p_n^r$, *then* $L\, p_n = q$ *if and only if for* $i = 1, 2, \ldots\ldots, r$ *we have* $L\, p_n^i = q^i$.

(b) *The element* q *is a limiting element of* $\mathfrak{R}$ *if and only if every* q^i $(i = 1, 2, \ldots\ldots, r)$ *is contained in* $\mathfrak{R}^i$ *or is a limiting element of* $\mathfrak{R}^i$, *and at least one of the* q^i *is a limiting element of the corresponding* $\mathfrak{R}^i$.

(c) $\mathfrak{R}$ *is closed if and only if every* $\mathfrak{R}^i$ *is closed* $(i = 1, 2, \ldots\ldots, r)$.

(d) $\mathfrak{R}$ *is compact if and only if every* $\mathfrak{R}^i$ *is compact* $(i = 1, 2, \ldots\ldots, r)$.

CHAPTER III.

A Theory of Functions Based on Properties of a System $(\mathfrak{P}; \mathfrak{U}; T)$.

§ 8. *Introductory.*

In this chapter we are concerned with a definite system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, which is assumed to satisfy the five postulates of § 5. In § 10 and § 12, in order to provide for the features of iterated limits, special hypotheses are made with regard to the composite character of the system. Subclasses of $\overline{\mathfrak{P}}$ are, in general, denoted by $\mathfrak{R}$; but a definite one of these subclasses, denoted by $\mathfrak{P}$, receives special consideration, being the range of the independent variable in our theory of functions. The derived class of $\mathfrak{P}$ is denoted by $\mathfrak{L}$, and $\mathfrak{Q}$ is the least common superclass of $\mathfrak{P}$ and $\mathfrak{L}$. The letters p, q and l denote elements of the respective classes $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{L}$, while a general element of $\overline{\mathfrak{P}}$ is written $\bar{p}$, and $\bar{q}$ is an element of the class $\overline{\mathfrak{Q}} = \overline{\mathfrak{P}} + \mathfrak{U}$.

Functions are denoted by the letters θ, ϕ, μ, etc. The notion of function, in general, involves two classes, one called the range and the other the class to which the function-values belong. If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of elements, then a function θ on $\mathfrak{X}$ to $\mathfrak{Y}$ is a correspondence between elements of $\mathfrak{X}$ and subclasses of $\mathfrak{Y}$, whereby to every element x of $\mathfrak{X}$ there corresponds uniquely a subclass θ_x of $\mathfrak{Y}$. If, for every x, θ_x consists of a single element y of $\mathfrak{Y}$, then θ is a single-valued function on $\mathfrak{X}$ to $\mathfrak{Y}$. In the present chapter we consider only single-valued functions on a subclass of $\overline{\mathfrak{Q}}$ to $\hat{\mathfrak{A}}$, where $\hat{\mathfrak{A}}$ is the class of all real numbers with the ideal elements $+\infty$ and $-\infty$ adjoined. The notation $\mathfrak{A}$ stands for the class of all real numbers, and the letter a invariably represents a real number, while the letters e and d always denote positive real numbers.*

§ 9. *The Character of a Function in the Neighborhood of an Element.*

We take as the subject of discussion a definite function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$. In the consideration of the character of the function μ with respect to a particular limiting element l of the range, three symbols play an important rôle:

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* The letters e and d replace the usual $\epsilon > 0$ and $\delta > 0$. There can be no doubt that convenience and economy are conserved by these and other special notations. We find sufficient precedent in the work of Professor E. H. Moore on "General Analysis."

(a) $\overline{\lim\limits_{p \to l}}\, \mu_p$: upper limit of μ_p as p approaches l.

(b) $\underline{\lim\limits_{p \to l}}\, \mu_p$: lower limit of μ_p as p approaches l.

(c) $\lim\limits_{p \to l} \mu_p$: limit of μ_p as p approaches l.

Following are the conditions under which these symbols may represent definite finite numbers:

Def. 1. The relation $\overline{\lim\limits_{p \to l}}\, \mu_p = a$ is equivalent to the following two conditions* on μ, l and a:

$$(1) \quad e : \supset : \exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l . \supset . \mu_p \leqq a + e,$$
$$(2) \quad e . \mathfrak{R}^l : \supset : \exists\, p^{\mathfrak{R}} \neq l \ni \mu_p \geqq a - e.$$

Def. 2. The relation $\underline{\lim\limits_{p \to l}}\, \mu_p = a$ is equivalent to the following two conditions on μ, l and a:

$$(1) \quad e : \supset : \exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l . \supset . \mu_p \geqq a - e,$$
$$(2) \quad e . \mathfrak{R}^l : \supset : \exists\, p^{\mathfrak{R}} \neq l \ni \mu_p \leqq a + e.$$

Def. 3. The relation $\lim\limits_{p \to l} \mu_p = a$ is equivalent to the following condition on μ, l and a:

$$e : \supset : \exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l . \supset . |\mu_p - a| \leqq e.$$

These definitions lead to the obvious theorem:

THEOREM I. *The limit of μ_p as p approaches a limiting element l is a number a if and only if the upper limit and the lower limit of μ_p as p approaches l are both equal to this same number a.* In symbols:

$$\lim_{p \to l} \mu_p = a : \backsim : \overline{\lim_{p \to l}}\, \mu_p = \underline{\lim_{p \to l}}\, \mu_p = a.$$

For a given limiting element l of $\mathfrak{P}$ we have also the following analogue of the Cauchy condition for convergence:

THEOREM II. *The following two conditions are necessary and sufficient for the existence of a finite limit of μ_p as p approaches l:*

(1) *If $\mathfrak{R}^l$ there exists a p in $\mathfrak{R}$ distinct from l such that μ_p is finite.*

(2) $e : \supset : \exists\, \mathfrak{R}_e^l \ni p_1^{\mathfrak{R}_e} \neq l . p_2^{\mathfrak{R}_e} \neq l . \supset . |\mu_{p_1} - \mu_{p_2}| \leqq e.$

Proof: That the conditions are necessary is quite obvious. As to their being sufficient, we observe that since l is a limiting element of $\mathfrak{P}$ we have a sequence $\{p_n\}$ of distinct elements of $\mathfrak{P}$ such that

* Condition (1) may be read: "For every e there exists a class $\mathfrak{R}_e$ such that $\mathfrak{R}_e^l$ ($\mathfrak{R}_e$ has the relation T to l) and such that for every p in $\mathfrak{R}_e$ distinct from l the function-value μ_p is less than or equal to $a+e$"; and (2) may be read: "For every e and every $\mathfrak{R}$ such that $\mathfrak{R}^l$ there exists a p in $\mathfrak{R}$ distinct from l such that the function-value μ_p is greater than or equal to $a-e$." The conditions in definitions 2 and 3 may be read in similar fashion.

(1) $$\mathfrak{R}^l : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\, \supset \,.\, p_n^{\mathfrak{R}}.$$

If for a given e we consider the class $\mathfrak{R}_e$ required to exist by condition (2) of the theorem, we see that

$$\exists\, n_e \ni n_1 > n_e \,.\, n_2 > n_e \,.\, \supset \,.\, |\mu_{p_{n_1}} - \mu_{p_{n_2}}| \leqq e.$$

By condition (1) of the theorem the terms of the sequence $\{\mu_{p_n}\}$, after a finite number of terms, are all real numbers; therefore the sequence is convergent to a finite limit by the condition just written. Let this limit be a, then

(2) $$e : \supset : \exists\, n_e \ni n > n_e \,.\, \supset \,.\, |\mu_{p_n} - a| \leqq \frac{e}{2},$$

and by condition (2) of the theorem we have for this same e

(3) $$\exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l \,.\, p_n^{\mathfrak{R}_e} \neq l \,.\, \supset \,.\, |\mu_p - \mu_{p_n}| \leqq \frac{e}{2}.$$

Since, by (1), this conclusion is fulfilled for some value of n, we see that, by (2) and (3),

$$p^{\mathfrak{R}_e} \neq l \,.\, \supset \,.\, |\mu_p - a| \leqq e;$$

that is, the limit of μ_p as p approaches l is a.

Theorem III. *If for a given l there exist a number a and a class $\mathfrak{R}$ such that $\mathfrak{R}^l$ and such that, for every p in R, $|\mu_p| \leqq a$, then there exist numbers $\bar{a}$ and $\underline{a}$ such that*

$$\overline{\lim_{p \to l}}\, \mu_p = \bar{a} \text{ and } \underline{\lim_{p \to l}}\, \mu_p = \underline{a}.$$

Proof: By use of postulate (III) we secure a sequence $\{\mathfrak{R}_n\}$ such that for every n we have $\mathfrak{R}_n^l$ and $\mathfrak{R}_n^{\mathfrak{R}}$, and further, such that each term of the sequence is contained in the preceding term. For every n let $\bar{a}_n$ be the least upper bound of μ_p, where p belongs to $\mathfrak{R}_n$ and is distinct from l, and let $\underline{a}_n$ be the greatest lower bound of μ_p with the same restrictions on p; then the sequences $\{\bar{a}_n\}$ and $\{\underline{a}_n\}$ are respectively decreasing and increasing monotonic sequences of real numbers. Since $\bar{a}_n \geqq \underline{a}_n$, both sequences converge. Let $\bar{a}$ and $\underline{a}$ be the limits of these sequences, then it is clear that they are respectively the upper limit and the lower limit of μ_p as p approaches l.

In the following definition, in which we employ the symbol, $\equiv$, of definitional equivalence, we indicate the conditions under which the limit, upper limit and lower limit of μ_p as p approaches a limiting element l may be infinite elements of the class $\mathfrak{A}$.

Def. 4. With respect to an arbitrary limiting element l of $\mathfrak{P}$ the following are definitional identities:

(a) $\overline{\lim\limits_{p \to l}}\, \mu_p = +\infty : \equiv : a \,.\, \mathfrak{R}^l \,.\, \supset \,.\, \exists\, p^{\mathfrak{R}} \neq l \ni \mu_p > a,$

(b) $\underline{\lim\limits_{p \to l}}\, \mu_p = -\infty : \equiv : a \,.\, \mathfrak{R}^l \,.\, \supset \,.\, \exists\, p^{\mathfrak{R}} \neq l \ni \mu_p < a,$

(c) $\underline{\lim\limits_{p \to l}}\, \mu_p = +\infty \,.\!: \equiv :\!.\, \lim\limits_{p \to l} \mu_p = +\infty \,.\!: \equiv :\!.\, a : \supset : \exists\, \mathfrak{R}^l_a \ni p^{\mathfrak{R}_a} \neq l \,.\, \supset .\, \mu_p > a,$

(d) $\overline{\lim\limits_{p \to l}}\, \mu_p = -\infty \,.\!: \equiv :\!.\, \lim\limits_{p \to l} \mu_p = -\infty \,.\!: \equiv :\!.\, a : \supset : \exists\, \mathfrak{R}^l_a \ni p^{\mathfrak{R}_a} \neq l \,.\, \supset .\, \mu_p < a.$

THEOREM IV. *If $\{p_n\}$ is a sequence such that $Lp_n = l$, then*

$$\underline{\lim_{p \to l}}\, \mu_p \leqq \underline{\lim_{n \to \infty}}\, \mu_{p_n} \leqq \overline{\lim_{n \to \infty}}\, \mu_{p_n} \leqq \overline{\lim_{p \to l}}\, \mu_p .$$

These four symbols clearly always represent definite elements of the class $\hat{\mathfrak{A}}$; *i. e.*, "the quantities always exist, finite or infinite." The proof of the theorem follows immediately from theorem III and the foregoing definitions, with some use of well-known properties of sequences of real numbers.

THEOREM V. *For a given l there exist sequences $\{p_n\}$ of distinct elements such that $Lp_n = l$ satisfying each of the conditions:*

$$\text{(a)} \quad \lim_{n \to \infty} \mu_{p_n} = \overline{\lim_{p \to l}}\, \mu_p ,$$

$$\text{(b)} \quad \lim_{n \to \infty} \mu_{p_n} = \underline{\lim_{p \to l}}\, \mu_p .$$

Proof of (a): First, suppose that $\overline{\lim\limits_{p \to l}}\, \mu_p = +\infty$. Consider a sequence $\{a_n\}$ of real numbers such that $\lim\limits_{n \to \infty} a_n = +\infty$, and a sequence $\{\mathfrak{R}_n)$ as required to exist for l by postulate (III). We have by definition 4, (a),

$$n \,.\, \supset \,.\, \exists\, p_n^{\mathfrak{R}_n} \neq l \ni \mu_{p_n} > a_n .$$

A sequence $\{p_n\}$ so secured has the desired properties, namely, $Lp_n = l$ and $\lim\limits_{n \to \infty} \mu_{p_n} = +\infty$.

Next, suppose that $\overline{\lim\limits_{p \to l}}\, \mu_p = a$. Consider a sequence $\{e_n\}$ of positive real numbers such that $\lim\limits_{n \to \infty} e_n = 0$. By condition (1) of definition 1,

$$(1) \qquad n : \supset : \exists\, \mathfrak{R}^l_{e_n} \ni p^{\mathfrak{R}_{e_n}} \neq l \,.\, \supset .\, \mu_p \leqq a + e_n .$$

Take a sequence $\{\mathfrak{R}_n\}$ such that for every n $\mathfrak{R}^l_n$ and $\mathfrak{R}^{\mathfrak{R}_{e_n}}_n$, and such that if $\mathfrak{R}^l$ the classes $\mathfrak{R}_n$, after a finite number, are all contained in $\mathfrak{R}$, all of which may be done by means of postulate (III). By condition (2) of definition 1 we have

$$(2) \qquad n : \supset : \exists\, p_n^{\mathfrak{R}_n} \neq l \ni \mu_{p_n} \geqq a - e_n .$$

Clearly, $Lp_n = l$, and since p_n is an element of $\mathfrak{R}_{e_n}$, we have by (1) and (2)

$$n : \supset : | \mu_{p_n} - a | \leqq 2\, e_n ,$$

and, the limit of e_n being zero, it is seen that the limit of μ_{p_n} is a, so that the sequence $\{p_n\}$ is of the kind desired.

In case $\overline{\lim\limits_{p \to l}}\, \mu_p = -\infty$ the proof is entirely obvious, and an entirely analogous proof is available for the condition (b).

THEOREM VI. (a) *For a given limiting element l, $\lim\limits_{p \to l} \mu_p = a$ if and only if for every sequence $\{p_n\}$ of distinct elements such that $L\, p_n = l$ it is true that $\lim\limits_{n \to \infty} \mu_{p_n} = a$.*

(b) *Proposition* (a) *still holds if a be replaced by $+\infty$ or by $-\infty$.*

This theorem* is an immediate consequence of Theorems IV and V.

§ 10. *Iterated Limits at an Element of a Composite Range.*

Let the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ be the composite of two systems, $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$, and let $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, where $\mathfrak{P}'$ and $\mathfrak{P}''$ are subclasses of $\overline{\mathfrak{P}}'$ and $\overline{\mathfrak{P}}''$ respectively. Let $\mathfrak{L}'$ and $\mathfrak{L}''$ be the derived classes of $\mathfrak{P}'$ and $\mathfrak{P}''$, and let $\mathfrak{Q}' = \mathfrak{P}' + \mathfrak{L}'$ and $\mathfrak{Q}'' = \mathfrak{P}'' + \mathfrak{L}''$; then we have for the derived class of $\mathfrak{P}$ the class $\mathfrak{L} = \mathfrak{L}'\mathfrak{Q}'' + \mathfrak{Q}'\mathfrak{L}''$, while $\mathfrak{Q} = \mathfrak{P} + \mathfrak{L} = \mathfrak{Q}'\mathfrak{Q}''$.

We consider again a definite function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$. For every p' the symbol $\mu_{p'}$ represents a function on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$, having for every p'' the function-value $\mu_{p'p''}$. Similarly, for every p'' $\mu_{p''}$ is a function on $\mathfrak{P}'$ to $\hat{\mathfrak{A}}$. The symbols

$$\overline{\lim_{p' \to l'}}\, \mu_{p'p''}, \quad \underline{\lim_{p' \to l'}}\, \mu_{p'p''}, \quad \overline{\lim_{p'' \to l''}}\, \mu_{p'p''}, \quad \underline{\lim_{p'' \to l''}}\, \mu_{p'p''}$$

represent definite functions, the first two being defined on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ and the last two on $\mathfrak{P}'$ to $\hat{\mathfrak{A}}$.

THEOREM VII. *For a given limiting element $l = l'l''$ we have*

$$\text{(a)} \quad \underline{\lim_{p'' \to l''}}\ \underline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \underline{\lim_{p'' \to l''}}\ \overline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \overline{\lim_{p'' \to l''}}\ \overline{\lim_{p' \to l'}}\, \mu_{p'p''},$$

$$\text{(b)} \quad \underline{\lim_{p'' \to l''}}\ \underline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \overline{\lim_{p'' \to l''}}\ \underline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \overline{\lim_{p'' \to l''}}\ \overline{\lim_{p' \to l'}}\, \mu_{p'p''}.$$

It is clear that the symbols indicated always represent definite elements of the class $\hat{\mathfrak{A}}$, and the inequalities follow easily from theorem IV.

THEOREM VIII. *If $l = l'l''$, then there exists a sequence $\{p_n\}$ of distinct elements such that*

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* Compare Alfred Pringsheim, "Zur Theorie der zweifach unendlichen Zahlenfolgen," *Mathematische Annalen*, Vol. LIII (1900), p. 301; also Franz London, "Ueber Doppelfolgen und Doppelreihen," same volume, p. 330. Analogies between the theorems of § 9 and § 10 of the present paper and theorems by Pringsheim and London in the papers here cited, analogies extending in some instances to the very details of the proofs, are so apparent that they call for no special notice. A method of specializing the present general theory to secure a theory of multiple sequences is shown in § 14.

$$L\,p_n = l \textit{ and } \lim_{n\to\infty} \mu_{p_n} = \overline{\lim_{p''\to l''}}\ \overline{\lim_{p'\to l'}}\ \mu_{p'p''},$$

and the proposition remains true if the iterated upper limit be replaced by any of the three quantities

$$\underline{\lim_{p''\to l''}}\ \overline{\lim_{p'\to l'}}\ \mu_{p'p''}, \qquad \overline{\lim_{p''\to l''}}\ \underline{\lim_{p'\to l'}}\ \mu_{p'p''}, \qquad \underline{\lim_{p''\to l''}}\ \underline{\lim_{p'\to l'}}\ \mu_{p'p''}.$$

We indicate the proof for the case of the iterated upper limit when this is a finite number a. Let θ be a function on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that for every p'' $\theta_{p''} = \overline{\lim\limits_{p'\to l'}}\,\mu_{p'p''}$; then $\overline{\lim\limits_{p''\to l''}}\,\theta_{p''} = a$, and by theorem V there is a sequence $\{p''_n\}$ of distinct elements such that $L\,p''_n = l''$ and such that

(1) $$e : \supset : \exists\, n_e \ni n > n_e \,.\supset.\, |\,\theta_{p''_n} - a\,| \leqq \frac{e}{2}.$$

We may regard the sequence $\{p''_n\}$ as chosen so that $\theta_{p''_n}$ is finite for every n; then again, by theorem V, there exists for every n a sequence $\{p'_{mn}\}$ of distinct elements such that $L\,p'_{mn} = l'$ and such that $\lim\limits_{m\to\infty} \mu_{p'_{mn}p''_n} = \theta_{p''_n}$. Take a sequence $\{e_n\}$ such that the limit of e_n is zero. We have

(2) $$n : \supset : \exists\, m_{e_n} \ni m > m_{e_n} \,.\supset.\, |\,\mu_{p'_{mn}p''_n} - \theta_{p''_n}\,| \leqq e_n.$$

Consider a sequence $\{\mathfrak{R}'_n\}$ such as is required to exist for l' by postulate (III). Take a sequence $\{m_n\}$ such that for every n $p'_{m_n n}$ is in $\mathfrak{R}'_n$ and $m_n > m_{e_n}$; then clearly $L\,p'_{m_n n} = l'$ and by (2) we have

(3) $$n \,.\supset \qquad {}_{m_n n}{}_{p''_n} - \theta_{p''_n}\,| \leqq e_n.$$

For a given e we may take $\bar{n}_e$ so that $e_n \leqq \frac{e}{2}$ for $n > \bar{n}_e$; therefore we see by (1) and (2) that $\lim\limits_{n\to\infty} \mu_{p'_{m_n n}p''_n} = a$. Take a sequence $\{p_n\}$ such that for all values of n $p_n = p'_{m_n n}\,p''_n$; then by theorem VIII, (a), of Chapter II, we have $L\,p_n = l$, and also $\lim\limits_{n\to\infty} \mu_{p_n} = a$.

The proofs of the remaining cases offer no new difficulties, and are omitted in the interests of brevity.

THEOREM IX. *If* $l = l'\,l''$, *then*

(a) $$\underline{\lim_{p\to l}}\ \mu_p \leqq \underline{\lim_{p''\to l''}}\ \underline{\lim_{p'\to l'}}\ \mu_{p'p''} \leqq \overline{\lim_{p''\to l''}}\ \overline{\lim_{p'\to l'}}\ \mu_{p'p''} \leqq \overline{\lim_{p\to l}}\ \mu_p,$$

(b) *If the limit of* μ_p *as* p *approaches* l *exists,* then*

$$\lim_{p''\to l''}\ \overline{\lim_{p'\to l'}}\ \mu_{p'p''} = \lim_{p''\to l''}\ \underline{\lim_{p'\to l'}}\ \mu_{p'p''} = \lim_{p\to l} \mu_p.$$

* The term "exists" is employed in the usual sense, to indicate that the limit is defined, *i. e.*, represents a definite real number or $+\infty$ or $-\infty$. The theorem may be regarded as comparing iterated limits with multiple limits; we can not, however, adopt with consistency any special notation to indicate that a limit is "multiple," since no class is assumed to be linear, and the limits studied in § 9 may be regarded as multiple limits of any order desired.

The propositions of this theorem are easy deductions from theorems V, VI, VII and VIII.

We have just derived, in terms of iterated limits, a necessary condition for the existence of the limit of μ_p as p approaches a limiting element l. To derive sufficient conditions on the iterated limits for the existence of the limit is a matter calling for additional restrictions. For this purpose it is convenient to introduce certain uniformity features. We indicate that a condition is satisfied uniformly with respect to the elements of a class by placing the notation for the class in parentheses following the symbol for the condition. Thus, if $\overline{\mathfrak{R}}''$ is a subclass of $\mathfrak{P}''$, and θ is a function on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$, finite for every p'' in $\overline{\mathfrak{R}}''$, and such that $\lim\limits_{p' \to l'} \mu_{p'p''} = \theta_{p''}$ for every p'', then we may indicate that the limit function θ is approached uniformly on $\overline{\mathfrak{R}}''$ as in the following definition:

Def. 5. $\lim\limits_{p' \to l'} \mu_{p'} = \theta(\overline{\mathfrak{R}}'') .:\equiv:. e : \supset : \exists\, \mathfrak{R}_e'^{\,l'} \ni p'^{\mathfrak{R}_e'} \neq l' . p''^{\overline{\mathfrak{R}}''} . \supset . |\mu_{p'p''} - \theta_{p''}| \leqq e.$

It will be observed that the uniformity of the condition consists in the existence for a given e of a single $\mathfrak{R}_e'$ effective for every p'' in $\overline{\mathfrak{R}}''$. Similarly we write:

Def. 6. (a) $\lim\limits_{p' \to l'} \mu_{p'} = +\infty\ (\overline{\mathfrak{R}}'') .:\equiv:. a : \supset : \exists\, \mathfrak{R}_a'^{\,l'} \ni p'^{\mathfrak{R}_a'} \neq l' . p''^{\overline{\mathfrak{R}}''} . \supset . \mu_{p'p''} > a,$

(b) $\lim\limits_{p' \to l'} \mu_{p'} = -\infty\ (\overline{\mathfrak{R}}'') .:\equiv:. a : \supset : \exists\, \mathfrak{R}_a'^{\,l'} \ni p'^{\mathfrak{R}_a'} \neq l' . p''^{\overline{\mathfrak{R}}''} . \supset . \mu_{p'p''} < a.$

THEOREM X. *If $l = l'l''$ and $\mathfrak{R}''$ is a class such that $\mathfrak{R}''^{\,l''}$, and if $\overline{\mathfrak{R}}''$ is the greatest common subclass of $\mathfrak{R}''$ and $\mathfrak{P}''$, then the following propositions hold:*

(a) $\lim\limits_{p' \to l'} \mu_{p'} = \theta(\overline{\mathfrak{R}}'') . \supset . \overline{\lim\limits_{p \to l}}\, \mu_p = \overline{\lim\limits_{p'' \to l''}}\, \theta_{p''} . \underline{\lim\limits_{p \to l}}\, \mu_p = \underline{\lim\limits_{p'' \to l''}}\, \theta_{p''},$

(b) $\lim\limits_{p' \to l'} \mu_{p'} = +\infty\ (\overline{\mathfrak{R}}'') . \supset . \lim\limits_{p \to l} \mu_p = +\infty,$

(c) $\lim\limits_{p' \to l'} \mu_{p'} = -\infty\ (\overline{\mathfrak{R}}'') . \supset . \lim\limits_{p \to l} \mu_p = -\infty;$

provided that in case l' is a p' we have in the respective cases the additional hypotheses

(a) $\underline{\lim\limits_{p'' \to l''}}\, \theta_{p''} \leqq \underline{\lim\limits_{p'' \to l''}}\, \mu_{l'p''}$ *and* $\overline{\lim\limits_{p'' \to l''}}\, \mu_{l'p''} \leqq \overline{\lim\limits_{p'' \to l''}}\, \theta_{p''},$

(b) $\lim\limits_{p'' \to l''} \mu_{l'p''} = +\infty,$

(c) $\lim\limits_{p'' \to l''} \mu_{l'p''} = -\infty;$

and in case l'' is a p'', in (a) *the further hypothesis*

$$\underline{\lim_{p'' \to l''}}\, \theta_{p''} \leqq \theta_{l''} \leqq \overline{\lim_{p'' \to l''}}\, \theta_{p''}.$$

We prove proposition (a), supposing that l' and l'' are improper limiting elements of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively. Suppose first that $\overline{\lim\limits_{p''\to l''}}\, \theta_{p''} = a$; then for a given e we have

$$\text{(1)} \qquad \exists\, \mathfrak{R}_e''^{\,l''} \ni p''^{\mathfrak{R}_e''} . \supset . \theta_{p''} \leqq a + \frac{e}{2},$$

$$\text{(2)} \qquad \mathfrak{R}_1''^{\,l''} . \supset . \exists\, p''^{\mathfrak{R}_1''} \ni \theta_{p''} \geqq a - \frac{e}{2},$$

and by definition 5,

$$\text{(3)} \qquad \exists\, \mathfrak{R}_e'^{\,l'} \ni p'^{\mathfrak{R}_e'} . \, p''^{\bar{\mathfrak{R}}''} . \supset . \,|\mu_{p'p''} - \theta_{p''}| \leqq \frac{e}{2}.$$

Let $\mathfrak{R}_2''$ be a common subclass of $\mathfrak{R}_e''$ and $\mathfrak{R}''$, such that $\mathfrak{R}_2''^{\,l''}$, since such classes exist by postulate (III); then by (1) and (3) we have

$$p'^{\mathfrak{R}''} . \supset . \theta_{p''} \leqq a + \frac{e}{2},$$

$$p'^{\mathfrak{R}_e'} . \, p''^{\mathfrak{R}_2''} . \supset . \,|\mu_{p'p''} - \theta_{p''}| \leqq \frac{e}{2};$$

therefore, if $\mathfrak{R}_e = \mathfrak{R}_e'\mathfrak{R}_2''$, we see that

$$\text{(4)} \qquad p^{\mathfrak{R}_e} . \supset . \mu_p \leqq a + e.$$

Any class $\mathfrak{R}$ such that $\mathfrak{R}^l$ must be of the form $\mathfrak{R} = \mathfrak{R}_3'\mathfrak{R}_3''$, where $\mathfrak{R}_3'^{\,l'}$ and $\mathfrak{R}_3''^{\,l''}$. Since $\mathfrak{R}_3'$ and $\mathfrak{R}_e'$ have elements in common, and since $\mathfrak{R}_3''$ and $\mathfrak{R}''$ have a common subclass which fulfils the hypothesis of (2), we have from (2) and (3)

$$\text{(5)} \qquad \exists\, p^{\mathfrak{R}} \ni \mu_p \geqq a - e.$$

We see, then, from (4) and (5), that $\overline{\lim\limits_{p\to l}}\, \mu_p = a$.

Now suppose that $\overline{\lim\limits_{p''\to l''}}\, \theta_{p''} = +\infty$; then for a given a we have

$$\text{(6)} \qquad \mathfrak{R}_4''^{\,l''} . \supset . \exists\, p''^{\mathfrak{R}_4} \ni \theta_{p''} > a + 1,$$

and, taking $e \leqq 1$, we have by (3),

$$p'^{\mathfrak{R}_e'} . \, p''^{\mathfrak{R}''} . \supset . \,|\mu_{p'p''} - \theta_{p''}| \leqq 1.$$

Let $\mathfrak{R}^l$ and let $\mathfrak{R} = \mathfrak{R}_5'\mathfrak{R}_5''$, then since $\mathfrak{R}_5'$ and $\mathfrak{R}_e'$ have elements in common, and since $\mathfrak{R}_5''$ and $\mathfrak{R}''$ have a common subclass which fulfils the hypothesis of (6), there is a p in $\mathfrak{R}$ such that $\mu_p > a$, and thus $\overline{\lim\limits_{p\to l}}\, \mu_p = +\infty$.

The proof in case $\overline{\lim\limits_{p''\to l''}}\, \theta_{p''} = -\infty$ is not difficult. Under the same hypotheses with respect to l' and l'' we easily see that $\underline{\lim\limits_{p''\to l''}}\, \theta_{p''} = \underline{\lim\limits_{p\to l}}\, \mu_p$.

If we remove the restrictions on l' and l'', the additional hypotheses then available easily lead to the desired conclusion.

The proofs of (b) and (c), which offer no new difficulties, are omitted.

We obtain conclusions similar to those of theorem X under more mild hypotheses as follows:

THEOREM XI. *If* $l = l' l''$ *and if* $\bar{\theta}$ *and* $\underline{\theta}$ *are functions on* $\mathfrak{P}''$ *to* $\hat{\mathfrak{A}}$, *then*

(a) *A sufficient condition for the relation* $\overline{\lim\limits_{p \to l}}\, \mu_p \leq \overline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''}$ *is*

$$\exists\, \mathfrak{R}''^{l''} \ni [e : \supset : \exists\, \mathfrak{R}'^{l'}_e \ni p'^{\mathfrak{R}'_e} . \, p''^{\mathfrak{R}''} . \supset . \, \mu_{p'p''} \leqq \bar{\theta}_{p''} + e],$$

(b) *A sufficient condition for the relation* $\underline{\lim\limits_{p \to l}}\, \mu_p \geq \underline{\lim\limits_{p'' \to l''}}\, \underline{\theta}_{p''}$ *is*

$$\exists\, \mathfrak{R}''^{l''} \ni [e : \supset : \exists\, \mathfrak{R}'^{l'}_e \ni p'^{\mathfrak{R}'_e} . \, p''^{\mathfrak{R}''} . \supset . \, \mu_{p'p''} \geqq \underline{\theta}_{p''} - e];$$

provided that if l'' *is a* p'' *we have for* (a) *and* (b) *the respective conditions,* $\overline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''} \geqq \bar{\theta}_{l''}$ *and* $\underline{\lim\limits_{p'' \to l''}}\, \underline{\theta}_{p''} \leqq \underline{\theta}_{l''}$.

We observe that if $\bar{\theta}_{p''}$ is finite for every p'' in $\mathfrak{R}''$, and if $\bar{\theta} = \underline{\theta}$, then the combined hypotheses of (a) and (b) are equivalent to the hypothesis of (a) of theorem X, while the combined conclusions are equivalent to the conclusion of X, (a) only in case $\overline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''} = \underline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''} = \overline{\lim\limits_{p'' \to l''}}\, \theta_{p''} = \underline{\lim\limits_{p'' \to l''}}\, \theta_{p''}$. If for every p'' in $\mathfrak{R}''$ we suppose $\bar{\theta}_{p''} = -\infty$, then (a) is equivalent to theorem X, (c), both in hypothesis and in conclusion, and similarly X, (b) is a corollary of XI, (b).

§ 11. *Continuity of Functions.*

We return now to the general situation considered in § 9. Making use of the notations there employed, we turn attention to questions of the continuity of the function μ on the range $\mathfrak{P}$.

Def. 7. A function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ is *continuous* on $\mathfrak{P}$ if and only if μ_p is finite for every p, and for every proper limiting element l of $\mathfrak{P}$ $\lim\limits_{p \to l} \mu_p = \mu_l$.

The definition here given is analogous to the usual definition of continuity of a function of a real variable. With the present postulates on our system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ we are not able to define an analogue of uniform continuity on a range; but by requiring, in addition to the conditions of definition 7, that there shall exist a finite limit for the function μ at every improper limiting element of $\mathfrak{P}$, we obtain a form of continuity that in most applications is equivalent to uniform continuity. We call this extensible continuity, and a function having this property is said to be extensibly continuous. We have then:

Def. 8. μ is *extensibly continuous* on $\mathfrak{P}$ if and only if it is continuous on $\mathfrak{P}$ and for every improper limiting element l of $\mathfrak{P}$ there exists an a_l such that $\lim\limits_{p \to l} \mu_p = a_l$.

The following theorem is obvious:

THEOREM XII. *If $\mathfrak{P}$ is closed, then μ is continuous on $\mathfrak{P}$ if and only if μ is extensibly continuous on $\mathfrak{P}$.*

With every function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ there are two associated functions, ϕ and ψ, called respectively the upper and lower limiting functions of μ on $\mathfrak{P}$.

Def. 9. ϕ, the upper limiting function of μ on $\mathfrak{P}$, is a function on the range $\mathfrak{Q}$ such that, for every l, $\phi_l = \overline{\lim\limits_{p \to l}}\, \mu_p$ and, for every p not in $\mathfrak{L}$, $\phi_p = \mu_p$.

Def. 10. ψ, the lower limiting function of μ on $\mathfrak{P}$, is a function on the range $\mathfrak{Q}$ such that, for every l, $\psi_l = \underline{\lim\limits_{p \to l}}\, \mu_p$ and, for every p not in $\mathfrak{L}$, $\psi_p = \mu_p$.

The functions ϕ and ψ are then functions on $\mathfrak{Q}$ to $\hat{\mathfrak{A}}$. They lead to greater economy in the statement of propositions on the function μ.

THEOREM XIII. (a) *μ is continuous on $\mathfrak{P}$ if and only if for every element p the function-values ϕ_p, ψ_p and μ_p are equal and finite.*

(b) *μ is extensibly continuous on $\mathfrak{P}$ if and only if for every q the function-values ϕ_q and ψ_q are equal and finite, and for every p they are equal to μ_p.*

(c) *If μ is extensibly continuous on $\mathfrak{P}$ and $\overline{\mathfrak{R}}$ is the greatest common subclass of $\overline{\mathfrak{P}}$ and $\mathfrak{Q}$, then ϕ is extensibly continuous on $\overline{\mathfrak{R}}$ and for every u in $\mathfrak{Q}$ $\lim\limits_{\bar{r} \to u} \phi_{\bar{r}} = \phi_u$.*

The truth of (a) and (b) is obvious. As to (c), it should be remarked that since $\overline{\mathfrak{R}}$ is a subclass of $\mathfrak{Q}$, ϕ is defined on $\overline{\mathfrak{R}}$, and since $\overline{\mathfrak{R}}$ is a subclass of $\overline{\mathfrak{P}}$, continuity and extensible continuity are defined for functions on $\overline{\mathfrak{R}}$ to $\hat{\mathfrak{A}}$. Further, in view of theorem II, (b), of § 6, and theorem VII, of § 7, we see that every limiting element of $\overline{\mathfrak{R}}$ is a limiting element of $\mathfrak{P}$. Since by (b) ϕ is finite for every $\bar{r}$ (element of $\overline{\mathfrak{R}}$), it remains, for the proof of (c), merely to show that for every l we have $\lim\limits_{\bar{r} \to l} \phi_{\bar{r}} = \phi_l$. Since μ is extensibly continuous on $\mathfrak{P}$, we have for every l, $\lim\limits_{p \to l} \mu_p = \phi_l$; that is,

$$(1) \qquad e \,.\, l : \supset : \exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l \,.\, \supset .\, |\mu_p - \phi_l| \leqq \frac{e}{2}.$$

By postulate (IV) we see that

$$(2) \qquad \exists\, \mathfrak{R}_1^l \ni \bar{p}^{\mathfrak{R}_1} \,.\, \supset .\, \exists\, \mathfrak{R}_2^{\bar{p}} \ni \mathfrak{R}_2^{\mathfrak{R}_e}.$$

We wish now to show that

$$(3) \qquad \bar{r}^{\mathfrak{R}_1} \neq l \,.\, \supset .\, |\phi_{\bar{r}} - \phi_l| \leqq e.$$

If $\bar{r}$ is a p, this follows from the fact that $\mathfrak{R}_1$ is necessarily a subclass of $\mathfrak{R}_e$. If $\bar{r}$ is not a p, it is an improper limiting element of $\mathfrak{P}$, and we have by an application of (1),

(4) $$\exists\, \overline{\mathfrak{R}}_e' \ni p^{\overline{\mathfrak{R}}_e} \,.\supset.\, |\phi_{\bar{r}} - \mu_p| \leqq \frac{e}{2}.$$

By use of (2) and postulates (III) and (V) we may show that $\overline{\mathfrak{R}}_e$ contains an element p of $\mathfrak{R}_e$ distinct from l; then (3) follows from (1) and (4).

THEOREM XIV. (a) *If $\mathfrak{P}$ is compact and μ is extensibly continuous on $\mathfrak{P}$, then μ is bounded on $\mathfrak{P}$, by finite bounds.*

(b) *If $\mathfrak{P}$ is compact, then there exist q_1 and q_2 such that the least upper bound of μ on $\mathfrak{P}$ is either ϕ_{q_1} or μ_{q_1}, and the greatest lower bound of μ on $\mathfrak{P}$ is either ψ_{q_2} or μ_{q_2}.*

(c) *If $\mathfrak{P}$ is compact and closed and μ is continuous on $\mathfrak{P}$, then there exist p_1 and p_2 such that the least upper bound and the greatest lower bound of μ on $\mathfrak{P}$ are respectively μ_{p_1} and μ_{p_2}.*

Proof of (a): Suppose that μ is not bounded from $+\infty$, and consider a sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty} a_n = +\infty$. We have then

(1) $$n \,.\supset.\, \exists\, p_n \ni \mu_{p_n} > a_n.$$

Since $\mathfrak{P}$ is compact, and since the number of distinct elements in the sequence $\{p_n\}$ can not be finite, there exists a subsequence $\{p_{n_m}\}$ of distinct elements of the sequence $\{p_n\}$ which has some element l as limit. Since μ is extensibly continuous, there is a number a_l such that

(2) $$e :\supset: \exists\, m_e \ni m > m_e \,.\supset.\, |\mu_{p_{n_m}} - a_l| \leqq e.$$

Now we may take an n_1 such that for $n > n_1$ we have $a_n > a_l + e$, and since there exist values of m greater than m_e such that n_m is greater than n_1, we see that (1) and (2) are contradictory. Thus μ is bounded from $+\infty$, and in similar manner we may show that μ is bounded from $-\infty$.

Proof of (b): The least upper bound of μ on $\mathfrak{P}$ may be a finite number a or $+\infty$. In either case there exists an element p such that μ_p is this least upper bound, or there is a sequence $\{p_n\}$ such that the limit of μ_{p_n} is this least upper bound. If the former is true, then $q_1 = p$ meets the requirements of the theorem; if the latter is true, then there is a subsequence $\{p_{n_m}\}$ of distinct elements of $\{p_n\}$ which has a limit l, and clearly we have ϕ_l as the least upper bound of μ on $\mathfrak{P}$. Similarly q_2 exists, fulfilling the conditions of the theorem.

Proof of (c): Since $\mathfrak{P}$ is closed, every q is a p, and since μ is continuous, $\phi_p = \mu_p = \psi_p$ for every p; therefore (c) is a corollary of (b).

§ 12. *Functions on a Composite Range.*

Returning to the special case when the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is the composite of the two systems $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$, and using the notations $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{L}'$ and $\mathfrak{P}''$, $\mathfrak{Q}''$, $\mathfrak{L}''$, as in § 10, we discuss the character of a function μ

on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ with regard to continuity and related properties. Use is made also of the upper and lower limiting functions, ϕ and ψ, of μ on $\mathfrak{P}$. We notice that for every q' $\phi_{p'}$ and $\psi_{q'}$ are definite functions on $\mathfrak{Q}''$, and for every q'' $\phi_{p''}$ and $\psi_{q''}$ are definite functions on $\mathfrak{Q}'$.

THEOREM XV. (a) *If l' is such that for every l of the form $l = l' p''$ ϕ_l is finite and $\lim\limits_{p \to l} \mu_p = \phi_l$, then $\phi_{l'}$ is continuous on $\mathfrak{P}''$.*

(b) *If l' is such that for every l of the form $l = l' q''$ ϕ_l is finite and $\lim\limits_{p \to l} \mu_p = \phi_l$, then $\phi_{l'}$ is extensibly continuous on $\mathfrak{P}''$ and for every l'' $\lim\limits_{p'' \to l''} \phi_{l' p''} = \phi_{l' l''}$.*

(c) *If μ is continuous on $\mathfrak{P}$, then for every p'' $\mu_{p''}$ is continuous on $\mathfrak{P}'$ and for every p' $\mu_{p'}$ is continuous on $\mathfrak{P}''$.*

(d) *If μ is extensibly continuous on $\mathfrak{P}$, then for every p'' $\mu_{p''}$ is extensibly continuous on $\mathfrak{P}'$ and for every p' $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$.*

The propositions of this theorem are easy deductions from theorems I and IX, (b).

In the following theorem we employ the notation for uniform approach to a limit that was introduced in definition 5.

THEOREM XVI. (a) *If l' is such that in every $\mathfrak{R}'$ such that $\mathfrak{R}'^{l'}$ there is a p' distinct from l' such that $\mu_{p'}$ is continuous on $\mathfrak{P}''$, and if there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p' \to l'} \mu_{p'} = \theta\,(\mathfrak{P}'')$, then θ is continuous on $\mathfrak{P}''$.*

(b) *If l' is such that in every $\mathfrak{R}'$ such that $\mathfrak{R}'^{l'}$ there is a p' distinct from l' such that $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$, and if there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p' \to l'} \mu_{p'} = \theta\,(\mathfrak{P}'')$, then θ is extensibly continuous on $\mathfrak{P}''$.*

(c) *With the additional hypothesis that l' is an improper limiting element of $\mathfrak{P}'$, or, in case l' is a p', that $\mu_{l' p''} = \theta_{p''}$ for every p'', we have for both* (a) *and* (b) *the additional conclusion that for every p'' $\theta_{p''} = \phi_{l' p''} = \psi_{l' p''}$, and for* (b) *the further conclusion that for every l'' $\lim\limits_{p'' \to l''} \theta_{p''} = \phi_{l' l''} = \psi_{l' l''}$.*

We prove first the lemma:

LEMMA. *If $l = l' l''$ and $\mathfrak{R}''^{l''}$ and $\overline{\mathfrak{R}}''$ is the greatest common subclass of $\mathfrak{P}''$ and $\mathfrak{R}''$, and if for every $\mathfrak{R}'$ such that $\mathfrak{R}'^{l'}$ there is a p' distinct from l' in $\mathfrak{R}'$ and a number a such that $\lim\limits_{p'' \to l''} \mu_{p' p''} = a$, and if there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p' \to l'} \mu_{p'} = \theta\,(\overline{\mathfrak{R}}'')$, then there is an a_1 such that $\lim\limits_{p'' \to l''} \theta_{p''} = a_1$.*

By the uniform approach to the function θ, we have for a given e,

$$(1) \qquad \exists\, \mathfrak{R}_e'^{\,l'} \ni p'^{\mathfrak{R}'_e} \neq l' \,.\, p''^{\overline{\mathfrak{R}}''} \,.\supset .\, | \mu_{p' p''} - \theta_{p''} | \leqq \frac{e}{3}\,;$$

and by the remaining hypothesis on μ there is a p_1' in $\mathfrak{R}_e'$ such that, taking account of theorem I, we have

$$(2) \qquad \exists\, \mathfrak{R}_1''\, l'' \ni p_1''\, \mathfrak{R}_1'' \neq l'' \,.\, p_2''\, \mathfrak{R}_1'' \neq l'' \,.\supset .\, |\mu_{p_1' p_1''} - \mu_{p_1' p_2''}| \leqq \frac{e}{3}.$$

By postulate (III) it may be shown that there exists a common subclass $\mathfrak{R}_2''$ of $\mathfrak{R}_1''$ and $\mathfrak{R}''$ such that $\mathfrak{R}_2''\, l''$. If p_1'' and p_2'' are any two elements of $\mathfrak{R}_2''$ distinct from l'', we have by (1) and (2) the three conditions

$$|\theta_{p_1''} - \mu_{p_1' p_1''}| \leqq \frac{e}{3}, \quad |\mu_{p_1' p_2''} - \theta_{p_2''}| \leqq \frac{e}{3} \quad \text{and} \quad |\mu_{p_1' p_1''} - \mu_{p_1' p_2''}| \leqq \frac{e}{3},$$

from which we obtain $|\theta_{p_1''} - \theta_{p_2''}| \leqq e$. Since condition (1) of theorem II is obviously fulfilled, we conclude that there exists an a_1 such that $\lim\limits_{p'' \to l''} \theta_{p''} = a_1$.

From the lemma and theorem X, (a), the present theorem should now be evident.

The language "μ is continuous on $\mathfrak{P}'$" might conveniently be used to indicate that for every p'' the function $\mu_{p''}$ is continuous on $\mathfrak{P}'$; and this manner of speaking is especially advantageous if the continuity on $\mathfrak{P}'$ is uniform on $\mathfrak{P}''$, *i. e.*, uniform with respect to p''. Thus we have

Def. 11. (a) μ is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ if and only if for every p the function value μ_p is finite, and for every proper limiting element l' of $\mathfrak{P}'$ $\lim\limits_{p' \to l'} \mu_{p'} = \mu_{l'}\ (\mathfrak{P}'')$.

(b) μ is extensibly continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ if and only if μ is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$, and for every improper limiting element l' of $\mathfrak{P}'$ there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p' \to l'} \mu_{p'} = \theta\ (\mathfrak{P}'')$.

The following theorem is a result of an easy application of the propositions of theorem XVI.

THEOREM XVII. (a) *If μ is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ and if for every p' $\mu_{p'}$ is continuous on $\mathfrak{P}''$, then μ is continuous on $\mathfrak{P}$.*

(b) *If μ is extensibly continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ and if for every p' $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$, then μ is extensibly continuous on $\mathfrak{P}$.*

The following theorem, the notations of which may easily be interpreted by analogy with those previously defined, is not without interest, and is found convenient in some applications that follow.

THEOREM XVIII. *If $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is the composite of three systems, $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$, $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ and $(\overline{\mathfrak{P}}'''; \mathfrak{U}'''; T''')$, and if μ is defined on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''\mathfrak{P}'''$, and $l = l'\,l''\,l'''$, then the following propositions hold:*

(a) *If there exists a function* θ *on* $\mathfrak{P}'\mathfrak{P}''$ *to* $\mathfrak{A}$ *such that* $\lim\limits_{p'''\to l'''} \mu_{p'''} = \theta(\mathfrak{P}'\mathfrak{P}'')$, *and if there exists a function* α *on* $\mathfrak{P}'''$ *to* $\mathfrak{A}$ *such that for every* p''' $\lim\limits_{p''\to l''} \lim\limits_{p'\to l'} \mu_{p'p''p'''} = \alpha_{p'''}$, *then there exists a number* a *such that* $\lim\limits_{p''\to l''} \lim\limits_{p'\to l'} \theta_{p'p''} = \lim\limits_{p'''\to l'''} \alpha_{p'''} = a$.

(b) *If, in addition to the hypotheses of* (a), *there exists a function* ξ *on* $\mathfrak{P}''\mathfrak{P}'''$ *to* $\mathfrak{A}$ *such that for every* p''' $\lim\limits_{p'\to l'} \mu_{p'p'''} = \xi_{p'''}(\mathfrak{P}'')$, *then there exists a function* γ *on* $\mathfrak{P}''$ *to* $\mathfrak{A}$ *such that* $\lim\limits_{p'\to l'} \theta_{p'} = \gamma(\mathfrak{P}'')$.

Proof: By the first hypothesis in (a) we see that

$$(1) \qquad p'' \,.\supset.\, \lim_{p'''\to l'''} \mu_{p''p'''} = \theta_{p''}(\mathfrak{P}'),$$

and by the second hypothesis we know that there exists a function ξ on $\mathfrak{P}''\mathfrak{P}'''$ such that

$$(2) \qquad p''\,.\,p''' \,.\supset.\, \lim_{p'\to l'} \mu_{p'p''p'''} = \xi_{p''p'''};$$

therefore, applying the lemma to theorem XVI and theorems X, (a) and IX, (b), we see that there exists a γ on $\mathfrak{P}''$ such that

$$(3) \qquad p'' \,.\supset.\, \lim_{p'\to l'} \theta_{p'p''} = \lim_{p'''\to l'''} \xi_{p''p'''} = \gamma_{p''}.$$

Now, for a given e, the first hypothesis in (a), (2) and (3) give respectively the conditions

$$(4) \qquad \exists\, \mathfrak{R}'''^{\,l'''}_{e} \ni p'\,.\,p''\,.\,p'''^{\mathfrak{R}'''_e} \neq l''' \,.\supset.\, |\mu_{p'p''p'''} - \theta_{p'p''}| \leqq \frac{e}{3},$$

$$(5) \qquad p''\,.\,p''' :\supset: \exists\, \mathfrak{R}'^{\,l'}_{e\,p''p'''} \ni p'^{\mathfrak{R}'_{e\,p''p'''}} \neq l' \,.\supset.\, |\xi_{p''p'''} - \mu_{p'p''p'''}| \leqq \frac{e}{3},$$

$$(6) \qquad p'' :\supset: \exists\, \mathfrak{R}'^{\,l'}_{e\,p''} \ni p'^{\mathfrak{R}'_{e\,p''}} \neq l' \,.\supset.\, |\theta_{p'p''} - \gamma_{p''}| \leqq \frac{e}{3}.$$

Since for every p'' and p''' the two classes $\mathfrak{R}'_{e\,p''p'''}$ and $\mathfrak{R}'_{e\,p''}$ have a common p' distinct from l', we see from (4), (5) and (6) that $\lim\limits_{p'''\to l'''} \xi_{p'''} = \gamma(\mathfrak{P}'')$, and since by the second hypothesis $\lim\limits_{p''\to l''} \xi_{p''p'''} = \alpha_{p'''}$ for every p''', we have the conclusion of (a) by another application of theorems X, (a), IX, (b) and the lemma to XVI.

As to proposition (b), it remains to prove that the approach of $\theta_{p'}$ to γ is uniform on $\mathfrak{P}''$. This follows from (4) and the following two conditions, which come respectively from the special hypothesis of (b) and the fact that $\lim\limits_{p'''\to l'''} \xi_{p'''} = \gamma(\mathfrak{P}'')$,

$$p''' : \supset : \exists\, \mathfrak{R}'^{\,l'}_{e\,p'''} \ni p'^{\mathfrak{R}_{e\,p'''}} \neq l' \cdot p'' \cdot \supset \cdot \left| \mu_{p'\,p''\,p'''} - \xi_{p''\,p'''} \right| \leq \frac{e}{3},$$

$$\exists\, \overline{\mathfrak{R}}'''^{\,l'''}_{e} \ni p'' \cdot p'''^{\overline{\mathfrak{R}}'''_e} \neq l''' \cdot \supset \cdot \left| \xi_{p''\,p'''} - \gamma_{p''} \right| \leq \frac{e}{3};$$

for, since $\overline{\mathfrak{R}}'''_e$ and $\mathfrak{R}'''_e$ have a common p''' distinct from l''', we have

$$p'^{\mathfrak{R}'_{e\,p'''}} \neq l' \cdot p'' \cdot \supset \cdot \left| \theta_{p'\,p''} - \gamma_{p''} \right| \leq e,$$

CHAPTER IV.

APPLICATIONS OF THE GENERAL THEORY BY DIRECT SPECIALIZATION.

§ 13. *Introductory.*

In developing the theory of Chapters I, II and III it has not been necessary to specify the character of the elements under consideration, and the nature of the conditions postulated is such as to provide great latitude in the matter of applications. Special theories are obtained by particular determination either of a system $(\mathfrak{P}; R)$ which satisfies the postulates of § 2, thus giving rise to a system $(\mathfrak{P}; \mathfrak{U}; T)$ of the required character, or directly of a system $(\mathfrak{P}; \mathfrak{U}; T)$ which satisfies the postulates of § 5. In the present chapter we suggest, by means of chosen examples, certain methods of procedure to secure these special theories. The first instances used, viz., multiple sequences and functions of real variables, are chosen not because of any novelty of form or content of the results reached, but rather because of the interest that may be attached to the manner in which various familiar theorems, usually treated as independent, emerge as special cases of the same general theorem. The remaining examples are in the domain of general analysis, and are chosen to show the availability of the present method in certain fields already shown to be fruitful of interesting and useful theories.

§ 14. *Multiple Sequences.*

We may specify a system $(\mathfrak{P}; R)$ as follows: The class $\mathfrak{P}$ is the class of all positive integers; the relation $\mathfrak{R}_1 R\, \mathfrak{R}_2$ holds if and only if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are equal and consist of a single element, or there exist two positive integers, n_1 and n_2, such that $\mathfrak{R}_1$ consists of all integers greater than n_1 and $\mathfrak{R}_2$ consists of all integers greater than n_2.

This system clearly satisfies the postulates of § 2, and the resulting system $(\mathfrak{P}; \mathfrak{U}; T)$ therefore satisfies the postulates of § 5. This latter system is said to be of type A_1, and is as follows: The class $\mathfrak{P}$ is the class of all positive integers, $\mathfrak{U}$ is a singular class having only the element ∞; the relation $\mathfrak{R}^n$ holds if and only if $\mathfrak{R}$ consists of the single element n, and the relation $\mathfrak{R}^\infty$ holds if and only if $\mathfrak{R}$ consists of all integers greater than some given integer. The composite system of r systems of type A_1 is a system of type A_r.

To obtain a theory of multiple sequences, consider the special case when the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ of Chapter III is of the type A_r, and the class $\mathfrak{P}$ coincides with $\overline{\mathfrak{P}}$. A function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ then gives an r-fold sequence of function-values, every one of which is a real number or $+\infty$ or $-\infty$. Since the nature of the range in this instance renders it unnecessary to place in evidence the notation for limiting element, and since it is desired to emphasize the character of the limits as multiple limits, it is expedient to adopt notation which places all the variables concerned in evidence. Accordingly the notation $\lim\limits_{(n^1\dots n^r)} \mu_{n^1\dots n^r}$ is used to indicate the limit of the function-value $\mu_{n^1\dots n^r}$ as the variables $n^1\dots n^r$ simultaneously increase without limit. Similarly, the notations $\overline{\lim\limits_{(n^1\dots n^r)}} \mu_{n^1\dots n^r}$ and $\underline{\lim\limits_{(n^1\dots n^r)}} \mu_{n^1\dots n^r}$ indicate respectively the upper and lower limits under the same conditions. Explicit definition of these symbols in the light of definitions 1, 2, 3 and 4 of Chapter III should cause no difficulty. For example, $\overline{\lim\limits_{(n^1\dots n^r)}} \mu_{n^1\dots n^r} = a$ is equivalent to the conditions: (a) For every e there exists $n^1_e\dots n^r_e$ such that if $n^i > n^i_e$ $(i = 1, 2, \dots, r)$, then $\mu_{n^1\dots n^r} \leqq a + e$; and (b) For every e and every $n^1\dots n^r$ there exist $n^1_1\dots n^r_1$ such that $n^i_1 > n^i$ $(i = 1, 2, \dots, r)$ and such that $\mu_{n^1_1\dots n^r_1} \geqq a - e$.

Among the contributions of § 9 to the theory of multiple sequences are the following, which we record in the form of a theorem.

THEOREM I. (a) *The limit of the multiple sequence* $\{\mu_{n^1\dots n^r}\}$ *is a finite number* a *if and only if the upper and lower limits of the multiple sequence are both equal to* a.

(b) *The limit of the multiple sequence* $\{\mu_{n^1\dots n^r}\}$ *exists and is finite if and only if for every* $n^1\dots n^r$ *there exist* $n^1_1\dots n^r_1$ *such that* $n^i_1 > n^i$ $(i = 1, 2, \dots, r)$ *and such that* $\mu_{n^1_1\dots n^r_1}$ *is finite, and for every* e *there exist* $n^1_e\dots n^r_e$ *such that, if* $n^i_1 > n^i_e$ *and* $n^i_2 > n^i_e$ $(i = 1, 2, \dots, r)$, *then* $|\mu_{n^1_1\dots n^r_1} - \mu_{n^1_2\dots n^r_2}| \leqq e$.

(c) *If there exist* $n^1_1\dots n^r_1$ *such that, for* $n^i > n^i_1$ $(i = 1, 2, \dots, r)$, $\mu_{n^1\dots n^r}$ *is finitely bounded, then there exist numbers* $\bar{a}$ *and* a *such that*

$$\overline{\lim_{(n^1\ldots n^r)}}\ \mu_{n^1\ldots n^r} = \bar{a} \quad and \quad \underline{\lim_{(n^1\ldots n^r)}}\ \mu_{n^1\ldots n^r} = \underline{a}.$$

(d) *If the simple sequence* $\{\mu_{n^1_m\ldots n^r_m}\}$ *is such that* $\lim_{m\to\infty} n^i_m = \infty$ $(i = 1, 2, \ldots, r)$, *then*

$$\underline{\lim_{(n^1\ldots n^r)}}\ \mu_{n^1\ldots n^r} \leqq \underline{\lim}_{m\to\infty}\ \mu_{n^1_m\ldots n^r_m} \leqq \overline{\lim}_{m\to\infty}\ \mu_{n^1_m\ldots n^r_m} \leqq \overline{\lim_{(n^1\ldots n^r)}}\ \mu_{n^1\ldots n^r}.$$

(e) *There exists a simple sequence of the kind described in* (d) *having for limit* $\overline{\lim_{(n^1\ldots n^r)}}\ \mu_{n^1\ldots n^r}$, *and also one having for limit* $\underline{\lim_{(n^1\ldots n^r)}}\ \mu_{n^1\ldots n^r}$.

(f) *The limit of the multiple sequence exists, finite or infinite, if and only if all simple sequences of the kind described in* (d) *have the same limit; and the limit of the multiple sequence is the common limit of the simple sequences.*

In § 10 we may consider the systems $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ to be of types A_{r_1} and A_{r_2} respectively, where $r_1 + r_2 = r$; then our hypothesis with respect to $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is fulfilled. The real force of the theorems on iterated limits is here realized only by repeated application of the principles established, a process made available by the persistence, under composition of systems, of the conditions specified in our postulates. We may conveniently use the notation* $\lim\limits_{\overline{(n^1\ldots n^s)}\ldots\overline{(n^t\ldots n^r)}} \mu_{n^1\ldots n^r}$ to denote the result of taking the upper limits as the variables $n^1\ldots n^r$ tend to infinity in groups, the group $n^t\ldots n^r$ passing to the limit first, etc. Analogous notations, easy of interpretation, may be used for limits of other types. With a little reflection the theorems of § 10 are seen to yield the following results:

THEOREM II. (a) *For every expression of the type*

$$\lim_{\overline{(n^1\ldots n^s)}\ldots\underline{(n^t\ldots n^r)}} \mu_{n^1\ldots n^r},$$

where the grouping of the variables and the arrangement of upper and lower dashes are entirely arbitrary, there exists a simple sequence $\{\mu_{n^1_m\ldots n^r_m}\}$ *such that* $\lim\limits_{m\to\infty} n^i_m = \infty$ $(i = 1, 2, \ldots, r)$ *having the given expression for limit.*

(b) *An expression of the type mentioned in* (a) *is not less than the expression obtained from it by replacing any number of upper dashes by lower dashes; or by the subdivision of any group that has an upper dash and giving the subgroups either upper or lower dashes; or by combining any number of adjacent groups and giving the combined group a lower dash. In particular,*

* Compare Bromwich and Hardy, "Some Extensions to Multiple Series of Abel's Theorem on the Continuity of Power Series," *Proceedings of the London Mathematical Society*, Series 2, Vol. II, p. 161.

*if the limit of the multiple sequence exists, then every expression of the type discussed is equal to the limit of the multiple sequence.**

(c) *If there is an s-fold sequence* $\{\theta_{n^1 \ldots n^s}\}$ *such that for* $n^i > n_1^i$ $(i = 1, 2, \ldots, r)$ $\theta_{n^1 \ldots n^r}$ *is finite, and such that*

$$\lim_{(n^{s+1} \ldots n^r)} \mu_{n^{s+1} \ldots n^r} = \theta(\mathfrak{R}^1 \ldots \mathfrak{R}^s),$$

where $\mathfrak{R}^i$ *consists of all* n^i *greater than* n_1^i $(i = 1, 2, \ldots, r)$, *then*

$$\overline{\lim_{(n^1 \ldots n^r)}} \mu_{n^1 \ldots n^r} = \overline{\lim_{(n^1 \ldots n^s)}} \theta_{n^1 \ldots n^s} \quad \text{and} \quad \underline{\lim_{(n^1 \ldots n^r)}} \mu_{n^1 \ldots n^r} = \underline{\lim_{(n^1 \ldots n^s)}} \theta_{n^1 \ldots n^s};$$

and if for similarly chosen $\mathfrak{R}^1 \ldots \mathfrak{R}^s$ *we have*

$$\lim_{(n^{s+1} \ldots n^r)} \mu_{n^{s+1} \ldots n^r} = +\infty\ (\mathfrak{R}^1 \ldots \mathfrak{R}^s),$$

then we also have

$$\lim_{(n^1 \ldots n^r)} \mu_{n^1 \ldots n^r} = +\infty;$$

and this latter statement remains true if $+\infty$ *be replaced by* $-\infty$.

The theorems of § 11 and § 12 are clearly applicable to multiple sequences, although in some cases the results are trivial, and in some cases are identical with results already obtained from § 9 and § 10. We note here the fact that if the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is of type A_1, then an extensibly continuous function on $\overline{\mathfrak{P}}$ gives a convergent sequence of finite terms, while analogous statements hold for systems of type A_r. If we assume the system to be of type A_r and consider μ defined on the class $\mathfrak{P} = \overline{\mathfrak{P}}$, then the lemma to theorem XVI, § 12, may be interpreted as follows:

THEOREM III. *If the s-fold sequence* $\{\theta_{n^1 \ldots n^s}\}$ *is such that*

$$\lim_{(n^{s+1} \ldots n^r)} \mu_{n^{s+1} \ldots n^r} = \theta(\mathfrak{P}^1 \ldots \mathfrak{P}^r),$$

and if for every $n^{s+1} \ldots n^r$ *there are values* $n_1^{s+1} \ldots n_1^r$ *such that* $n_1^i > n^i$ $(i = s+1, \ldots, r)$ *and such that* $\mu_{n_1^{s+1} \ldots n_1^r}$ *is a convergent s-fold sequence, then* $\{\theta_{n^1 \ldots n^s}\}$ *is a convergent s-fold sequence.*

§ 15. *Functions of Real Variables.*

To obtain applications of the general theory to functions of a real variable we might, as in the previous section, specify a system $(\mathfrak{P}; R)$ in which $\mathfrak{P}$ should be the class of real numbers and R should be so defined as to secure ideal elements corresponding to $+\infty$ and $-\infty$. For the sake of simplicity, however, we proceed at once to the specification of a system $(\mathfrak{P}; \mathfrak{U}; T)$. $\mathfrak{P}$ is

* Compare the note by G. H. Hardy, *Proceedings of the London Mathematical Society*, Series 2, Vol. II, p. 190.

the class of all real numbers; $\mathfrak{U}$ consists of two elements, $+\infty$ and $-\infty$; a class $\mathfrak{R}$ has the relation T to a given element p_1 if and only if there is a number d such that $\mathfrak{R}$ consists of all elements p such that $|p-p_1| \leqq d$; $\mathfrak{R}$ has the relation T to $+\infty$ if and only if there is a number a such that $\mathfrak{R}$ consists of all elements p such that $p > a$; and $\mathfrak{R}$ has the relation T to $-\infty$ if and only if there is a number a such that $\mathfrak{R}$ consists of all elements p such that $p < a$.

Such a system, which obviously fulfils our postulates, is designated as a system of type B_1, and the composite of r such systems is a system of type B_r. Attention should be called to the fact that with this special determination of the system $(\mathfrak{P}; \mathfrak{U}; T)$ the definitions and theorems in Chapter II relative to limiting elements, and, pertaining to properties of subclasses of $\mathfrak{P}$, are in accordance with the usual treatment of these features of the range of a real variable.

It may be seen, without detailed discussion here, that in this instance the theory developed in Chapter III is a theory of multiple and iterated limits and continuity of functions of several real variables. The definitions and terminology employed render the interpretations of the various theorems immediate, except for the fact that the term "extensible continuity" has not been in use to denote a property of a function of a real variable. We show in theorem V that this property, for functions on a limited number set, is equivalent to the property "uniform continuity." For convenience in the proof of theorem V we prove first the following theorem:*

THEOREM IV. *If the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is of type B_r, then every subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ is compact.*

Proof: In view of theorems V, (a) and VII, (d) of § 7, it is sufficient to consider the special case when $r=1$ and $\mathfrak{P}=\overline{\mathfrak{P}}$. Let $\{p_n\}$ be a sequence of distinct elements of $\mathfrak{P}$. If any limited subclass of $\mathfrak{P}$ contains an infinite subsequence of $\{p_n\}$, then by a well-known property of the number system this subsequence gives rise to at least one limiting element. If no limited subclass of $\mathfrak{P}$ contains such a subsequence, then, considering a sequence $\{a_m\}$ of real numbers such that $\lim\limits_{m\to\infty} a_m = +\infty$, we see that for every m there is an n_m such that $p_{n_m} > a_m$ or $p_{n_m} < -a_m$. Clearly at least one of the infinite ideal elements is then a limit of a subsequence of $\{p_n\}$.

THEOREM V. *If the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is of type B_r, and if μ is defined on the subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$, then we have the propositions:*

* There is a difference in the force of the term "compact" as employed here and as employed by Fréchet (*Rendiconti del Circolo Matematico di Palermo*, Vol. XXII, p. 6), due to the fact that we recognize ideal limiting elements while Fréchet does not.

(a) *If μ is extensibly continuous on $\mathfrak{P}$, then μ is uniformly continuous on $\mathfrak{P}$.*

(b) *If $\mathfrak{P}$ is limited and μ is uniformly continuous on $\mathfrak{P}$, then μ is extensibly continuous on $\mathfrak{P}$.*

Proof of (a): By the definition of extensible continuity we see that for a given limiting element p of $\mathfrak{P}$ and for an arbitrary positive number e we have

$$(1) \qquad \exists\, d_{ep} \ni p = p^1 \ldots p^r \cdot p_1 = p_1^1 \ldots p_1^r \cdot |p_1^i - p^i| \leqq d_{ep} \cdot \supset \cdot |\mu_{p_1} - \mu_p| \leqq e \quad (i = 1, 2, \ldots, r);$$

while for an element p that is not a limiting element of $\mathfrak{P}$ we have

$$(2) \qquad \exists\, d_p \ni p = p^1 \ldots p^r \cdot p_1 = p_1^1 \ldots p_1^r \cdot |p_1^i - p^i| \leqq d_p \cdot \supset \cdot p_1 = p \quad (i = 1, 2, \ldots, r).$$

Consider a function δ on $\mathfrak{P}$ defined as follows: For every limiting element p of $\mathfrak{P}$ let δ_p be one-half of the least upper bound of the set of values effective as d_{ep} in (1). For every element p that is not a limiting element of $\mathfrak{P}$ let δ_p be one-half of the least upper bound of the set of values effective as d_e in (2). This function δ is positive for every p, and we proceed to show that the greatest lower bound of δ on $\mathfrak{P}$ is positive. Let a_e denote this greatest lower bound of δ on $\mathfrak{P}$, for the value of e in question, then since $\mathfrak{P}$ is compact by theorem IV, we see by theorem XIV, (b), of § 10 that either there is a p such that $\delta_p = a_e$ or there is a limiting element l of $\mathfrak{P}$ such that $\underline{\lim}_{p \to l}\, \delta_p = a_e$. In the former case a_e is clearly positive. In the latter case we have by the extensible continuity of μ on $\mathfrak{P}$, applying theorem II of § 9,

$$\exists\, \mathfrak{R}_e^l \ni p_1^{\mathfrak{R}_e} \cdot p_2^{\mathfrak{R}_e} \cdot \supset \cdot |\mu_{p_1} - \mu_{p_2}| \leqq e.$$

Now we clearly have $l = q^1 \ldots q^r$ and $\mathfrak{R}_e = \mathfrak{R}^1 \ldots \mathfrak{R}^r$, where the relation $\mathfrak{R}^i q^i$ holds for $i = 1, 2, \ldots, r$; therefore each of the $\mathfrak{R}^i$ must be of one of the three forms: (a) all $\bar{p}^i$ such that $|\bar{p}^i - q^i| \leqq d^i$; (b) all $\bar{p}^i$ such that $\bar{p}^i > \bar{a}^i$; (c) all $\bar{p}^i$ such that $\bar{p}^i < \underline{a}^i$. Consider now a class $\mathfrak{R}_1 = \mathfrak{R}_1^1 \ldots \mathfrak{R}_1^r$, such that $\mathfrak{R}_1^i$, defined as follows: If $\mathfrak{R}^i$ is of form (a), then $\mathfrak{R}_1^i$ consists of all $\bar{p}^i$ such that $|\bar{p}^i - q^i| \leqq d$; if $\mathfrak{R}^i$ is of form (b), then $\mathfrak{R}_1^i$ consists of all $\bar{p}^i$ such that $\bar{p}^i > \bar{a}^i + d$; if $\mathfrak{R}^i$ is of form (c), then $\mathfrak{R}_1^i$ consists of all $\bar{p}^i$ such that $\bar{p}^i < \underline{a}^i - d$. The number d is one-half of the least of the d^i, in case any of the $\mathfrak{R}^i$ are of form (a), and otherwise d is unity. For a given $\mathfrak{R}_e$, then, $\mathfrak{R}_1$ is a definite class, and for every p in $\mathfrak{R}_1$ we have $\delta_p \geqq \frac{1}{2} d$. Clearly $\underline{\lim}_{p \to l}\, \delta_p \geqq \frac{1}{2} d$, so that a_e is positive. Now referring to (1) and (2) we see that if $p_1 = p_1^1 \ldots p_1^r$, and $p_2 = p_2^1 \ldots p_2^r$, and $|p_1^i - p_2^i| \leqq a_e$, then $|\mu_{p_1} - \mu_{p_2}| \leqq e$; that is, μ is uniformly continuous on $\mathfrak{P}$.

Proof of (b): Since uniform continuity on $\mathfrak{P}$ implies the convergence of μ_p as p approaches any proper limiting element of $\mathfrak{P}$, and since, $\mathfrak{P}$ being limited, every limiting element of $\mathfrak{P}$ is finite, it is sufficient to show that μ is convergent at every finite improper limiting element of $\mathfrak{P}$. By hypothesis we have, if $p_1 = p_1^1 \ldots . p_1^r$ and $p_2 = p_2^1 \ldots . p_2^r$,

$$(3) \qquad e : \supset : \exists\, d_e \ni | p_1^i - p_2^i | \leqq d_e \, (i = 1, 2, \ldots, r) \,.\supset.\, | \mu_{p_1} - \mu_{p_2} | \leqq e.$$

If l is an improper limiting element of $\mathfrak{P}$, then for a given e we may take $\mathfrak{R}_e = \mathfrak{R}_e^1 \ldots . \mathfrak{R}_e^r$, where $\mathfrak{R}_e^i = [\text{all } \bar{p}^i \ni | \bar{p}^i - q^i | \leqq d_e/2]$ $(i = 1, 2, \ldots, r)$, where $l = q^1 \ldots . q^r$, and obviously, if p_3 and p_4 are both in $\mathfrak{R}_e$, we have by (3) $| \mu_{p_3} - \mu_{p_4} | \leqq e$. Thus, by theorem II of § 9, μ is convergent at l.

Interesting results are obtained if, in § 10 and § 12, we take one of the component systems to be of type A_r and the other of type B_r. We notice here a few special cases.

In theorem IX, (b), of § 10 let $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ be of type A_1 and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ of type B_1. Let $\mathfrak{P}' = \overline{\mathfrak{P}}'$, but let $\mathfrak{P}''$ be an arbitrary subclass of $\overline{\mathfrak{P}}''$. l' is necessarily the ideal element ∞, but we take l'' as a proper limiting element of $\mathfrak{P}''$, this class being assumed to have such a limiting element. If we replace the notation p' by n and p'' by x, and set $l'' = x_0$, the theorem yields the following:

THEOREM VI. *If* $\{\xi_n(x)\}$ *is a sequence of functions defined for every* x *of the set* $\mathfrak{P}''$, *and if* $\overline{\lim_{n \to \infty}}\, \xi_n(x) = \bar{\xi}(x)$ *and* $\underline{\lim_{n \to \infty}}\, \xi_n(x) = \underline{\xi}(x)$, *and if for every* e *there exist* n_e *and* d_e *such that* $| \xi_n(x) - a | \leqq e$ *for* $n > n_e$ *and* x *such that* $| x - x_0 | \leqq d_e$, *then we have*

$$\lim_{x \to x_0} \bar{\xi}(x) = \bar{\xi}(x_0) = \lim_{x \to x_0} \underline{\xi}(x) = \underline{\xi}(x_0) = a.$$

A sequence of functions satisfying for every e the condition of the hypothesis of this theorem is here designated as a sequence "totally* convergent at x_0." By similar specialization we obtain from theorem X, (a), of § 10 the following:

* If there is an interval $x_0 - d$ to $x_0 + d$ such that, for every x in the interval, $\lim_{n \to \infty} \xi_n(x) = \xi(x)$, and if for every e there is an n_e and a d_e such that for $n > n_e$ and $x_0 - d_e \leqq x \leqq x_0 + d$ we have the condition $| \xi_n(x) - \xi(x) | \leqq e$, then the sequence may be called "uniformly convergent at x_0," by analogy with the use of this term in the theory of series of functions (W. H. Young, *Proceedings of the London Math. Society*, Series 2, Vol. I, p. 90; also Vol. VI, p. 29). In case the limit function $\xi(x)$ exists for all values of x in an interval $x_0 - d$ to $x_0 + d$, total convergence at x_0 implies uniform convergence at x_0; and in case the number of functions of the sequence that are continuous at x_0 is not finite, uniform convergence at x_0 implies total convergence at x_0; but for an unconditioned sequence of functions total convergence at a point and uniform convergence at a point arc independent properties.

THEOREM VII. *If a sequence of functions is uniformly convergent on a set consisting of all elements x in the interval from $x_0 - d$ to $x_0 + d$ and if the limit function is continuous at x_0, then the sequence is totally convergent at x_0.*

And from the same theorem, but taking $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ to be of type B_1, and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ of type A_1, we have

THEOREM VIII. *If the functions of a sequence are equally* continuous at x_0 and if the sequence is convergent at x_0, then the sequence is totally convergent at x_0.*

By taking account of theorem V of the present section we have the following two theorems resulting from theorems XV and XVI of § 12:

THEOREM IX. *If $\{\xi_n(x)\}$ is a sequence of functions defined on $\mathfrak{P}''$, and if the sequence is convergent for every x in $\mathfrak{P}''$, then*

(a) *If the sequence is totally convergent at every proper limiting element of $\mathfrak{P}''$, the limit function is continuous on $\mathfrak{P}''$.*

(b) *If the sequence is totally convergent at every limiting element of $\mathfrak{P}''$, the limit function is uniformly continuous on $\mathfrak{P}''$.*

THEOREM X. *If $\{\xi_n(x)\}$ is a sequence of functions defined on the limited set $\mathfrak{P}''$, and if the sequence is uniformly convergent on $\mathfrak{P}''$, then*

(a) *If for every term of the sequence there is a subsequent term that is continuous on $\mathfrak{P}''$, then the limit function is continuous on $\mathfrak{P}''$.*

(b) *If for every term of the sequence there is a subsequent term that is uniformly continuous on $\mathfrak{P}''$, then the limit function is uniformly continuous on $\mathfrak{P}''$.*

The remaining two theorems of this section are seen to follow from theorem XVII of § 12, if we take account of theorem V of the present section and remember that in the general theorems the situation is symmetrical with respect to the two component systems.

THEOREM XI. (a) *If $\{\xi_n(x)\}$ is a sequence of functions continuous on $\mathfrak{P}''$, and if the sequence is uniformly convergent on $\mathfrak{P}''$, then at every proper limiting element of $\mathfrak{P}''$ the sequence is totally convergent.*

(b) *If $\mathfrak{P}''$ is limited and the functions of the sequence are uniformly continuous on $\mathfrak{P}''$, and if the sequence is uniformly convergent on $\mathfrak{P}''$, then at every limiting element of $\mathfrak{P}''$ the sequence is totally convergent.*†

* The functions of a sequence are equally continuous at x_0 if for every e there is a d_e such that for every n and for $|x - x_0| \leqq d_e$ we have $|\xi_n(x) - \xi_n(x_0)| \leqq e$. For a discussion of the term see Fréchet, *loc. cit.*, p. 11.

† Theorems XI, (a), and IX, (a), together give the well-known theorem: "A uniformly convergent sequence of continuous functions has a continuous function for limit." Propositions (b) of these two theorems give the corresponding theorem for a sequence of uniformly continuous functions. We notice that these theorems are corollaries of the two propositions of theorem X.

THEOREM XII. (a) *If the functions of the sequence* $\{\xi_n(x)\}$ *are equally continuous on* $\mathfrak{P}''$, *and if the sequence is convergent at every* x *in* $\mathfrak{P}''$, *then at every proper limiting element of* $\mathfrak{P}''$ *the sequence is totally convergent, and the limit function is continuous on* $\mathfrak{P}''$.

(b) *If* $\mathfrak{P}''$ *is limited and the functions of the sequence are equally uniformly continuous* on* $\mathfrak{P}''$, *and if the sequence is convergent at every* x *in* $\mathfrak{P}''$, *then at every limiting element of* $\mathfrak{P}''$ *the sequence is totally convergent and the limit function is uniformly continuous on* $\mathfrak{P}''$.

§ 16. *An R Relation in Terms of* K_1.

The K_1 relation used by Professor E. H. Moore in Part II of his memoir on "General Analysis" may be characterized as a relation on the composite class $\mathfrak{P}\,\mathfrak{J}$, where $\mathfrak{P}$ is a class of elements of any kind whatever, and $\mathfrak{J}$ is the class of positive integers.† In other words, K_1 may be considered defined for a class $\mathfrak{P}$ if a criterion exists by which it may be determined for every element p and integer m whether K_{pm} or $^-K_{pm}$; *i. e.*, whether the relation K_1 holds or does not hold for p and m. While Professor Moore defines certain properties for this K_1 relation, he does not permanently condition the relation by any fixed properties or postulates.

We define a relation R in terms of K_1 as follows: The relation $\mathfrak{R}_1 R\, \mathfrak{R}_2$ holds if and only if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ both consist of the same single element p, or there exist two integers m_1 and m_2 such that $\mathfrak{R}_1$ consists of all elements p such that K_{pm_1} and $\mathfrak{R}_2$ consists of all elements p such that K_{pm_2}.

In order that the system $(\mathfrak{P}; R)$ obtained in this way shall satisfy the seven postulates of § 2, it is necessary that some restrictions be placed on the K_1 relation. The following conditions are found to be sufficient:

(1) For every m there exists a p such that K_{pm}.
(2) For every p there exists an m for which K_{pm} does not hold.
(3) If $m_1 < m_2$ and if K_{pm_2}, then K_{pm_1}.

Assuming that these conditions are fulfilled, we see that a class v_p consists of a single class $\mathfrak{R}$ which contains the single element p. One ideal element exists, consisting of all classes $\mathfrak{R}$ of the type $\mathfrak{R} =$ [all $p \ni K_{pm}$], there being a class $\mathfrak{R}$ of this class of classes corresponding to each integer m.

* The functions of the sequence are "equally uniformly continuous" on P'' if, for every e, there is a d_e such that, if $|x_1 - x_2| \leq d_e$, then the relation $|\xi_n(x_1) - \xi_n(x_2)| \leq e$ holds for every n.

† E. H. Moore, "Introduction to a Form of General Analysis," p. 126.

§ 17. *Application to a System* $(\mathfrak{P}; K_2)$.

In this section we consider a system $(\mathfrak{P}; K_2)$, where $\mathfrak{P}$ is an arbitrary class of elements and K_2 is a relation on $\mathfrak{P}\mathfrak{P}\mathfrak{J}$. That is, we suppose a criterion provided by which we are able to say for every p_1, p_2 and m whether or not the relation $K_{p_1 p_2 m}$ holds. The notation K_2 is used by Professor Moore for a relation of this type,* and as in the case of the relation K_1, the relation is conditioned by various hypotheses to secure desired results in the theorems in which it is involved, but no permanent postulates or conditions are adopted.

For our purposes it is convenient to postulate the following conditions on the system† $(\mathfrak{P}; K_2)$:

(1) The relation K_{ppm} holds for every p and m.

(2) If $K_{p_1 p_2 m}$, then $K_{p_2 p_1 m}$.

(3) If $m_1 < m_2$ and if $K_{p_1 p_2 m_2}$, then $K_{p_1 p_2 m_1}$.

(4) For every m there exists an m_1 such that if $K_{p_2 p_1 m_1}$ and $K_{p_1 p m_1}$, then $K_{p_2 p m}$.

(5) If p_1 and p_2 are such that $K_{p_1 p_2 m}$ holds for every m, then $p_1 = p_2$.

Still further restrictions‡ on the system $(\mathfrak{P}; K_2)$ would be required to enable us to derive from it a system $(\mathfrak{P}; R)$ which would fulfil our postulates. These assumptions, however, furnish sufficient basis for a special definition of ideal elements, and for the determination of a system§ $(\mathfrak{P}; \mathfrak{U}; T)$ which fulfils the postulates of § 5.

An ideal element of the system $(\mathfrak{P}; K_2)$ is a class s of sequences $\{p_n\}$ of elements of $\mathfrak{P}$, having the following properties:

1. If $\{p_n\}$ is a sequence belonging to the class s, then for every m there exists an n_m such that, if n_1 and n_2 are both greater than n_m, the relation $K_{p_{n_1} p_{n_2} m}$ holds.

2. If $\{p_n\}$ and $\{\hat{p}_n\}$ are sequences belonging to the class s, then for every m there exists an n_m such that, if n_1 and n_2 are both greater than n_m, the relation $K_{p_{n_1} \hat{p}_{n_2} m}$ holds.

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* *Loc. cit.*, p. 126.

† A system of the type here indicated forms the basis of T. H. Hildebrandt's "Contribution to the Foundations of Fréchet's Calcul Fonctionnel" (AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV, p. 237). He gives a "complete existential theory" for eight properties of the system, including the first three and the fifth of the properties postulated here. The first three are among the properties considered by E. H. Moore, *loc. cit.*, p. 127.

‡ A sufficient additional restriction would be the following assumption: If $p_1 \neq p_2$, then for every m there is a p such that $K_{p_1 p m}$ holds but $K_{p_2 p m}$ does not hold.

§ A system $(\mathfrak{P}; \mathfrak{U}; T)$ in which $\mathfrak{U}$ is the null class may be derived by omitting this definition of ideal elements. In this case assumption (4) may be made less restrictive by permitting m_1 to depend on p as well as on m.

3. For every p there exists a sequence $\{p_n\}$ of the class s and an integer m such that for every n there is an n_1 greater than n such that $K_{p_{n_1} p m}$ does not hold.

4. Any class s_1 which has properties 1, 2 and 3 and contains s, must coincide with s.

Let $\mathfrak{U}$ denote the class of all ideal elements arising by this definition and let $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$, then we may extend the definition of K_2, to make it a relation on $\mathfrak{P}\mathfrak{Q}\mathfrak{J}$, as follows: The relation K_{pum} holds if and only if there is a sequence $\{p_n\}$ of the class s such that for every n the relation $K_{pp_n m}$ holds.

To complete the specification of a system $(\mathfrak{P}; \mathfrak{U}; T)$ it remains to define a relation T for the classes $\mathfrak{P}$ and $\mathfrak{U}$. Let the relation $\mathfrak{R}^q$ hold if and only if there is an integer m such that $\mathfrak{R}$ consists of all elements p for which the relation K_{pqm} holds. That this system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the postulates of § 5 easily follows from five conditions which are easily deduced from the assumptions on the system $(\mathfrak{P}; K_2)$ and the special definition of ideal elements. These five conditions, the first of which is identical with the first assumption, may be written:

$(\bar{1})$ The relation K_{ppm} holds for every p and m.

$(\bar{2})$ For every u and m there exists a p such that K_{pum}.

$(\bar{3})$ If $m_1 < m_2$ and if K_{pqm_2}, then K_{pqm_1}.

$(\bar{4})$ For every m there exists an m_1 such that if $K_{p_2 p_1 m_1}$ and $K_{p_1 q m_1}$, then $K_{p_2 q m}$.

$(\bar{5})$ If $q_1 \neq q_2$ there exists an m such that no element p can fulfil both relations $K_{p q_1 m}$ and $K_{p q_2 m}$.

The theory of Chapters II and III is available for any system $(\mathfrak{P}; K_2)$ which satisfies the assumptions of this section, by the mediation of the associated special system $(\mathfrak{P}; \mathfrak{U}; T)$. Here, as in the case of a real variable, there is a close relation between the properties "uniform continuity" and "extensible continuity." If $(\overline{\mathfrak{P}}; K_2)$ is a system which fulfils the foregoing assumptions, and μ is a function defined on a subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$, then μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists an m_e such that if $K_{p_1 p_2 m_e}$ then $|\mu_{p_1} - \mu_{p_2}| \leqq e$.

THEOREM XIII. *If $(\overline{\mathfrak{P}}; K_2)$ is a system fulfilling assumptions* (1) *to* (5), *then a function μ is uniformly continuous on a compact subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ if and only if, with reference to the associated system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, μ is extensibly continuous on $\mathfrak{P}$.*

The proof of this theorem may be made similar to that of theorem V, § 15.

§ 18. *The Fréchet Voisinage.*

In his thesis M. Fréchet* denotes by (V) a class of elements for which there is defined, fulfilling certain conditions which he specifies, the notion of "voisinage." We may present his assumptions accurately, but in form adapted to our purposes, as follows: With the class $\mathfrak{P}$ of undefined elements is associated a function V, forming the system $(\mathfrak{P}; V)$. The function V is defined on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$; *i. e.*, it assigns to every pair of elements, p_1 and p_2, a real number, which is denoted by $(p_1 p_2)$. The conditions postulated for this system $(\mathfrak{P}; V)$ are:

(1) For every two elements p_1 and p_1 we have $(p_1 p_2) = (p_2 p_1) \geqq 0$.

(2) If $(p_1 p_2) = 0$, then $p_1 = p_2$.

(3) If $p_1 = p_2$, then $(p_1 p_2) = 0$.

(4) There exists a function $\phi(d)$ such that $\lim \phi(d) = 0$ and such that if $(p_1 p_2) \leqq d$ and $(p_2 p_3) \leqq d$, then $(p_1 p_3) \leqq \phi(d)$.

Proceeding now as in the case of a system $(\mathfrak{P}; K_2)$, we give attention to the introduction of ideal elements.† An ideal element of the system $(\mathfrak{P}; V)$ is a class s of sequences $\{p_n\}$ of elements of the class $\mathfrak{P}$ which fulfils the following conditions:

1. If $\{p_n\}$ is a sequence belonging to the class S, then for every d there exists an n_d such that, if n_1 and n_2 are both greater than n_d, then $(p_{n_1} p_{n_2}) \leqq d$.

2. If $\{p_n\}$ and $\{\hat{p}_n\}$ are sequences belonging to s, then for every d there exists an n_d such that, if n_1 and n_2 are both greater than n_d, then $(p_{n_1} \hat{p}_{n_2}) \leqq d$.

3. For every p there exists a sequence $\{p_n\}$ of the class s and a positive number d such that for every n there is an n_1 greater than n such that $(p_{n_1} p) > d$.

4. Any class s_1 which satisfies conditions 1, 2 and 3 and contains s must coincide with s.

We extend the definition of V so that it is a function on $\mathfrak{P}\mathfrak{Q}$, where, as before, $\mathfrak{Q}$ is the class $\mathfrak{P}$ with ideal elements u adjoined, by assigning to $(p\,u)$ the value d_1 of the greatest lower bound of the set of numbers d for each of which there exists a sequence $\{p_n\}$ of the class u such that $(p p_n) \leqq d$ for every n.

In terms of this extended function V a relation T is specified: The relation $\mathfrak{R}^q$ holds if and only if there exists a d such that $\mathfrak{R}$ consists of all elements p for which $(p\,q) \leqq d$. It may be shown without difficulty that the system

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* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII, p. 17.

† Here also we might specify a system $(\mathfrak{P}; R)$ in which R should be defined in terms of V in such manner as to fulfil our postulates, by the adoption of an additional condition on $(\mathfrak{P}; V)$. A condition effective for this purpose would be: If $p_1 \neq p_2$, then for every d there exists a p such that $(p\,p_1) \leqq d$, while $(p\,p_2) > d$.

$(\mathfrak{P}; \mathfrak{U}; T)$, now definitely determined by the system $(\mathfrak{P}; V)$, satisfies the postulates of § 5. As to the significance of the general theory of Chapters II and III, in this special case, we give attention here only to a feature of extensible continuity. Let $(\overline{\mathfrak{P}}; V)$ be a system satisfying conditions (1) to (4) and let μ be a function defined on the subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$. The function μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists a d_e such that if $(p_1 p_2) \leqq d_e$, then $|\mu_{p_1} - \mu_{p_2}| \leqq e$.

In strict analogy with theorem XIII of § 17 we have

THEOREM XIV. *If $(\overline{\mathfrak{P}}; V)$ is a system satisfying conditions (1) to (4), then a function μ is uniformly continuous on a compact subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ if and only if, with reference to the associated system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, μ is extensibly continuous on $\mathfrak{P}$.*

§ 10. *A Class of Functions as Range of the Independent Variable.*

By the mediation of our definition of a system $(\mathfrak{P}; \mathfrak{U}; T)$ in terms of a system $(\mathfrak{P}; V)$ our general theory is available for any class $\mathfrak{P}$ of elements for which there is defined a *voisinage*, or an *écart*, which may be considered as a special voisinage. Among these classes $\mathfrak{P}$, is the class of all real-valued, single-valued functions that are uniformly continuous on a given interval of the real number system.* Our theory is equally applicable, however, to a class $\mathfrak{P}$ consisting of all real-valued, single-valued functions on a range absolutely unconditioned.

Let $\mathfrak{K}$ denote a class of elements k, concerning which no hypotheses are required. We consider a system† $(\mathfrak{P}; \mathfrak{U}; T)$ in which $\mathfrak{P}$ is the class of all single-valued functions on $\mathfrak{K}$ to $\mathfrak{A}$, $\mathfrak{U}$ is the null class, and T is defined relative to a particular function σ on $\mathfrak{K}$ to $\mathfrak{A}$. For a given function σ the relation T is specified as follows: The relation $\mathfrak{R}^p$ holds if and only if there exists a positive number e such that $\mathfrak{R}$ consists of all functions p_1 such that for every k $|p_{1k} - p_k| \leqq e|\sigma_k|$. This system obviously satisfies the five postulates of § 5. Since in this special instance the class $\mathfrak{U}$ is not arbitrary, and since the system involves the arbitrary class $\mathfrak{K}$ and the arbitrary function σ, the notation $(\mathfrak{P}; T; \mathfrak{K}; \sigma)$ is adopted for a system of this special type.

A necessary and sufficient condition that a sequence $\{p_n\}$ shall have the limit p, by definition 2, § 7, is: For every e there exists an n_e such that, if $n > n_e$, then $|p_{nk} - p_k| \leqq e|\sigma_k|$ for every k. If this condition is fulfilled, the

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* M. Fréchet, *loc. cit.*, p. 36.

† We might just as readily set up a system $(\mathfrak{P}; R)$ which would give rise to this system $(\mathfrak{P}; \mathfrak{U}; T)$ by the process explained in § 5, but the present plan is more direct.

sequence $\{p_n\}$ of functions is said to approach the function p *relatively uniformly,** the relativity being with respect to the *scale function* σ. Theorem IV of § 7 now shows that a subclass $\mathfrak{N}$ of $\mathfrak{P}$ is closed (definition 4, § 7) if and only if every sequence of functions belonging to $\mathfrak{N}$ that converges relatively uniformly with respect to σ converges to a limit function† that is in $\mathfrak{N}$. For example, the class of all functions constant on $\mathfrak{K}$ is closed; or, the class of all functions p such that $a_1 \leqq p_k \leqq a_2$ for every k is closed.

If we make the special hypothesis that $\mathfrak{K}$ is a subclass of a class $\overline{\mathfrak{P}}$ for which there is defined a relation R so that $(\overline{\mathfrak{P}}; R)$ satisfies the postulates of § 2, then, through the associated system of the type $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, the theory of Chapter III is applicable to functions p defined on $\mathfrak{K}$. With this restriction on $\mathfrak{K}$ and the hypothesis that σ is bounded on $\mathfrak{K}$, we have the following examples of closed subclasses $\mathfrak{N}$ of $\mathfrak{P}$:

1. The class of all functions p that are convergent at a given limiting element of $\mathfrak{K}$.
2. The class of all functions p that are convergent to a given limit at a given limiting element of $\mathfrak{K}$.
3. The class of all functions p that are continuous on $\mathfrak{K}$.
4. The class of all functions p that are extensibly continuous on $\mathfrak{K}$.

With the further hypothesis that $(\overline{\mathfrak{P}}; R)$ is the composite of two systems, $(\overline{\mathfrak{P}}'; R')$ and $(\overline{\mathfrak{P}}''; R'')$, and that $\mathfrak{K} = \mathfrak{K}'\mathfrak{K}''$, where $\mathfrak{K}'$ and $\mathfrak{K}''$ are subclasses respectively of $\overline{\mathfrak{P}}'$ and $\overline{\mathfrak{P}}''$, we have the further examples of closed subclasses of $\mathfrak{P}$:

5. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$, $\lim\limits_{k''\to l''} \lim\limits_{k'\to l'} p_{k'k''}$ exists and is finite.
6. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$ and a given number a, $\lim\limits_{k''\to l''} \lim\limits_{k'\to l'} p_{k'k''} = a$.
7. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$, there is (for each p) a function α on $\mathfrak{K}''$, convergent at l'', such that $\lim\limits_{k'\to l'} p_{k'} = \alpha\,(\mathfrak{K}'')$.
8. The subclass of 7 containing every function p for which the corresponding α has the limit a at l''.

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* E. H. Moore, *loc. cit.*, p. 29; also *Atti del IV Congresso Internazionale dei Matematici* (Rome, 1908), Vol. II, p. 101.

† A class of functions that is closed in this sense, for a given σ, has the closure property $C_{\mathfrak{S}}$, "closed as to $\mathfrak{S}$," used by E. H. Moore ("General Analysis," p. 37; and *Atti, etc.*, p. 101), where $\mathfrak{S}$, the scale class, contains the single function σ. If there exist positive real numbers a_1 and a_2 such that $a_2 \leqq |\sigma_k| \leqq a_2$ for every k, then closure as to σ is equivalent to closure under extension by adjoining the limits of all uniformly convergent sequences of functions of the class. Important instances in which this equivalence is not effective are furnished by classes of functions defined for positive integers and giving rise to absolutely convergent series (E. H. Moore, "General Analysis," p. 38, as to $\mathfrak{M}^{\mathrm{III}_e}$; and *Atti, etc.*, p. 102, as to $\mathfrak{M}^{\mathrm{III}}$).

9. The class of all functions p such that for every k'' $p_{k''}$ is continuous on $\mathfrak{K}'$.

10. The class of all functions p such that for every k'' $p_{k''}$ is extensibly continuous on $\mathfrak{K}'$.

11. The class of all functions p continuous on $\mathfrak{K}'$ uniformly on $\mathfrak{K}''$.

12. The class of all functions p extensibly continuous on $\mathfrak{K}'$ uniformly on $\mathfrak{K}''$.

The proofs that these classes are closed* may be made to depend on the general theorems of Chapter III in the same way that the theorems on sequences of functions in § 15 are deduced from them.

Dropping now the special hypotheses on $\mathfrak{K}$ and σ, we observe that theorem† VII of § 7 reduces in this case to the familiar proposition, "Every derived class is closed."

Consider a system $(\overline{\mathfrak{P}}; T; \mathfrak{K}; \sigma)$ as defined above, where $\mathfrak{K}$ and σ are arbitrary, and consider a function μ defined on a subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$, finite for every p. The function μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists a d_e such that, if $|p_{1k}-p_{2k}| \leqq d_e |\sigma_k|$ for every k, then $|\mu_{p_1}-\mu_{p_2}| \leqq e$. We have again the relation between uniform continuity and extensible continuity:

THEOREM XV. *If $\mathfrak{P}$ is a compact subclass of a class $\overline{\mathfrak{P}}$ pertaining to a system $(\overline{\mathfrak{P}}; T; \mathfrak{K}; \sigma)$, then a function μ is uniformly continuous on $\mathfrak{P}$ if and only if μ is extensibly continuous on $\mathfrak{P}$.*

NOTE.—The investigations leading to the present paper were completed in June, 1911, and the manuscript left the hands of the author in April, 1912. These facts are offered in explanation of what might otherwise appear to be unwarranted disregard of certain recent contributions to the literature of this field. We have added foot-note references to papers by E. R. Hedrick, *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 285–294, and T. H. Hildebrandt, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 237–290. At this point we should mention a recent paper by E. V. Huntington, on "A Set of Postulates for Abstract Geometry, Expressed in Terms of the Simple Relation of Inclusion," *Mathematische Annalen*, Band 73 (1913), pp. 522–559, which obviously has a strong bearing in the field of the present paper. Mention may also be made of a more recent paper by the present writer, "Limits in Terms of Order, with the Example of Limiting Element not Approachable by a Sequence," *Transactions of the American Mathematical Society*, Vol. XV (1914), pp. 51–71, which pertains to the same general field and in which relationships of various systems of postulates receive further attention.

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* These classes, 1–12, have also the property "self-closure" defined by E. H. Moore, *Atti, etc.*, Vol. II, p. 102, and designated simply as "closure" in "General Analysis," p. 37. Other properties of general reference, *i. e.*, properties defined for classes of real-valued functions in general, and therefore applicable to the classes here enumerated, are the five *dominance properties*, D, D_0, D'_0, D_1 and D_2 ("General Analysis," p. 39). Classes 1, 3, 5, 7, 9, 10, 11 and 12 have properties D_0 and D_1; classes 2, 6 and 8 have property D_1, and, in case the given number a which enters in the definition of each class is positive or zero, they have property D_0, and property D'_0 in case a is zero; and the class 4 has properties D, D_0, D_1 and D_2.

† It is worthy of note that this general theorem, as applied in the special case now under consideration, is, by an application of theorem II, (b), of § 6, equivalent to a special case of theorem III, p. 52, of Professor Moore's memoir on "General Analysis." The special feature is, obviously, in the reduction of the scale class $\mathfrak{S}$ to the single function σ.

VITA.

Ralph Eugene Root was born near Trenton, Missouri, July 18, 1879. After elementary training in the schools in and near Akron, Iowa, he took preparatory and collegiate courses at Morningside College, Sioux City, Iowa, graduating in 1905. He taught mathematics and science in the high school at Forest City, Iowa, for one year, and became assistant instructor in mathematics and graduate student at the State University of Iowa in 1906, where he received the M. S. degree in 1909. After another year as instructor at the University of Iowa, he held a Fellowship in Mathematics at the University of Chicago for the year 1910–11. His residence at the University of Chicago covered half of the summer quarter, 1909, and the five quarters from June, 1910 to September, 1911, during which time he took courses in mathematics and astronomy under Professors Moore, Dickson, Moulton, Bliss, Wilczynski, Laves and MacMillan. To all of these he wishes to express sincere appreciation, and especially to Professor Moore, under whose inspiration and guidance his research work was undertaken, and to whose helpful criticisms and suggestions he is greatly indebted for progress made.