The University of Chicago
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LINEAR POLARS OF THE k-HEDRON IN n-SPACE

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

(DEPARTMENT OF MATHEMATICS)

BY
HARRIS FRANKLIN MACNEISH

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

Published March 1912

Composed and Printed By
The University of Chicago Press
Chicago, Illinois, U.S.A.

INTRODUCTION

The following general definition of the harmonic mean of a set of segments is given by C. MacLaurin in *A Treatise of Algebra, Appendix Concerning the General Properties of Geometrical Lines*, § 27: "A segment PQ is the harmonic mean of a set of segments PP_i, $i=1, 2, \ldots, n$; if $\frac{n}{PQ}=\sum_{i=1}^{n}\frac{1}{PP_i}$. Q is also called the harmonic center of P as to the set of points $\{P_i\}$."

This generalization and its application to polar theory were known to Roger Cotes, who gives in *Harmonia Mensurarum* (1722) the following general theorem called *Cotes's Theorem*: "If a transversal intersecting a curve C_n of the n^{th} order in n points P_i, revolves about a fixed point P, the harmonic center Q of P as to the set of points P_i describes a straight line." MacLaurin gives a proof of this theorem (*op. cit.*, § 28).

Poucelet, "Mémoire sur les centres des Moyennes Harmoniques," *Journal für Mathematik*, Vol. III, 1828, gives a treatment of the harmonic mean based upon MacLaurin's definition.

E. de Jonquières, "Mémoire sur la théorie des pôles et polaires," *Liouville's Journal*, sér. 2, Vol. II (1857), p. 249, applies the theory of the harmonic mean to the polar theory of curves of the third and fourth order.

L. Cremona, "Introduzione ad una Teoria Geometrica delle Curve Piane," *Memorie della Accademia delle Scienze dell' Istituto di Bologna*, ser. 1, Vol. XII (1861), pp. 305–436, gives a résumé of the preceding theory and an extensive treatment of the properties of curves and surfaces of the n^{th} order from a purely synthetic standpoint, including a treatment of polar theory based upon the idea of harmonic mean.

The geometric definition of Linear Polar (see § 3, Definition $\text{IV}_{k,1}$) used in the following treatment occurs in the *Collected Memoirs* of E. Caporali, pp. 258–66. Caporali also considers the quadrangle-quadrilateral configuration which is generalized in Part IV of this paper. The generalized configuration is considered by F. Morley in a paper on "Projective Co-ordinates," *Transactions of the American Mathematical Society*, IV (1905), 288.

For a bibliography of the subject I refer to the *Encyklopädie der mathematischen Wissenschaften*, III, AB, 4*a*, §§ 24, 25, 26, and III, C, 4, § 5.

CONTENTS

I. SYNTHETIC TREATMENT[1]

§ 1. The treatment is based on the following assumptions for general projective geometry from Veblen and Young, "A Set of Assumptions for Projective Geometry," *American Journal*, XXX, 376, § 9.

The point is an undefined element, and the line is regarded as an undefined class of points.

A_1. *If* A *and* B *are distinct points, there is at least one line containing both* A *and* B.

A_2. *If* A *and* B *are distinct points, there is not more than one line containing both* A *and* B.

A_3. *If* A, B, C *are points not belonging to the same line, and if a line* l *contains a point* D *of a line joining* B *and* C *and a point* E, *distinct from* D, *of a line joining* C *and A, then the line* l *contains a point* F *of a line joining* A *and* B.

E_0. *There are at least three points on every line.*

E_1. *There exists at least one line.*

H. For any three collinear points A, B, C there exists a unique harmonic conjugate[2] point D (distinct from A, B, C) of point A as to the pair of points B, C.

Definition of an i-space: S^i, $i=2, 3, \ldots$.

If F^0 and F^{i-1} represent a point and an $(i-1)$-space, respectively, (F^0 not on F^{i-1}), an i-space S^i is the set of all points $\{S^0\}$ collinear with F^0 and the points of F^{i-1}.

A 0-space is a point.

A 1-space is a line.

E_{i+1} $(i=1, 2, 3 \ldots, n-1)$. *It is not true that every point lies on every* i-*space.*

§ 2. In this treatment we consider a set of definitions and theorems concerning r s-spaces in $(s+1)$-space which shall be numbered $\mathrm{I}_{r,s}$, $\mathrm{II}_{r,s}$, etc. The principal definition is the recursion definition $\mathrm{IV}_{r,s}$ of the polar s-space of a point as to an r-hedron in $(s+1)$-space and the theorems $\mathrm{I}_{r,s}$, $\mathrm{II}_{r,s}$, etc., lead up to this definition.

[1] The substance of §§ 1, 2, 3, 4, 5 was developed in connection with Dr. Veblen's projective geometry course (Princeton, 1908–9).

[2] The harmonic conjugate point is defined by the usual complete quandrangle construction.

DEFINITION $I_{2,1}$: *The polar line of a point as to a pair of lines* is the harmonic conjugate line of the point as to the pair of lines.

THEOREM $I_{3,1}$: The three polar lines of a point as to the pairs of lines of a triangle form a triangle perspective to the given triangle.

Let P be a point and p_1, p_2, p_3 a triangle with vertices P_{23}, P_{31}, P_{12}. Let q_1, q_2, q_3 be the polar lines of P as to $p_2\, p_3$, $p_3\, p_1$, $p_1\, p_2$ respectively. q_1 and q_2 intersect on PP_{12}, since the harmonic conjugate point of P as to the points $P_3=(PP_{12},\ p_3)$ and P_{12} is unique. Similarly q_2, q_3 meet on PP_{23} and q_3, q_1 on PP_{31}.

The triangle q_1, q_2, q_3 is called the *cogredient triangle* of P as to triangle p_1, p_2, p_3.

THEOREM $II_{3,1}$: *The Desargues Theorem.* The intersection points of the pairs of homologous sides of two perspective triangles are collinear.

DEFINITION $IV_{3,1}$: *The polar line of a point as to a triangle* is the line of perspective of the given triangle and the cogredient triangle.

DEFINITION $IV_{3,0}$: *The linear polar point of a point as to a linear point triad.* Given points P, P_1, P_2, P_3 on the line p. Through P_1, P_2, P_3 pass three non-concurring coplanar lines p_1, p_2, p_3 distinct from p. The polar line q of P as to the triangle p_1, p_2, p_3 intersects p in the point Q, called the linear polar point[1] of P as to the point triad P_1, P_2, P_3.

THEOREM $III_{3,1}$: If two triangles are perspective, the two polar lines of a point on their line of perspective meet on their line of perspective.

Let the corresponding sides of the perspective triangles p_1, p_2, p_3 and p_1', p_2', p_3' meet in the points P_1, P_2, P_3 of their line of perspective p.

q_1 the polar line of P (any point on p) as to $p_2\, p_3$ meets q_1' the polar line of P as to $p_2'\, p_3'$ in Q_1 on p, since the harmonic conjugate point of P as to P_2, P_3 is unique. Similarly q_2, q_2' meet in Q_2 and q_3, q_3' in Q_3 on p.

Quadrangles (p_1p_2), (p_1q_1), (q_1q_2), (p_2q_2), and $(p_1'p_2')$, $(p_1'q_1')$, $(q_1'q_2')$, $(p_2'q_2')$ have five pairs of corresponding sides meeting on p; therefore the sixth pair of sides, i.e., q, the polar of P as to p_1, p_2, p_3 and q' the polar of P as to p_1', p_2', p_3' meet on p.[2]

Points Q_1, Q_2, Q_3 are a fixed point triad associated with P, P_1, P_2, P_3 called the cogredient point triad of P as to P_1, P_2, P_3.

THEOREM $IV_{3,0}$: The linear polar point of a point as to a linear point triad is unique.

[1] By Theorem $IV_{3,0}$ the linear polar point is independent of the auxiliary triangle and of the plane of the triangle.

[2] Veblen and Young, *op. cit.*, Theorem 7.

From Theorem $III_{3,1}$, the cogredient point triad Q_1, Q_2, Q_3 are fixed points and the linear polar point Q is determined uniquely as the sixth point of the quadrangular set $(P_1, P_2, P;\ Q_2, Q_1, Q)$.

The sixth point of a quadrangular set of which five points are given is independent of the plane of the quadrangle, therefore, in finding the linear polar point the auxiliary triangle may be taken in any plane whatever passing through the given line.

§ 3. In order to generalize inductively in the plane the theorems and definitions given in § 2 for the 3-line and linear 3-ad, the following definitions and theorems are assumed for the $(k-1)$-line and the linear point $(k-1)$-ad and are proved for the k-line and the linear point k-ad for $k \geqq 4$.

THEOREM $I_{k,1}$: The k polar lines of a point as to the k $(k-1)$-line figures of a k-line form a k-line perspective to the given k-line.

For point P and k-line $\{p_i\}$, $(i=1, 2, \ldots, k)$ let q_i be the polar line of P as to the $(k-1)$-line figure $\{p_h\}$, $(h=1, 2, \ldots, k;\ h \neq i)$.

The k-line $\{q_i\}$ is called the *cogredient* k-line to $\{p_i\}$, $(i=1, 2, \ldots, k)$ as to point P.

Let $R_{i,st}$ be the points of intersection of lines p_i and PP_{st}, $(i=1, 2, \ldots, k;\ i \neq s, t)$ where $P_{st}=(p_s p_t)$.

Then q_s the polar line of $\{p_i\}$, $(i=1, 2, \ldots, k;\ i \neq s)$ as to P and q_t the polar line of $\{p_i\}$, $(i=1, 2, \ldots, k;\ i \neq t)$ as to P intersect in Q_{st} which is on PP_{st}, because the linear polar point of P as to the $(k-1)$-ad P_{st}, $R_{i,st}$, $(i=1, 2, \ldots, k;\ i \neq s, t)$ is unique (Theorem $IV_{k-1,0}$), and the two k-lines $\{p_i\}$ and $\{q_i\}$ are perspective from P.

THEOREM $II_{k,1}$: If two k-lines are perspective from a point, the points of intersection of corresponding sides are collinear.

Given two k-lines $\{p_i\}$ and $\{q_i\}$, $(i=1, 2, \ldots, k)$.

Triangles p_j, p_{j+1}, p_{j+2} and q_j, q_{j+1}, q_{j+2} are perspective from P, so corresponding sides meet in points A_j, A_{j+1}, A_{j+2} on a line a_j, $(j=1, 2, \ldots, k-2)$.

Successive lines a_j and a_{j+1} have in common two points A_{j+1}, A_{j+2}, $(j=1, 2, \ldots, k-2)$, so that all the lines a_j coincide and the intersection points of corresponding sides of the two given k-lines are collinear on a line called *the line of perspective*.

DEFINITION $IV_{k,1}$: *The polar line of a point as to a* k*-line* is the line of perspective of the k-line and its cogredient k-line as to the given point.[1]

[1] Cremona, *op. cit.*, p. 364.

THEOREM $III_{k,1}$: If two k-line figures are perspective from a point, the two polar lines of a point on their line of perspective meet on their line of perspective.

Let p be the line of perspective of the k-lines $\{p_i\}$, $\{p_i'\}$. For a point P on p let $\{q_i\}$ and $\{q_i'\}$ be the cogredient k-lines and q and q' the polar lines of P as to $\{p_i\}$ and $\{p_i'\}$ respectively $(i=1, 2, \ldots, k)$.

q_i and q_i' $(i=1, 2, \ldots, k)$ meet on p, for they are the polar lines of P as to the $(k-1)$-lines $\{p_j\}$ and $\{p_j'\}$, $(j=1, 2, \ldots, k;\ j\neq i)$ (Theorem $III_{k-1,1}$), therefore the cogredient k-lines $\{q_i\}$ and $\{q_i'\}$ have p as line of perspective. Then q_i and q_i' meet in Q_i on p and $\{Q_i\}$ is called the *cogredient point k-ad* of P as to $\{P_i\}$, $(i=1, 2, \ldots, k)$.

The quadrangles P_{rs}, $(p_r q_r)$, Q_{rs}, $(p_s q_s)$ and P'_{rs}, $(p'_r q'_r)$, Q'_{rs}, $(p'_s q'_s)$ have five pairs of sides meeting on line p, therefore the sixth pair of sides q and q' meet on p, $(r, s=1, 2, \ldots, k;\ r\neq s)$.

DEFINITION $IV_{k,0}$: *The linear polar point of a point as to a linear point k-ad.*[1] Given points $P, P_1, P_2, \ldots, P_k$ on line p. Through P_i draw coplanar lines p_i distinct from p, no three concurring. The lines p_i determine a k-line and the polar line q of P as to the k-line $\{p_i\}$ intersects p in the point Q which is the polar point of P as to the linear k-ad P_i, $(i=1, 2, \ldots, k)$.

THEOREM $IV_{k,0}$: *The polar point of a point as to* a linear point k-ad is unique.

Given $P, P_1, P_2, \ldots, P_k$ on line p. The cogredient point set $\{Q_i\}$ of P as to $\{P_i\}$, $(i=1, 2, \ldots, k)$ is determined by Theorem $III_{k,1}$, and the polar point Q of P as to $\{P_i\}$ is determined uniquely as the sixth point of any one of the quadrangular sets $(P_r\, P_s\, P;\ Q_s\, Q_r\, Q)$ $(r, s=1, 2, \ldots, k;\ r\neq s)$.

§ 4. *In space of 3 dimensions.*

DEFINITION $I_{2,2}$: *The polar plane of a point as to a pair of planes* is the harmonic conjugate plane of the point as to the pair of planes.

DEFINITION $I_{k,2}$: *A* k-*hedron in space* is a set of k-planes no 4 of which have a common point.

The following definitions and theorems are assumed for the $(k-1)$-hedron and given in full for the k-hedron, $k\geqq 4$.

[1] Poucelet, *op. cit.*, p. 231, defines Q by the equation $\frac{n}{PQ}=\sum_{i=1}^{n}\frac{1}{PP_i}$. This definition is the usual basis of treatments of linear polar theory.

THEOREM $I_{k,2}$: The k polar planes of a point as to the k $(k-1)$-hedrons of a k-hedron form a k-hedron perspective to the given k-hedron.

For point P^0 and k-plane $\{P_i^2\}$, $(i=1, 2, \ldots, k)$ let Q_j^2 be the polar plane of P^0 as to the $(k-1)$-hedron $\{P_h^2\}$, $(h=1, 2, \ldots, k; h \neq j)$.

$\{Q_j^2\}$, $(j=1, 2, \ldots, k)$ is called the *cogredient* k-*hedron* to the k-hedron $\{P_i^2\}$, $(i=1, 2, \ldots, k)$ as to P^0.

Let $R_{i,st}^1$ be the line of intersection of planes P_i^2 and P^0 P_{st}^1, $(i=1, 2, \ldots, k; i \neq s, t)$ where $P_{st}^1 = (P_s^2 P_t^2)$.

Then Q_s^2 is the polar plane of the $(k-1)$-hedron $\{P_j^2\}$, $(j=1, 2, \ldots, k; j \neq s)$ as to P^0 and Q_t^2 is the polar plane of the $(k-1)$-hedron $\{P_j^2\}$, $(j=1, 2, \ldots, k; j \neq t)$ as to P^0.

Q_s^2 and Q_t^2 intersect in line Q_{st}^1 which is on plane P^0 P_{st}^1 because the polar line of P^0 as to the $(k-1)$-line P_{st}^1, $R_{j,st}^1$, $(j=1, 2, \ldots, k; j \neq s, t)$ is uniquely defined, and the two k-hedrons $\{P_i^2\}$ and $\{Q_i^2\}$ are perspective from P^0.

THEOREM $II_{k,2}$: If two k-hedrons are perspective from a point the lines of intersection of corresponding planes are coplanar.

For $k=2$, the theorem is evident.

For $k \geqq 3$. Any plane (not through a vertex of either k-hedron) through the point of perspective intersects the k intersection lines of pairs of homologous faces in collinear points by Theorem $II_{k,1}$, therefore the k intersection lines of pairs of corresponding faces are coplanar and the plane is called the *plane of perspective.*

DEFINITION $IV_{k,2}$: *The polar plane of a point as to a* k-*hedron* in space is the plane of perspective of the k-hedron and its cogredient k-hedron as to the given point.

§ 5. *In space of* n *dimensions.*

In order to prove inductively the theorems of § 4 in n-space we assume in $(n-1)$-space Theorems $I_{k-1,n-2}$ and $II_{k-1,n-2}$, leading to the Definition $IV_{k-1,n-2}$: The polar $(n-2)$-space of a point as to a $(k-1)$-hedron in $(n-1)$-space is the $(n-2)$-space of perspective of the $(k-1)$-hedron and its cogredient $(k-1)$-hedron.

DEFINITION $I_{2,n-1}$: *The polar* $(n-1)$-*space of a point as to a pair of* $(n-1)$-*spaces* is the harmonic conjugate $(n-1)$-space of the point as to the pair of $(n-1)$-spaces and is determined as follows: Any line through the given point and not through the $(n-2)$-space of intersection of the two given $(n-1)$-spaces intersects each $(n-1)$-space in a point. The harmonic conjugate point of the given point as to this pair of points and the $(n-2)$-space of intersection of the two given $(n-1)$-spaces determine the harmonic conjugate $(n-1)$-space of the given point as to the pair of $(n-1)$-spaces. This determination can be proved to be unique.

DEFINITION $\mathrm{I}_{k,n-1}$: *An* n-*space* k-*hedron* is a set of k $(n-1)$-spaces, no $n+1$ of which have a common point.

DEFINITION $\mathrm{II}_{k,n-1}$: *Two* k-*hedrons are perspective from a point* if the $(n-2)$-space edges, in corresponding pairs, lie in $(n-1)$-spaces which pass through the point of perspectivity.

THEOREM $\mathrm{I}_{k,n-1}$: In n-space the k polar $(n-1)$-spaces of a given point as to the k $(k-1)$-hedrons of a k-hedron form a k-hedron perspective to the given k-hedron.

For point P^0 and k-hedron $\{P_i^{n-1}\}$, $(i=1, 2, \ldots, k)$ let Q_j^{n-1} be the polar $(n-1)$-space of P^0 as to the $(k-1)$-hedron $\{P_h^{n-1}\}$, $(h=1, 2, \ldots, k;\ h\neq j)$.

$\{Q_j^{n-1}\}$, $(j=1, 2, \ldots, k)$ is called the cogredient k-hedron of k-hedron $\{P_i^{n-1}\}$, $(i=1, 2, \ldots, k)$ as to P^0.

Let $R_{i,st}^{n-2}$ be the $(n-2)$-space of intersection of $(n-1)$-spaces P_i^{n-1} and $P^0P_{st}^{n-2}$, $(i=1, 2, \ldots, k;\ i\neq s, t)$ where $P_{st}^{n-2}=(P_s^{n-1}P_t^{n-1})$.

Then Q_s^{n-1} is the polar $(n-1)$-space of $(k-1)$-hedron $\{P_j^{n-1}\}$, $(j=1, 2, \ldots, k;\ j\neq s)$ as to P^0.

And Q_t^{n-1} is the polar $(n-1)$-space of $(k-1)$-hedron $\{P_j^{n-1}\}$, $(j=1, 2, \ldots, k;\ j\neq t)$ as to P^0.

Q_s^{n-1} and Q_t^{n-1} intersect in $(n-2)$-space Q_{st}^{n-2} which is on $(n-1)$-space $P^0\ P_{st}^{n-2}$ since the polar $(n-2)$-space of P^0 as to $(n-1)$-space $(k-1)$-hedron P_{st}^{n-2}, $R_{j,st}^{n-2}$, $(j=1, 2, \ldots, k;\ j\neq s, t)$ is uniquely defined, and the two k-hedrons $\{P_i^{n-1}\}$ and $\{Q_i^{n-1}\}$ are perspective from P^0.

DEFINITION $\mathrm{III}_{k,n-1}$: A complete k $(n-1)$-space is a k $(n-1)$-space with no $(n+1)$ $(n-1)$-spaces through the same point such that each $(n-1)$-space cuts every other in an $(n-2)$-space.

LEMMA: A complete k $(n-1)$-space is an n-space k-hedron, i.e., has all of its elements in an n-space.

Every $(n-1)$-space of a complete k $(n-1)$-space intersects every other $(n-1)$-space in an $(n-2)$-space, therefore the n-space determined by one pair of $(n-1)$-spaces contains all the remaining $(n-1)$-spaces, since it contains two distinct $(n-2)$-space of every one that remains.

THEOREM $\mathrm{II}_{k,n-1}$: *The Desargues Theorem for n-space.* If two k-hedrons are perspective from a point, corresponding $(n-1)$-space faces meet in $(n-2)$-spaces of the same $(n-1)$-space.

If $\{A_i^{n-1}\}$ and $\{B_i^{n-1}\}$, $(i=1, 2, \ldots, k)$ are perspective k-hedrons, any pair of corresponding $(n-2)$-space edges $A_{1,2}^{n-2}=A_1^{n-1}$, A_2^{n-1} and $B_{1,2}^{n-2}=B_1^{n-1}$, B_2^{n-1} lie in the same $(n-1)$-space $C_{1,2}^{n-1}$ and therefore intersect in an $(n-3)$-space $C_{1,2}^{n-3}$. Then $C_1^{n-2}=A_1^{n-1}$, B_1^{n-1}, and $C_2^{n-2}=A_2^{n-1}$, B_2^{n-1} contain $C_{1,2}^{n-3}$ and in general any pair of $(n-2)$-spaces C_r^{n-2}, C_s^{n-2} which are intersections of corresponding pairs of $(n-1)$-space faces of the given k-hedrons have a common $(n-3)$-space, therefore the whole intersection figure is a complete k $(n-2)$-space and hence must lie in an $(n-1)$-space (by the Lemma) which is called the *$(n-1)$-space of perspective.*

DEFINITION $\mathrm{IV}_{k,n-1}$: *The polar $(n-1)$-space of a point as to a* k*-hedron in* n*-space* is the $(n-1)$-space of perspective of the k-hedron and its cogredient k-hedron as to the given point.

Thereoms $\mathrm{III}_{k,n-1}$ and $\mathrm{IV}_{k,n-2}$ are unnecessary for $n>2$, as the uniqueness of the linear polar is evident from the construction except in the case of linear polars of linear point sets.

All the theorems and constructions .of this section may be dualized.

II. ANALYTIC TREATMENT

§ 6. It is possible to extend the set of assumptions given in § 1 to form a sufficient basis for a system of homogeneous co-ordinates and to proceed analytically (Veblen and Young, *op. cit.*, § 2, p. 352).

For an n-ary linear form we use the Clebsch notation:

$$a_x^{(n)} = \sum_{g=1}^{n} a_g x_g$$

and we indicate the factored n-ary k-ic

$$f_x^{n,k} = \prod_{i=1}^{k} a_{i,x}^{(n)}$$

where[1]

$$a_{i,x}^{(n)} = \sum_{g=1}^{n} a_{ig} x_g$$

The polar operator[2] is written

$$\left(x' \frac{\partial}{\partial x}\right)_n = \sum_{i=1}^{n} x'_i \frac{\partial}{\partial x_i}$$

and the polar operator repeated r times is indicated

$$\left(x' \frac{\partial}{\partial x}\right)_n^r$$

The $(k-1)^{st}$ polar or linear polar of the point $x' = (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$ with respect to $f_x^{n,k}$ where $f_{x'}^{n,k} \neq 0$ may be written in the form

$$\left(x' \frac{\partial}{\partial x}\right)_n^{k-1} f_x^{n,k} = \underline{|k-1}\, f_{x'}^{n,k} \sum_{j=1}^{k} \frac{a_{j,x}^{(n)}}{a_{j,x'}^{(n)}}$$

§ 7. *The polar line of a point as to a 2-line.*
Given point $P: (x'_1, x'_2, x'_3)$ and lines $p_1: a_{1,x}^{(3)} = 0$; $p_2: a_{2,x}^{(3)} = 0$.
The line $PP_{12} = p$ is

$$\frac{a_{1,x}^{(3)}}{a_{1,x'}^{(3)}} - \frac{a_{2,x}^{(3)}}{a_{2,x'}^{(3)}} = 0\,.$$

[1] The superscript (n) is omitted when no ambiguity arises.

[2] The subscript n is omitted when no ambiguity arises.

The line $x_3=0$ intersects p, p_1, p_2 in P_0, P_1, P_2.

$$P_0:\frac{a^{(2)}_{1,x}}{a^{(3)}_{1,x'}}-\frac{a^{(2)}_{2,x}}{a^{(3)}_{2,x'}}=0;\quad P_1: a^{(2)}_{1,x}=0;\quad P_2: a^{(2)}_{2,x}=0$$

The harmonic conjugate of the point given by $\lambda_1 a^{(2)}_{1,x}+\lambda_2 a^{(2)}_{2,x}=0$ as to the points $a^{(2)}_{1,x}=0$ and $a^{(2)}_{2,x}=0$ is given by the equation

$$\lambda_1 a^{(2)}_{1,x}-\lambda_2 a^{(2)}_{2,x}=0.$$

Then the harmonic conjugate Q of P_0 as to P_1, P_2 is

$$\frac{a^{(2)}_{1,x}}{a^{(3)}_{1,x'}}+\frac{a^{(2)}_{2,x}}{a^{(3)}_{2,x'}}=0,$$

then the line $q=QP_{12}$ is

$$\frac{a^{(3)}_{1,x}}{a^{(3)}_{1,x'}}+\frac{a^{(3)}_{2,x}}{a^{(3)}_{2,x'}}=0,$$

the linear polar of P as to lines p_1 and p_2.

§ 8. *The polar line of a point as to a* k-*line.*[1]

Given point P (x_1', x_2', x_3') and lines $\{p_i\}$: $a_{i,x}=0$, $(i=1, 2, \ldots, k)$. For purposes of an inductive development we assume that the polar line of a point P (x_1', x_2', x_3') as to a $(k-1)$-line $\{l_i\}$: $b_{i,x}=0$, $(i=1, 2, \ldots, k-1)$ is given by the equation

$$\sum_{i=1}^{k-1}\frac{b_{i,x}}{b_{i,x'}}=0$$

then the cogredient k-line of k-line $\{p_i\}$ as to P will be

$$\{q_i\}:\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{a_{j,x}}{a_{j,x'}}=0,\quad (i=1, 2, \ldots, k).$$

Any line through the point of intersection of p_i and q_i is given by the equation

$$\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{a_{j,x}}{a_{j,x'}}+\alpha_i a_{i,x}=0,\quad (i=1, 2, \ldots, k)$$

where α_i are arbitrary constants.

These equations are all identical for $\alpha_i=\dfrac{1}{a_{i,x'}}$ so that all the points

[1] Cf. Cayley, "Sur quelques théorèmes de la géométrie de position," *Collected Works*, I, 360.

$(p_i\, q_i)$, $(i=1, 2, \ldots, k)$ are collinear on the line

$$\sum_{i=1}^{k} \frac{a_{i,x}}{a_{i,x'}}=0$$

which is the equation of the polar line of P as to the k-line $\{p_i\}$.

§ 9. *The linear polar point of a point as to a linear point* k-*ad.*

Given point $P:(x_1', x_2', 0)$ and points $P_i : a^{(2)}_{i,x}=0$, $(i=1, 2, \ldots, k)$ on line $p : x_3=0$.

Pass the lines $p_i : a^{(3)}_{i,x}=0$ through the points P_i.

The polar line of P as to the k-line $\{p_i\}$ is the line

$$q : \sum_{i=1}^{k} \frac{a^{(3)}_{i,x}}{a^{(2)}_{i,x'}}=0$$

and q intersects p in the point

$$Q : \sum_{i=1}^{k} \frac{a^{(2)}_{i,x}}{a^{(2)}_{i,x'}}=0,$$

which is the equation of the polar point of P as to the k-ad $\{P_i\}$.

§ 10. *The MacLaurin generalized definition of harmonic mean.*

Let $OQ=y=\dfrac{x_1}{x_2}$ represent the distance from some fixed point O taken as origin on a given line to any point Q of the line

For the point $P_i : a^{(2)}_{i,x}=0 \quad y=-\dfrac{a^{(2)}_{i,2}}{a^{(2)}_{i,1}}$.

For the point P (x_1', x_2') $OP=\dfrac{x_1'}{x_2'}$.

For a general point Q (x_1, x_2) $OQ=\dfrac{x_1}{x_2}$.

$$QP_i=OQ-OP_i=\frac{a^{(2)}_{i,x}}{a^{(2)}_{i,1}x_2}$$

$$PP_i=OP-OP_i=\frac{a^{(2)}_{i,x'}}{a^{(2)}_{i,1}x_2'}$$

If Q is the polar point of P as to $\{P_i\}$, $(i\neq 1, 2, \ldots, k)$,

$$\frac{a^{(2)}_{i,x}}{a^{(2)}_{i,x'}}=0$$

which reduces to

$$\sum \frac{QP_i}{PP_i}=0$$

Whence

$$\sum \frac{PP_i - PQ}{PP_i} = 0$$

or

$$\sum \left(\frac{1}{PQ} - \frac{1}{PP_i} \right) = 0,$$

or

$$\frac{n}{PQ} = \sum \frac{1}{PP_i},$$

so that PQ is the harmonic mean of the segments PP_i according to the MacLaurin generalized definition.

§ 11. *The polar plane of the* k-*hedron in space as to a given point.*

Given point P° (x_1', x_2', x_3', x_4') and k-plane $\{P_i^2\}$: $a_{i,x}^{(4)} = 0$, $(i = 1, 2, \ldots, k)$.

For purposes of an inductive development we assume that the polar plane of a point P° (y_1, y_2, y_3, y_4) as to a $(k-1)$-plane $\{R_i^2\}$: $b_{i,x}^{(4)} = 0$, $(i = 1, 2, \ldots, k-1)$ is given by the equation:

$$\sum_{i=1}^{k-1} \frac{b_{i,x}^{(4)}}{b_{i,y}^{(4)}} = 0$$

Then the cogredient k-plane of the k-plane $\{P_i^2\}$ as to point P° is

$$\{Q_i^2\} : \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{a_{j,x}^{(4)}}{a_{j,x'}^{(4)}} = 0$$

Any plane through the line $(P_i^{(2)} Q_i^{(2)})$ is given by the equation:

$$\sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{a_{j,x}^{(4)}}{a_{j,x'}^{(4)}} + a_i \, a_{i,x}^{(4)} = 0$$

where a_i are arbitrary constants. These planes are all identical for $a_i = \frac{1}{a_{i,x'}^{(4)}}$, so that the polar plane of P° as to the k-plane $\{P_i^2\}$ is given by the equation:

$$\sum_{i=1}^{k} \frac{a_{i,x}^{(4)}}{a_{i,x'}^{(4)}} = 0$$

§ 12. *The polar* $(n-1)$-*space of a* k-*hedron in* n-*space as to a given point.*

Given point P° $(x_1', x_2', \ldots, x_{n+1}')$ and k-hedron $\{P_i^{n-1}\}$: $a_{i,x}^{(n+1)} = 0$, $(i = 1, 2, \ldots, k)$.

For purposes of an inductive development we assume that the polar line of a point P° $(y_1, y_2, \ldots, y_{n+1})$ as to a $(k-1)$-hedron in n-space $\{R_i^{n-1}\} : b_{i,x}^{(n+1)}=0$ is given by the equation

$$\sum_{i=1}^{k} \frac{b_{i,x}^{(n+1)}}{b_{i,y}^{(n+1)}}=0$$

Then the cogredient k-hedron of $\{P_i^{n-1}\}$ as to P° is

$$\{Q_i^{n-1}\} : \sum_{\substack{j=1 \\ j\neq i}}^{k} \frac{a_{j,x}^{(n+1)}}{a_{j,x'}^{(n+1)}}=0$$

Any $(n-1)$-space through the $(n-2)$-space $(P_i^{n-1}\ Q_i^{n-1})$ is given by the equation

$$\sum_{\substack{j=1 \\ j\neq i}}^{k} \frac{a_{j,x}^{(n+1)}}{a_{j,x'}^{(n+1)}}+a_i\, a_{i,x}^{(n+1)}=0$$

where a_i are arbitrary constants and these $(n-1)$-spaces are all identical for $a_i=\frac{1}{a_{i,x'}^{n+1}}$, so that the polar $(n-1)$-space of P° as to the k-hedron $\{P_i^{n-1}\}$ is given by the equation

$$\sum_{i=1}^{k} \frac{a_{i,x}^{(n+1)}}{a_{i,x'}^{(n+1)}}=0$$

III. ALGEBRAIC LOCI

§ 13. From Section II we can prove "Cotes's Theorem."[1]

THEOREM I: "*Any transversal* line through a point intersects its polar line as to a curve of the n^{th} order in the polar point of the linear point n-ad determined by the curve on the transversal";
and the generalization to n-space:

THEOREM II: Any transversal line through a point intersects its polar $(n-1)$-space as to an n-space spread of the k^{th} order in the polar point of the linear point k-ad determined by the spread on the transversal.

From Theorem I we obtain the following method for constructing the polar line[2] of a point P as to a curve of the n^{th} order C_n.

Through P pass any two transversals p_1, p_2 intersecting C_n in points $P_{i,1}$, $P_{i,2}$, $(i=1, 2, \ldots, n)$. Connect the points $P_{i,1}$ and $P_{i,2}$ by the lines p_i forming n-line $\{p_i\}$. (This can be done in n^2 ways by changing the notation for the points.) Then the polar line q of P as to the n-line $\{p_i\}$ is the polar line of P as to the curve C_n, because q has two points, one on each transversal common with the polar line of C_n, by Theorem I.

Likewise from Theorem II we obtain the general method of constructing the polar $(n-1)$-space of a point P^0 as to a spread Q_k of the k^{th} order in n-space.

Through P^0 pass any n transversal lines P^1_j $(j=1, 2, \ldots, n)$ not in the same $(n-1)$-space, intersecting Q_k in points $P^0_{i,j}$, $(i=1, 2, \ldots, k)$.

Let the points $P^0_{i,j}$, $(j=1, 2, \ldots, n)$ determine $(n-1)$-space P^{n-1}_i, whence for $(i=1, 2, \ldots, k)$ we get the n-space k-hedron $\{P^{n-1}_i\}$. (This can be done in k^n ways by changing the notation for the points $P^0_{i,j}$.) Then the polar $(n-1)$-space Q^{n-1} of P^0 as to the k-hedron $\{P^{n-1}_i\}$ is the polar $(n-1)$-space of P^0 as to the spread Q_k, since Q^{n-1} has n points, one on each transversal common with the polar $(n-1)$-space of P^0 as to Q_k, by Theorem II.

[1] MacLaurin, *op. cit.*, § 28.

[2] For the cubic see Salmon, *Higher Plane Curves*, 3d ed., p. 143; Durege, *Curven Dritten Ordnung*, pp. 167, 168.

IV. CERTAIN CONFIGURATIONS WITH POLARITY PROPERTIES

a) THE ASSOCIATED 4-POINT AND 4-LINE IN THE PLANE

§ 14. Let p_i be the polar line of the point P_i of a given 4-point figure $\{P_i\}$ in a plane taken with respect to the triangle formed by the other three points ($i=1, 2, 3, 4$).

We then have associated with the 4-point $\{P_i\}$ the 4-line $\{p_i\}$. The two figures form a complete quadrangle and complete quadrilateral with a common diagonal triangle.

In homogeneous co-ordinates with the common diagonal triangle as triangle of reference, if one of the four points P_i is taken as unity point, the corresponding points and lines of the two figures have the same co-ordinates:

$$\begin{array}{llll} p_1: & -x_1+x_2+x_3=0 & P_1: & (-1, 1, 1) \\ p_2: & x_1-x_2+x_3=0 & P_2: & (1, -1, 1) \\ p_3: & x_1+x_2-x_3=0 & P_3: & (1, 1, -1) \\ p_4: & x_1+x_2+x_3=0 & P_4: & (1, 1, 1) \end{array}$$

From the duality of these equations it is evident that the configuration is self-reciprocal.

In supernumerary co-ordinates with $\Sigma x_i=0$

$$p_i: x_i=0 \quad P_i: (x_i=-3,\ x_j=1), \ (j=1, 2, 3, 4;\ j\neq i)$$

for $i=1, 2, 3, 4$.

The group of collineations under which the configuration is invariant is the permutation group $G_{4!}$ and the 24 transformations are given by the following equations in supernumerary co-ordinates

$$x_i'=x_{r_i}, \quad (i=1, 2, 3, 4)$$

$$(r_1, r_2, r_3, r_4 \text{ distinct}=1, 2, 3, 4)$$

b) THE ASSOCIATED $(n+2)$-POINT AND $(n+2)$-FLAT IN n-SPACE[1]

§ 15. *The* n*-space configuration.*

An l-space is incident with an m-space if, for $l<m$ the l-space lies in the m-space, for $l>m$ the l-space contains the m-space. An n-space

[1] The contents of Sections IV*a* and IV*b* are in substance given in MacNeish, *A Self Dual Configuration in* n*-Space*, Master's Thesis, University of Chicago, 1904 (written in connection with Dr. Moore's projective geometry course, 1902), deposited in Library of the Department of Mathematics of the University of Chicago.

configuration is a system of n sets of k-spaces $(k=0, 1, \ldots, n-1)$; a_0 points, a_1 lines, and in general a_n n-spaces such that every g-space is incident with the same number a_{gh} of h-spaces $(g, h=0, 1, 2, \ldots, n-1; g\neq h)$.

For k-spaces we use the notation:

$$A^k_{i_1 i_2 \ldots i_{k'}} \quad \begin{array}{l}(k=0, 1, \ldots, n-1)\\ (i_j=1, 2, \ldots, a_k \text{ for } j=1, 2, \ldots, k';\\ i_j\neq i_{j'} \text{ for } j\neq j')\end{array}$$

The numbers a_i, a_{gh} are written as a square matrix called the *configuration specification*[1] as follows:

$$(a_{gh}), \quad (g, h=0, 1, \ldots, n-1; \quad a_{gg}=a_g)$$

The elements of the main diagonal $a_{gg}=a_g$ specify the number of g-spaces, and any element a_{gh} specifies the number of g-spaces incident with each h-space.

It can be proved that between the numbers of a configuration specification, the following relations hold:

$$a_{ij}\, a_{jj}=a_{ji}\, a_{ii}\,, \quad (i, j=0, 1, \ldots, n-1)$$

The *dual configuration* to a given configuration in n-space is defined by interchanging the words g-space and $(n-g-1)$-space $(g=0, 1, \ldots, n-1)$ in the definition of the given configuration.

An n-space configuration dual to itself is called a *self* n-*space dual configuration.*

GENERAL DEFINITION OF CIRCUMSCRIPTION IN n-SPACE

In n-space one configuration (a_{ij}) circumscribes another (b_{ij}) index $n-k$, $(n-1 \geq k \geq 1)$ if the a_r r-spaces $A^r_{i_1 i_2, \ldots, i_{r'}}$ of the first for $r=k$, $k+1, \ldots, n-1$ are in one-to-one correspondence with the $b_{\bar r}$ $\bar r$-spaces $B^{\bar r}_{j_1 j_2, \ldots, j_{\bar r'}}$, of the second for $\bar r=r-$k in such a way that corresponding r-spaces and $\bar r$-spaces are incident.

§ 16. *The associated $(n+2)$-point and $(n+2)$-flat in* n-*space.*

Given $n+2$ points A^0_i, $(i=1, 2, \ldots, n+2)$ in n-space (no $k+2$ of them in a k-space for $k=1, 2, \ldots, n-1$). Let $\bar A^{n-1}_i$ be the polar $(n-1)$-space of A^0_i, taken with respect to the $(n+1)$-hedron $\{A^0_j\}$, $(j=1, 2, \ldots, n+2;\ j\neq i)$ whose vertices are the remaining given points (see § 5, Definition $\mathrm{III}_{k, n-1}$). We then have associated with the $(n+2)$-point an $(n+2)$-flat.

[1] Cf. E. H. Moore, "Tactical Memoranda I," *American Journal of Mathematics*, XVIII (1896), 264.

In the $(n+2)$-point figure $\{A_i^0\}$ any k-space is denoted $A^k_{i_1 i_2, \ldots, i_{k+1}}$ and it contains every element of lower dimensions whose subscripts are all of the set $i_1, i_2, \ldots, i_{k+1}$.

In the $(n+2)$-flat figure $\{\bar{A}_i^{n-1}\}$ any k-space is denoted $\bar{A}^k_{i_1 i_2, \ldots, i_{n-k}}$ and it lies in every element of higher dimensions whose subscripts are all of the set $i_1, i_2, \ldots, i_{n-k}$.

In supernumerary co-ordinates in n-space, where $\sum_{i=1}^{n+1} x_i = 0$

$\bar{A}_i^{n-1} : x_i = 0$

$A_i^0 \quad : (x_i = -(n+1), x_j = 1), \quad (j = 1, 2, \ldots, n+2;\ j \neq i)$

for $i = 1, 2, \ldots, n+2$.

From the duality of the co-ordinates (i.e., point co-ordinates and $(n-1)$-space co-ordinates) since $\sum_{i=1}^{n+2} x_i = 0$, it follows that the configuration is self-reciprocal.

The group[1] of $(n+1)$-ary collineations under which the configuration is invariant is simply isomorphic to the symmetric group on $n+2$ letters and the equations of the collineations are of the form:

$$r : x_i' = x_{r_i} \ (i = 1, 2, \ldots, n+2)$$

where $r = (r_1, r_2, \ldots, r_{n+2})$ is a permutation of $(1, 2, \ldots, n+2)$.

§ 17. THEOREM: The $(n+2)$-flat is inscribed index $n-1$ in the $(n+2)$-point.

$$A^{n-1}_{i_1 i_2, \ldots, i_n} \text{ is represented by } \quad x_{i_{n+1}} = x_{i_{n+2}}$$

$$\bar{A}^{n-2}_{i_{n+1}, i_{n+2}} \text{ is represented by } \begin{cases} x_{i_{n+1}} = 0 \\ x_{i_{n+2}} = 0 \end{cases}$$

Therefore $A^{n-1}_{i_1, \ldots, i_n}$ of the $(n+2)$-gon contains $\bar{A}^{n-2}_{i_{n+1} i_{n+2}}$ of the $(n+2)$-flat.

And in general

$$A^{n-k}_{i_1 i_2, \ldots, i_{n-k+1}} \text{ of the } (n+2)\text{-gon}$$

is represented by

$$x_{i_{n-k+2}} = x_{i_{n-k+3}} = \ldots = x_{i_{n+2}}$$

[1] See E. H. Moore, "Concerning Klein's Group of $(n+1)!$ n-ary Collineations," *American Journal of Mathematics*, XXII (1900), 336.

and $\bar{A}^{n-k-1}_{i_{n-k+2}\, i_{n-k+3}, \ldots, i_{n+2}}$ of the $(n+2)$-flat is determined by the $k+1$ equations:

$$\begin{cases} x_{i_{n-k+2}}=0 \\ x_{i_{n-k+3}}=0 \\ \cdots\cdots \\ x_{i_{n+2}}\quad =0 \end{cases}$$

so that $A^{n-k}_{i_1\, i_2, \ldots, i_{n-k+1}}$ of the $(n+2)$-point contains $\bar{A}^{n-k-1}_{i_{n-k+2}\, i_{n-k+3}, \ldots, i_{n+2}}$ of the $(n+2)$-flat, and the $(n+2)$-point circumscribes the $(n+2)$-flat index $n-1$.

§ 18. The associated $(n+2)$-point and $(n+2)$-flat in n-space form a configuration whose specification is the matrix:

$$(a_{gh}),\quad (g,\ h=0,\ 1,\ \ldots,\ n-1)$$

where

$$a_{gg}=\binom{n+2}{g+1}+\binom{n+2}{n-g}$$

and

$$a_{gh}=\binom{n-g+1}{h-g}\ \text{for}\ h>g$$

and

$$a_{gh}=\binom{g+2}{g-h}\ \text{for}\ g>h$$

where $\binom{u}{v}$ denotes ${}_uC_v$, the number of combinations of v things taken from u things. $u>v$.

Theorem: In the polar $(n-1)$-space $\bar{A}^{n-1}_j$ of the point A^0_j, as to the $(n+1)$-point $\{A^0_i\}$, $(i=1,\ 2,\ \ldots,\ n+2;\ i\neq j)$ in n-space, the section of the $(n+2)$-point $\{A^0_i\}$, $(i=1,\ 2,\ \ldots,\ n+2)$ is the $(n+1)$-point $(n+1)$-flat configuration in $(n-1)$-space.

In a supernumerary co-ordinate system

A^0_i is represented by $(x_i=-(n+1),\ x_j=1$ for $j=1,\ 2,\ \ldots,\ n+2;\ j\neq i)$ and

$$\bar{A}^{n-1}_i \text{ is represented by } x_i=0$$

For simplicity consider the section of the configuration in $(n-1)$-space $\bar{A}^{n-1}_1$: $x_1=0$ and in order to have a supernumerary system in this $(n-1)$-space we will omit the variable x_1 and call $x_i=y_{i-1}$, $(i=2,\ 3,\ \ldots,\ n+2)$ whence $\sum_{i=1}^{n+1} y_i=0$.

Line A^1_{1k} intersects $\bar{A}^{n-1}_1$ in point B^0_{k-1} $(k=2, 3, \ldots, n+2)$ with co-ordinates $(y_{k-1}=-n, y_j=1$ *for* $j=1, 2, \ldots, n+1; j\neq k-1)$ $\bar{A}^{n-1}_k$ $k=2, 3, \ldots, n+2$ intersects $\bar{A}^{n-1}_1$ in $\bar{B}^{n-2}_{k-1}$ given by the pair of equations $x_1=0$, $x_{k-1}=0$ or simply by $x_{k-1}=0$ $(k=2, 3, \ldots, n+2)$.

Then $(n+1)$-point $\{B^0_{k-1}\}$ and $(n+1)$-flat $\{\bar{B}^{n-2}_{k-1}\}$ have precisely the co-ordinates of the associated $(n+1)$-point $(n+1)$-flat configuration in $(n-1)$-space (see § 17). The same can be proved of the sections in the $(n-1)$-spaces $\bar{A}^{n-1}_r$, $(r=2, 3, \ldots, n+2)$.

c) THE ASSOCIATED r-POINT AND r-FLAT IN n-SPACE

§ 19. (1) For $r=n+2$:

Given $(n+2)$-point $\{P^0_i\}$, $(i=1, 2, \ldots, n+2)$ in n-space. Call P^{n-1}_i the polar $(n-1)$-space of point P^0_i, as to $(n+1)$-point $\{P^0_{i'}\}$, $(i'=1, 2, \ldots, n+2; i'\neq i)$. An $(n+1)$-point in n-space is also an $(n+1)$-flat. Then $\{P^{n-1}_i\}$ is the $(n+2)$-flat associated with the $(n+2)$-point $\{P^0_i\}$. The properties of this configuration are discussed in §§ 16, 17, 18.

(2) For $r=n+3$:

Given $(n+3)$-point $\{P^0_i\}$, $(i=1, 2, \ldots, n+3)$ in n-space. With $(n+2)$-point $\{P^0_{i'}\}$, $(i'=1, 2, \ldots, n+3; i'\neq i)$ is associated $(n+2)$-flat $P^{n-1}_{i,i'}\}$ by § 18 (1). Call P^{n-1}_i the polar $(n-1)$-flat of the point P^0_i, as to the $(n+2)$-flat $\{P^{n-1}_{i,i'}\}$, $(i'=1, 2, \ldots, n+3; i'\neq i)$. Then $\{P^{n-1}_i\}$ is the $(n+3)$-flat associated with the $(n+3)$-point $\{P^0_i\}$ $(i=1, 2, \ldots, n+3)$.

(3) In general:

Given r-point $\{P^0_i\}$, $(i=1, 2, \ldots, r)$ in n-space $r\geqq n+2$. With $(r-1)$-point $\{P^0_{i'}\}$, $(i'=1, 2, \ldots, r; i'\neq i)$ is associated an $(r-1)$-flat $\{P^{n-1}_{i,i'}\}$ obtained by successive application of the method of § 19 (2) above. Call P^{n-1}_i the polar $(n-1)$-space of point P^0_i, as to $(r-1)$-flat $\{P^{n-1}_{i,i'}\}$, $(i'=1, 2, \ldots, r; i'\neq i)$. Then $\{P^{n-1}_i\}$ is the r-flat associated with r-point $\{P^0_i\}$.

d) ASSOCIATED POINT SETS ON A LINE

Given r-point $\{P^0_i\}$, $(i=1, 2, \ldots, r)$ on a line P^1. To any subset of $r-1$ of these points $\{P^0_j\}$, $(j=1, 2, \ldots, r; j\neq k)$ there is a cogredient set (see § 3, Theorem $\mathrm{IV}_{k,0}$) of $r-1$ points $\{\bar{P}^0_j\}$, $(j=1, 2, \ldots, r; j\neq k)$ as to the point P^0_k. Call the polar point of P^0_k as to the $(r-1)$-point $\{\bar{P}^0_j\}$, Q^0_k. The r-point $\{Q^0_i\}$ is the associated r-point to $\{P^0_i\}$, $(i=1, 2, \ldots, r)$.

V. THE RECIPROCITY OF CERTAIN ASSOCIATED LINEAR SETS OF POINTS

§ 20. Let the linear equation $a_x = a_1x_1 + a_2x_2 = 0$ represent the point $(a_2, -a_1)$ on some fundamental line. We use the co-ordinates (u_1, u_2) to represent a point in a manner analogous to the method of writing point and line co-ordinates in the plane. Then $a_u = a_1u_1 + a_2u_2 = 0$ represents the point (a_1, a_2) and the equations $a_1x_1 + a_2x_2 = 0$ and $a_2u_1 - a_1u_2 = 0$ represent the same point. We will consider certain sets of points given by their co-ordinates and write their equations in u_1, u_2; while certain sets of points associated with them will be given by equations in x_1, x_2.

Throughout Section V, the notation for the concomitants of Binary Forms will be that of Clebsch, *Theorie der binären algebraischen Formen.*

§ 21. *Associated linear 3-points.*

For a linear point triple represented by a binary cubic $f_u = 0$, we designate as the associated point triple, the triple consisting of the harmonic conjugate points of each point as to the remaining pair. The associated point triple is represented by the cubic covariant of f_u, i.e., $Q_u = 0$ (cf. Clebsch, *op. cit.*, pp. 115, 134), or by the contravariant $Q_x = 0$ obtained by changing u_1 to $-x_2$, u_2 to x_1 in Q_u. Now $Q_u(Q_x) = -R_4^2 f_u$ where R is the Discriminant of f_u (cf. Clebsch, *op. cit.*, p. 123); therefore the two point triples are reciprocal. The two point triples form 3 pairs of points belonging to a quadratic involution and the double points are represented by the Hessian H_u of f_u.

§ 22. *Associated linear 4-points.*

Let P' be the linear polar point of $P(y_1, y_2)$ as to the point triple A_p, B_p, C_p associated with the triple A, B, C represented by a binary cubic $f_u = 0$. A_p, B_p, C_p (cf. § 21) are represented by $Q_x = 0$. Then P' is given by the equation:

$$y_1^2 \frac{\partial^2 Q_x}{\partial x_1^2} + 2y_1y_2 \frac{\partial^2 Q_x}{\partial x_1 \partial x_2} + y_2^2 \frac{\partial^2 Q_x}{\partial x_2^2} = 0 \qquad (1)$$

Given the four points

$$\begin{aligned} A &: a_u = 0 \\ B &: u_1 = 0 \\ C &: u_2 = 0 \\ D &: u_1 + u_2 = 0 \end{aligned}$$

then $f_u = a_u u_1 u_2(u_1+u_2) = a_1 u_1^3 u_2 + (a_1+a_2)u_1^2 u_2^2 + a_2 u_1 u_2^3 = 0$ represents the four points A, B, C, D.

By formula (1) we can obtain the equation for point A', the polar point of A as to the triple B_A, C_A, D_A associated with B, C, D. Similarly points B', C', D' can be obtained. A', B', C', D' form the 4-point associated with 4-point A, B, C, D.

$$\begin{aligned}
A' &: x_1(2a_1^2 - 2a_1a_2 - a_2^2) - x_2(a_1^2 + 2a_1a_2 - 2a_2^2) = 0\\
B' &: x_1(a_1+a_2)\,(a_1-2a_2)\,(2a_1-a_2) - a_1x_2(a_1^2 - 4a_1a_2 + a_2^2) = 0\\
C' &: a_2x_1(a_1^2 - 4a_1a_2 + a_2^2) - x_2(a_1+a_2)\,(a_1-2a_2)\,(2a_1-a_2) = 0\\
D' &: a_2x_1(a_1^2 + 2a_1a_2 - 2a_2^2) - a_1x_2(2a_1^2 - 2a_1a_2 - a_2^2) = 0
\end{aligned}$$

Then if f'_x is the product of these four linear expressions $f'_x = 0$ represents the four points A', B', C', D'.

From f_u we obtain:

$$\begin{aligned}
f_x &= -a_2x_1^3x_2 + (a_1+a_2)x_1^2x_2^2 - a_1x_1x_2^3 = 0\\
H_x &= -\tfrac{1}{24}[3a_2^2x_1^4 - 4a_2(a_1+a_2)x_1^3x_2 + 2(2a_1^2 + a_1a_2 + 2a_2^2)x_1^2x_2^2 - 4a_1(a_1+a_2)x_1x_2^3\\
&\qquad + 3a_1^2x_2^4]
\end{aligned}$$

where H_x is the Hessian of f_x.

The two invariants I and J of f_x are:

$$\begin{aligned}
I &= \tfrac{1}{6}(a_1^2 - a_1a_2 + a_2^2)\\
J &= -\tfrac{1}{72}(a_1+a_2)\,(a_1-2a_2)\,(2a_1-a_2)
\end{aligned}$$

Then f'_x is expressible as a function of f_x, H_x, I, J:

$$f'_x = 8 \cdot 6^4\{24J(12J^2 - I^3)H_x + I^2(I^3 + 42J^2)f_x\} \tag{2}$$

§ 23. *The self-reciprocal* 4-*point.*

If either $J=0$ or $12J^2 - I^3 = 0$; $f'_x = 0$ represents the same 4 points as $f_u = 0$ and the 4-point is self-reciprocal.

For $J=0$, the 4 points are harmonic and each point goes into itself, so that the 4-point is *identically self-reciprocal.*

For $12J^2 - I^3 = 0$, the 4 points are operated on by the substitutions (AB) (CD); (AC) (DB); (AD) (BC).

Therefore the two cases in which the 4-point is self-reciprocal constitute the substitutions of the subgroup G_4 of the symmetric group $G_{4!}$ on 4 letters.

It can be proved that $12J^2 - I^3 = 0$ is the necessary and sufficient condition that A,A'; B,B'; C,C'; D,D' are pairs of a quadratic involution.

$12J^2 - I^3 = 36J_H$ (Clebsch, *op. cit.*, p. 141, note); therefore $12J^2 - I^3 = 0$ is the condition that the 4 points represented by the Hessian of f_u are harmonic; therefore the necessary and sufficient condition that a 4-point be self-reciprocal is that either $f_u=0$ or $H_u=0$ shall represent harmonic points.

$J=-\frac{1}{72}(k+1)\ (k-2)\ (2k-1)$ where k is the cross ratio of the four roots of $f_u=0$. If k is rationally expressible in the coefficients of f_u, then J is rationally factorable into factors linear in the coefficients of f_u.

$12J^2-I^3=J_H=-\frac{1}{72}(h+1)\ (h-2)\ (2h-1)$ where h is the cross ratio of the four roots of $H_x=0$. If h is rationally expressible in terms of the coefficients of H_x, $J_H=0$ will be rationally factorable into three factors linear in the coefficients of H_x and therefore quadratic in the coefficients of f_u.

§ 24. *Cubic covariant theory connected with the self-reciprocal 4-point.*

We will consider what function of the concomitants of the cubic representing three given distinct points, determines a set of points any one of which taken with the original set of three points constitutes a self-reciprocal 4-point.

Suppose $g_u=0$ is the cubic representing three given distinct points. $Q_u=0$ represents the three 4th harmonic points to the triple represented by $g_u=0$; this corresponds to $J=0$ for the quartic $f_u=0$ (cf. § 23). Therefore there are precisely three points which may be taken with a given point triple to form an identically self-reciprocal 4-point.

In § 23, $\frac{a_2}{a_1}=k$ is the cross ratio of the four points A, B, C, D, therefore:

$$12J^2-I^3=(2k^2-2k-1)\ (k^2+2k-2)\ (k^2-4k+1)$$

Let three given points be $P: u_1=0$; $Q: u_2=0$; $R: b_u=b_1u_1+b_2u_2=0$, then:

$$g_u=3b_1u_1^2u_2+3b_2u_1u_2^2$$

For any 4th point $X: (x_1, x_2)$ the cross ratio of the 4 points P, Q, R, X is $k=\frac{x_1b_2}{x_2b_1}$.

Then $(2k^2-2k-1)\ (k^2+2k-2)\ (k^2-4k+1)$ reduces to $2b_2^6x_1^6-6b_1b_2^5x_1^5x_2-15b_1^2b_2^4x_1^4x_2^2+40b_1^3b_2^3x_1^3x_2^3-15b_1^4b_2^2x_1^2x_2^4-6b_1^5b_2x_1x_2^5+2b_1^6x_2^6$, which in terms of the concomitants of g_u is equal to $-17Rg_x^2+14Q_x^2-5\Delta_x^3$.

Then the 6 *points represented by*

$$-17Rg_x^2+14Q_x^2-5\Delta_x^3=0 \tag{3}$$

have the property that any one of them taken with the three points represented by $g_u=0$ form a non-identically self-reciprocal four point.

If 4 points so obtained are represented by $f_u=0$ then if the cross ratio of the 4 points represented by $H_u=0$ is rationally expressible in terms of the coefficients of H_u, the sextic equation (3) will be factorable into rational quadratic factors (cf. § 23).

§ 25. *The linear* 4-*point and its associated* 4-*point are reciprocal for* $I=0$.

Let $f_u=0$ represent A, B, C, D.

Then $f'_x=24J(12J^2-I^3)H_x+I^2(I^3+42J^2)f_x=0$ represents A', B', C', D' (cf. § 22).

For $I=0$, $f'_x=0$ is equivalent to $H_x=0$.

Since I of H_x is $\frac{I^2}{6}$ (cf. Clebsch, *op. cit.*, p. 141), if $I=0$, I of H_x is zero.

Therefore A'', B'', C'', D'' will be represented by the Hessian of H_x, i.e., $f''_x=\frac{J}{3}f_x-\frac{I}{6}H_x=0$ (cf. Clebsch, *op. cit.*, p. 139).

But since $I=0$ $f''_x=0$ reduces to $f_x=0$ and the sets A, B, C, D and A', B', C', D' are reciprocal.

If $g_u=0$ *is a cubic representing three distinct points,* $\Delta_u=0$, *its Hessian represents the two points either of which taken with the original three given points form a quartic for which* $I=0$, i.e., *form a reciprocal four-point.*

§ 26. *A* 4-*point and its associated* 4-*point are not in general reciprocal.*

The 4-point $A : a_u=0$; $B : u_1=0$; $C : u_2=0$; $D : u_1+u_2=0$ is represented by:

$$f_u=u_1u_2(u_1+u_2)\,(a_1u_1+a_2u_2)=0$$

and the associated 4-point A', B', C', D' is represented by:

$$f'_x=I^2(I^3+42J^2)f_x+24J(12J^2-I^3)H_x=0$$

Let

$$k=I^2(I^3+42J^2)$$
$$l=24J(12J^2-I^3)$$

Then the 4-point A'', B'', C'', D'' associated with A', B', C', D' is represented by:

$$f''_u=k'f'_u+l'H'_u=0$$

where

$$k'=I'^2(I'^3+42J'^2)$$

and

$$l'=24J'(12J'^2-I'^3)$$

$$H'_u=\left(\frac{I}{3}kl+\frac{J}{3}l^2\right)f_u+\left(k^2-\frac{I}{6}l^2\right)H_u \text{ (cf. Clebsch, } \textit{op. cit.}\text{, p. 139).}$$

Therefore

$$f_u'' = \Big\{ I^2 I'^2 (I'^3 + 42 J'^2)(I^3 + 42 J^2)$$
$$+ 24 J'(12 J'^2 - I'^3)\left(\frac{I}{3}kl + \frac{J}{3}l^2\right) \Big\} f_u + \Big\{ 24 I'^2 J (I'^3 + 42 J'^2)(12 J^2 - I^3)$$
$$+ 24 J'^2 (12 J'^2 - I'^3)(k^2 - \frac{I}{6}l^2) \Big\} H_u = 0$$

If $f_u'' = 0$ reduces to $f_u = 0$, the coefficient of H_u must vanish identically, since I and J are independent.

We shall therefore consider the relation:

$$24 I'^2 J (I'^3 + 42 J'^2)(12 J^2 - I^3) + 24 J'(12 J'^2 - I'^3)(k^2 - \frac{I}{6}l^2) \equiv 0$$

or

$$k'l + l'(k^2 - \frac{I}{6}l^2) \equiv 0$$

$$I' = Ik^2 + 2Jkl + \frac{I^2}{6}l^2 \text{ (cf. Clebsch, } op.\ cit.\text{, p. 141, note).}$$

Then

$$I' = I^2[I^9 + 132 I^6 J^2 - 1980 I^3 J^4 + 38016 J^6]$$
$$k^2 - \frac{I}{6}l^2 = I[I^9 - 12 I^6 J^2 + 4068 I^3 J^4 - 13824 J^6]$$

$$J' = Jk^3 + \frac{I^2}{2}k^2 l + \frac{IJ}{2}kl^2 + \left(\frac{J^2}{3} - \frac{I^3}{36}\right)l^3 \text{ (cf. Clebsch, } op.\ cit.\text{, p. 141, note).}$$

Then in J' the term of highest degree in I is $-11 I^{15} J$.
The term of $(I'^3 + 42 J'^2)$ of highest degree in I is I^{33}.
The term of $(12 J'^2 - I'^3)$ of highest degree in I is $-I^{33}$.

Then the term of $k'l + l'(k^2 - \frac{I}{6}l^2)$ of highest degree in I is $10 I^{58} J$, therefore the coefficient of H_u in f_u'' does not vanish identically and $f_u'' = 0$ is not equivalent to $f_u = 0$, i.e., the 4-points A, B, C, D and A', B', C', D' are not in general reciprocal.

VI. CONCOMITANT THEORY OF THE ASSOCIATED 4-POINT AND 4-LINE IN THE PLANE

Let

$$\begin{aligned} A &: a_u = a_1u_1 + a_2u_2 + a_3u_3 = 0 \\ B &: u_1 = 0 \\ C &: u_2 = 0 \\ D &: u_3 = 0 \end{aligned}$$

be four distinct coplanar points, then

$$f_u = a_1u_1^2u_2u_3 + a_2u_1u_2^2u_3 + a_3u_1u_2u_3^2 = 0$$

represents the 4-point A, B, C, D.

By taking the polar line of each point as to the triangle formed by the remaining three points we obtain the associated 4-line a', b', c', d'.

$$\begin{aligned} a' &: \quad x_1a_2a_3 + x_2a_3a_1 + x_3a_1a_2 = 0 \\ b' &: -3x_1a_2a_3 + x_2a_3a_1 + x_3a_1a_2 = 0 \\ c' &: \quad x_1a_2a_3 - 3x_2a_3a_1 + x_3a_1a_2 = 0 \\ d' &: \quad x_1a_2a_3 + x_2a_3a_1 - 3x_3a_1a_2 = 0 \end{aligned}$$

Then

$$f'_x = -3\sum x_1^4a_2^4a_3^4 + 4\sum x_1^3x_2a_1a_2^3a_3^4 + 14\sum x_1^2x_2^2a_1^2a_2^2a_3^4 - 20\sum x_1^2x_2x_3a_1^2a_2^3a_3^3 = 0$$

For the general ternary point quartic $f_u = 0$ in symbolic notation, let:

$$f_u = a_u^4 = b_u^4 = \ldots.$$

$$\text{Invariant } A = (abc)^4$$

$$\text{Contravariant } I_x = (abx)^4 = x_\alpha^4 = x_\beta^4 = \ldots.$$

$$\text{Covariant } S_u = (\alpha\beta u)^4 = s_u^4 = t_u^4 = \ldots.$$

and

$$\text{Contravariant } W_x = (stx)^4.$$

Then

$$81W_x - 181A^2I_x = 2^9 \cdot 3^6 \cdot f'_x$$

For the ternary line quartic $f'_x = 0$ representing 4-line a', b', c', d' let the corresponding concomitants be denoted $\overline{A}$, $\overline{I}_u$, $\overline{W}_u$, $\overline{S}_x$.

$$\overline{A} = 3^2 \cdot 2^{12} \cdot a_1^8a_2^8a_3^8$$

$$\overline{I}_u = 2^7 \cdot 3a_1^4a_2^4a_3^4 \left\{ \sum a_1^4u_1^4 + 2\sum a_1^3a_2u_1^3u_2 + 3\sum a_1^2a_2^2u_1^2u_2^2 - 8\sum a_1^2a_2a_3u_1^2u_2u_3 \right\}$$

$$\bar{S}_x = 2^{13} \cdot 3a_1^8a_2^8a_3^8 \left\{ 9\sum a_1^4a_2^4x_3^4 - 12\sum a_1a_2^3a_3^4x_1^3x_2 + 214\sum a_1^2a_2^2a_3^4x_1^2x_2^2 \right.$$
$$\left. -196\sum a_1^2a_2^3a_3^3x_1^2x_2x_3 \right\}$$
$$\bar{W}_u = 2^{33} \cdot 3a_1^{20}a_2^{20}a_3^{20} \left\{ 181\sum a_1^4u_1^4 + 362\sum a_1a_2^3u_1u_2^3 + 543\sum a_1^2a_2^2u_1^2u_2^2 \right.$$
$$\left. -680\sum a_1^2a_2a_3u_1^2u_2u_3 \right\}$$

Then

$$f_u'' = 81\bar{W}_u - 181\bar{A}^2\bar{I}_u = 2^{41} \cdot 3^6a_1^{21}a_2^{21}a_3^{21}\sum a_1u_1^2u_2u_3 = 2^{41} \cdot 3^6 \cdot a_1^{21}a_2^{21}a_3^{21}f_u$$

This verifies analytically the fact that the quadrangle quadrilateral configuration is reciprocal.