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A Comparison of Different Line-Geometric Representations for Functions of a Complex Variable

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INTRODUCTION

Wilczynski[1] has recently given two methods for constructing a congruence of lines determined by a functional relation between two complex variables, which enable one to visualize the properties of the function by studying the properties of the resulting congruence. In the first method, the two complex variables are represented upon two distinct planes, parallel to each other and a unit apart, the corresponding coordinate axes for the two planes being chosen parallel to each other in such a way that the two origins lie upon a line perpendicular to the two planes. If the points of the first plane are joined to the points of the second plane which correspond to them by means of a given functional relation $w=F(z)$, a two-parameter family, or a congruence, of straight lines is obtained. These congruences have certain characteristic properties which hold for the totality of analytic functions, and in addition, of course, special properties which depend upon the choice of the particular function $F(z)$. The developables and focal sheets of such congruences are always imaginary, except in a trivial special case, but some interesting real surfaces are closely associated with them.

The second method of representation makes use of a Riemann sphere. The two complex variables are projected upon the same sphere, and points of the sphere corresponding to each other by means of the function $w=F(z)$ are joined by lines. The congruences obtained in this way always have real focal sheets and developables, and are therefore more interesting than those obtained by the first method.

It can be seen at once that there are other methods of constructing congruences of lines in connection with a relation $w=F(z)$, these other

[1] E. J. Wilczynski: "Line-geometric representations for functions of a complex variable," Transactions of the American Mathematical Society, Vol. XX (1919), pp. 271-298.

methods imposing less drastic restrictions upon the planes or spheres upon which the variables are represented. It is the purpose of the present paper to consider properties of congruences which are obtained from such generalizations of the above methods. In section I we shall study the congruences resulting when the planes of reference are kept parallel, but when the coordinate axes in the two planes are given arbitrary positions. We shall find, as might be expected, that these congruences are not essentially distinct from those corresponding to the special case. In fact, if a congruence is constructed by considering one of the complex variables as a given function of the other, then a projectively equivalent congruence may be obtained by keeping the coordinate axes parallel, and considering in place of the given function one closely related to it, a *rotated function*, the angle of rotation being the angle between corresponding axes.

In section II we shall consider the general properties of those congruences obtained by the general conformal correspondence between two planes, the relative positions of the two planes in space being left arbitrary. It will be found that this general theory is included essentially in that special case in which the two planes are perpendicular to each other, and the axes occupy certain special positions. Such a method of representation has a serious disadvantage. For an arbitrary function, it is impossible to predict whether the developables and focal sheets of the congruence are real or imaginary. Therefore from the point of view of the general theory of functions, such a method is far less useful than the method of parallel planes, though it may be of value in special instances.

Section III deals with an extension of the method of the Riemann sphere. The two complex variables will be projected upon two distinct but concentric spheres. As in section II, the simpler method furnishes the more valuable results. The method of concentric spheres does not permit us to make a general statement as to the reality of the developables and focal sheets for all possible functions $w=F(z)$, as in the case when we use a single sphere. In the more general case, the properties of the individual function play an essential rôle in answering such questions.

The author wishes to express her gratitude to Professor Wilczynski for his constant interest and helpful suggestions to her in the writing of this thesis.

I. EXTENSION OF THE METHOD OF PARALLEL PLANES

Let us denote by

$$(1) \qquad \begin{cases} z=x+iy, & z_0=x-iy \\ w=u+iv, & w_0=u-iv \end{cases}$$

the two complex variables and their conjugates, and assume the functional relation

$$w=F(z)$$

which implies that

$$w_0=F_0(z_0),$$

where F_0 is the function conjugate to F.

Let us represent the point P_z upon the $\xi\eta$-plane of a system of $\xi\eta\zeta$-axes in ordinary cartesian space, letting the x- and y-axes coincide with the ξ- and η-axes respectively. Then the space coordinates of P_z are

$$(2) \qquad \xi_1=x, \qquad \eta_1=y, \qquad \zeta_1=0.$$

Now let us represent the variable w upon a plane parallel to the $\xi\eta$-plane, a unit above it, but allow the real and imaginary axes in this plane to be in an arbitrary position. If the angle between the ξ- and u-axes is θ, and if the coordinates of the origin of the complex numbers $u+iv$ are $(a, b, 1)$, then the point P_w will have the coordinates

$$(3) \qquad \xi_2=u\cos\theta-v\sin\theta+a, \quad \eta_2=u\sin\theta+v\cos\theta+b, \quad \zeta_2=1.$$

Let R be a region of the z-plane in which the function $w=F(z)$ is analytic, and let us join each point of this region to the corresponding points P_w. If the function is n-valued, where n is finite, there will be n lines of the congruence through each point of R. The projective properties of the congruence defined in this way will be studied by means of a system of differential equations of the type[2]

[2] E. J. WILCZYNSKI: "One parameter families and nets of ruled surfaces and a new theory of congruences," Transactions of the American Mathematical Society, Vol. XXI (1920), pp. 157-206. This paper will hereafter be referred to as "Ruled surfaces and congruences."

$$(4)\quad \begin{cases} \dfrac{\partial^2\lambda}{\partial z^2}+p_{11}\dfrac{\partial\lambda}{\partial z}+p_{12}\dfrac{\partial\mu}{\partial z}+q_{11}\lambda+q_{12}\mu=0 \\ \dfrac{\partial^2\mu}{\partial z^2}+p_{21}\dfrac{\partial\lambda}{\partial z}+p_{22}\dfrac{\partial\mu}{\partial z}+q_{21}\lambda+q_{22}\mu=0 \\ \dfrac{\partial\lambda}{\partial z_0}=a_{11}\dfrac{\partial\lambda}{\partial z}+a_{12}\dfrac{\partial\mu}{\partial z}+b_{11}\lambda+b_{12}\mu \\ \dfrac{\partial\mu}{\partial z_0}=a_{21}\dfrac{\partial\lambda}{\partial z}+a_{22}\dfrac{\partial\mu}{\partial z}+b_{21}\lambda+b_{22}\mu \end{cases}$$

Clearly the relation which we have assumed between the space coordinate system and the real and imaginary axes of the z-plane involves no loss of generality. We may also assume that $a=b=0$, so that the origin of coordinates for the variable w is on the ζ-axis, without changing the projective properties of the congruence. For make the projective transformation of space

$$(5)\qquad \bar{\xi}=\xi-a\,\zeta,\ \bar{\eta}=\eta-b\,\zeta,\ \bar{\zeta}=\zeta$$

then the coordinates of P_z and P_w,

$$(6)\qquad \begin{cases} \xi_1=x & ;\ \eta_1=y & ;\ \zeta_1=0 \\ \xi_2=u\cos\theta-v\sin\theta+a; & \eta_2=u\sin\theta+v\cos\theta+b; & \zeta_2=1 \end{cases}$$

will become

$$(7)\qquad \begin{cases} \bar{\xi}_1=x & ,\ \bar{\eta}_1=y & \bar{\zeta}_1=0 \\ \bar{\xi}_2=u\cos\theta-v\sin\theta, & \eta_2=u\sin\theta+v\cos\theta, & \bar{\zeta}_2=1 \end{cases}$$

which makes $a=b=0$.

By means of (1), we can introduce into (7) the variables z, z_o, w, w_0. We find the following homogeneous cartesian coordinates for P_z and P_w:

$$(8)\qquad \begin{cases} \lambda_1=\dfrac{1}{2}(z+z_0), & \mu_1+\dfrac{1}{2}\cos\theta(w+w_0)-\dfrac{1}{2i}\sin\theta(w-w_0), \\ \lambda_2=\dfrac{1}{2i}(z-z_0), & \mu_2=\dfrac{1}{2}\sin\theta\,(w+w_0)+\dfrac{1}{2i}\cos\theta(w-w_0), \\ \lambda_3=0, & \mu_3=1 \\ \lambda_4=1, & \mu_4=1. \end{cases}$$

If we write

$$(9) \qquad \begin{cases} a = \cos\theta + i \sin\theta \\ a_0 = \cos\theta - i \sin\theta, \end{cases}$$

(8) can be written in the form

$$(10) \qquad \begin{cases} P_z: & & P_w: \\ \lambda_1 = \frac{1}{2}(z+z_0), & & \mu_1 = \frac{1}{2}(aw + a_0 w_0), \\ \lambda_2 = \frac{1}{2i}(z-z_0), & & \mu_2 = \frac{1}{2i}(aw - a_0 w_0), \\ \lambda_3 = 0, & & \mu_3 = 1 \\ \lambda_4 = 1, & & \mu_4 = 1. \end{cases}$$

If in particular, $\theta = 0^0$, so that the u- and x- axes are parallel, (10) reduces to the special case considered by Wilczynski,

$$(11) \qquad \begin{cases} P_z: & & P_w: \\ \lambda_1 = \frac{1}{2}(z+z_0), & & \mu_1 = \frac{1}{2}(w + w_0), \\ \lambda_2 = \frac{1}{2i}(z-z_0), & & \mu_2 = \frac{1}{2i}(w - w_0), \\ \lambda_3 = 0, & & \mu_3 = 1, \\ \lambda_4 = 1, & & \mu_4 = 1. \end{cases}$$

A comparison of (10) and (11) will show that the two situations are equivalent. For in studying the totality of analytic functions $w = F(z)$, in (11), among them will be included those derived from a particular one by multiplying it by the rotating factor $a = e^{i\theta}$, giving the function which appears in (10). Thus *the projective properties of the class of congruences which is defined by the totality of all analytic functions $w = F(z)$ by the method of parallel planes, are independent of the relative position of the origins, of the angle between the real axes of the two complex variables, and, of course, of the distance between the two planes. The congruence which corresponds to an individual function $F(z)$ in any particular representation of this sort corresponds not to the same function but to the function $e^{i\theta}w = F(z)$, if the angle between the real axes of the two planes be changed by θ.*

II. THE METHOD OF NON-PARALLEL PLANES

Let us consider now the case in which the planes upon which the two complex variables are represented are not parallel. Then the line of intersection of the two planes will be a proper line, which we may choose as the ξ-axis. We may identify the $\xi\eta$- and z-planes, and choose as the η-axis a line which passes through the origin of the xy-system. Let us use the following notations:

$$
(12)\quad \begin{cases}
\varphi \equiv \text{the angle between the two planes.} \\
\theta_1 \equiv \text{the angle between the } \xi\text{- and } x\text{-axes.} \\
\theta_2 \equiv \text{the angle between the } \xi\text{- and } u\text{-axes.} \\
(o, b_1) \equiv \text{the coordinates of the origin } O_1 \text{ of the } xy\text{-system,} \\
\qquad \text{with respect to the } \xi\eta\text{-axes.} \\
(a_2, b_2) \equiv \text{the coordinates of the origin } O_2 \text{ of the } uv\text{-system,} \\
\qquad \text{with respect to a system composed of the } \xi\text{-axis} \\
\qquad \text{and the line of intersection of the } \eta\zeta\text{-plane with} \\
\qquad \text{the } w\text{-plane.}
\end{cases}
$$

Then the cartesian coordinates of P_z and P_w are

$$
(13)\quad
\begin{array}{ll}
\quad P_z: & \quad P_w: \\
\left\{\begin{array}{l}
\xi_1 = x\cos\theta_1 - y\sin\theta_1, \\
\eta_1 = x\sin\theta_1 + y\cos\theta_1 + b_1, \\
\zeta_1 = 0
\end{array}\right. &
\begin{array}{l}
\xi_2 = u\cos\theta_2 - v\sin\theta_2 + a_2, \\
\eta_2 = \cos\varphi[u\sin\theta_2 + v\cos\theta_2 + b_2], \\
\zeta_2 = \sin\varphi[u\sin\theta_2 + v\cos\theta_2 + b_2]:
\end{array}
\end{array}
$$

We change to a projectively equivalent, but simpler form by means of the transformation

$$
\bar{\xi} = \xi, \qquad \bar{\eta} = \eta - \frac{\cos\varphi}{\sin\varphi}\zeta; \qquad \bar{\zeta} = \frac{1}{\sin\varphi}\zeta
$$

which is admissible since the z- and w- planes are assumed to be non-parallel. The new coordinates, expressed in homogeneous cartesian form, are

$$
(14)\quad
\begin{array}{ll}
\quad P_z: & \quad P_w: \\
\left\{\begin{array}{l}
\lambda_1 = x\cos\theta_1 - y\sin\theta_1, \\
\lambda_2 = x\sin\theta_1 + y\cos\theta_1 + b_1, \\
\lambda_3 = 0, \\
\lambda_4 = 1,
\end{array}\right. &
\begin{array}{l}
\mu_1 = u\cos\theta_2 - v\sin\theta_2 + a_2, \\
\mu_2 = 0, \\
\mu_3 = u\sin\theta_2 + v\cos\theta_2 + b_2, \\
\mu_4 = 1.
\end{array}
\end{array}
$$

But the values of λ_k, μ_k, as given by (13), would reduce to these same values for $\varphi=90°$. We have shown therefore, that *if the planes of reference are non-parallel, a congruence of this sort is projectively equivalent to one obtained from it by rotating the w-plane around the line of intersection of the two planes until the z- and w-planes are perpendicular to each other.*

If we use the notation

$$(15)\qquad \begin{cases} a=\cos\theta_1+i\sin\theta_1, & \beta=\cos\theta_2+i\sin\theta_2, \\ a_0=\cos\theta_1-i\sin\theta_1, & \beta_0=\cos\theta_2-i\sin\theta_2, \end{cases}$$

and introduce into (14) the complex variables given by (1), we have the following coordinates for P_z and P_w:

$$(16)\qquad \begin{cases} & P_z: & & P_w: \\ & \lambda_1=\frac{1}{2}(az+a_0z_0), & & \mu_1=\frac{1}{2}(\beta w+\beta_0w_0+2a_2), \\ & \lambda_2=\frac{1}{2i}(az-a_0z_0+2ib_1), & & \mu_2=0, \\ & \lambda_3=0, & & \mu_3=\frac{1}{2i}(\beta w-\beta_0w_0+2ib_2), \\ & \lambda_4=1, & & \mu_4=1. \end{cases}$$

By an argument similar to that used in section I, we see that it is not necessary to consider this general situation. Equations (16) should also have been obtained if, in the two perpendicular planes of reference, the x- and u- axes had been taken parallel to the line of intersection of the planes, while the variables from which the congruence was constructed were

$$Z=az,$$
$$W=\beta w.$$

This amounts to a transformation of both independent and dependent variables, rotating them through angles corresponding to the angles between the individual real axes, and the line of intersection of their planes. Since our point of view is the study of the totality of all such functional relations, the more special case will suffice. We may assume therefore, without loss of generality,

$$a=a_0=\beta=\beta_0=1$$

and (16) may be written

$$
(17)\qquad
\begin{array}{ll}
P_z: & P_w: \\
\left\{\begin{array}{l}
\lambda_1=\frac{1}{2}(z+z_0), \\
\lambda_2=\frac{1}{2i}(z-z_0+2ib_1) \\
\lambda_3=0, \\
\lambda_4=1,
\end{array}\right. &
\begin{array}{l}
\mu_1=\frac{1}{2}(w+w_0+2a_2), \\
\mu_2=0, \\
\mu_3=\frac{1}{2i}(w-w_0+2ib_2), \\
\mu_4=1.
\end{array}
\end{array}
$$

If we follow the line $z_0=$ const. in the z-plane, we obtain a ruled surface of the congruence. If this ruled surface is a developable,[3] the pairs $(\lambda_i,\mu_i,i=1,\ .\ .\ .4)$ of (17) must satisfy the relation

$$
(18)\qquad \left|\frac{\partial\lambda_i}{\partial z},\quad \frac{\partial\mu_i}{\partial z},\quad \lambda_i,\quad \mu_i\right|=0\quad (i=1,2,3,4).
$$

Similarly, if the family of ruled surfaces $z=$const. consists of developables the relation

$$
(18')\qquad \left|\frac{\partial\lambda_i}{\partial z_0},\quad \frac{\partial\mu_i}{\partial z_0},\quad \lambda_i,\quad \mu_i\right|=0\quad (i=1,\ 2,\ 3,\ 4)
$$

must hold. These reduce to a single condition

$$
w_0'(z-w-a_2+i(b_1-b_2))=0.
$$

If $w_0'=0$, or $w=$const., the congruence reduces to a bundle of lines through a point on the w-plane. If the second factor vanishes, then

$$
w+a_2+ib_2=z+ib_1,
$$

which is a special linear function, and represents a bundle of parallel lines, perpendicular to the line of intersection of the two planes, and cutting them at equal distances from this line. For all other functions, the two families of ruled surfaces $z=$const., $z_0=$const. are not developables.

Let us now derive the system of differential equations (4) which the coordinates (17) are to satisfy. Since the coordinates are linear in the variables, the second order equations can be found at once. They are

$$
(19)\qquad \frac{\partial^2\lambda}{\partial z^2}=0,\qquad\qquad \frac{\partial^2\mu}{\partial w^2}=0.
$$

If we introduce z as independent variable in the latter equation it becomes

[3] "Ruled surfaces and congruences," p. 158.

$$\frac{\partial^2\mu}{\partial z^2}-\frac{w''}{w'}\,\frac{\partial\mu}{\partial z}=0.$$

The coefficients of the first order equations may be found by the method of undetermined coefficients. The complete system is

$$(20)\begin{cases}\dfrac{\partial^2\lambda}{\partial\mu^2}=0, \qquad \dfrac{\partial^2\mu}{\partial z^2}-\dfrac{w''}{w'}\,\dfrac{\partial\mu}{\partial z}=0,\\ \dfrac{\partial\lambda}{\partial z_0}=\dfrac{w_0-z\,+a_2-i(b_1+b_2)}{z_0-w_0-a_2-i(b_1-b_2)}\,\dfrac{\partial\lambda}{\partial z}+\dfrac{1}{w'}\,\dfrac{w-w_0+2ib_2}{z_0-w_0-a_2-i(b_1-b_2)}\,\dfrac{\partial\mu}{\partial z}+\\ \qquad\dfrac{\lambda-\mu}{z_0-w_0-a_2-i(b_1-b_2)}\\ \dfrac{1}{w_0'}\,\dfrac{\partial\mu}{\partial z_0}=\dfrac{z_0-z-2i\,b_1}{z_0-w_0-a_2-i(b_1-b_2)}\,\dfrac{\partial\lambda}{\partial z}+\dfrac{1}{w'}\cdot\dfrac{w-z_0+a_2+i(b_1+b_2)}{z_0-w_0-a_2-i(b_1-b_2)}+\\ \qquad\dfrac{-\mu\lambda}{z_0-w_0-a_2-i(b_1-b_2)}\end{cases}$$

where w' and w_0' indicate the derivatives of these functions with respect to z and z_0 respectively.

The first step in the reduction of system (20) to the canonical form shows the disadvantage of this method of constructing a congruence. Instead of the two given planes of reference, the two focal sheets of the congruence could be introduced as new surfaces of reference. This involves a change of dependent variables, the new ones being obtained from the linear factors of the quadratic covariant[4]

$$(21) \qquad a_{21}\,\lambda^2-(a_{11}-a_{22})\,\lambda\mu-a_{12}\mu^2.$$

In the parallel plane representation, these two points on the lines of the congruence were always imaginary, for all non-trivial functional relations, and therefore the focal loci were always imaginary surfaces. A single example will suffice to show that in the present representation, for this function, the focal points are real on some lines of the congruence, and imaginary on others. Hence no general statement can be made about the reality of the focal surfaces for an arbitrary functional relation. In our case the covariant (21) has the value

$$(22)\begin{cases}w'w_0'(z-z_0+2ib_1)\lambda^2+\{w'[w_0-z+a_2-i(b_1+b_2)]\\ -w_0'[w-z_0+a_2+i(b_1+b_2)]\}\lambda\mu+(w-w_0+2ib_2)\mu^2.\end{cases}$$

This is a quadratic form with imaginary coefficients. Then the

[4] "Ruled surfaces and congruences," p. 183.

character of its factors will be determined by the sign of its discriminant

$$(23)\begin{cases}\Delta=\{w'[w_0-z+a_2-i(b_1+b_2)]-w_0'[w-z_0+a_2+i(b_1+b_2)]\}^2\\ \quad -4w'w_0'[z-z_0+2ib_1][w-w_0+2ib_2].\end{cases}$$

Now let us consider the function

$$w=e^z$$

and set $a_2=b_1=b_2=0$. Then Δ becomes

$$\Delta=[z_0e^{z_0}-ze^z]^2-4e^{z+z_0}(z-z_0)(e^z-e^{z_0}).$$

Let z vary over pure imaginary values, $z=iy$. Then we have

$$\Delta=-4y[y\cos^2 y-4\sin y]$$

which has the special values

$$\Delta=8\pi \text{ for } y=\frac{\pi}{2}$$

$$\Delta=-4\pi^2 \text{ for } y=\pi.$$

Since the sign of Δ changes from positive to negative, the factors of (21) are sometimes real and sometimes imaginary. Therefore some lines of the congruence have real focal points while others have imaginary focal points.

A further disadvantage of this method appears when we try to introduce as new independent parameters those which correspond to the developables of the congruence. This involves the integration of the partial differential equation[5]

$$(24)\quad \left(\frac{\partial\theta}{\partial z_0}\right)^2-(a_{11}+a_{22})\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial z_0}+(a_{11}a_{22}-a_{12}a_2)\left(\frac{\partial\theta}{\partial z}\right)^2=0$$

which in the present case assumes the form

$$(25)\begin{cases} w'[z_0-w_0-a_2-i(b_1-b_2)]\left(\dfrac{\partial\theta}{\partial z_0}\right)^2-\{w'[w_0-z+a_2-i(b_1+b_2)]\\ +w_0'[w-z_0+a_2+i(b_1+b_2)]\}\dfrac{\partial\theta}{\partial z}\dfrac{\partial\theta}{\partial z_0}+w_0'[z-w-a_2+i(b_1-b_2)]\\ \qquad\left(\dfrac{\partial\theta}{\partial z_0}\right)=0\end{cases}$$

which does not seem to admit of any simple method of integration.

From the point of view of obtaining geometric properties common to all analytic functions, the method of parallel planes seems to be

[5] "Ruled surfaces and congruences," p. 187.

more powerful, though the present more general method may be advantageous for special functions. In fact when $a_2 = b_1 = b_2 = 0$, for such simple functions as $w = z^n$ with n real; $w = cz$ where c is a complex constant; and for the class of linear functions[6] $w = \dfrac{az+b}{cz+d}$ when a, b, c, d are real with $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} < 0$, the focal points are always real, while in the parallel plane representation they are always imaginary.

[6] Compare A. Emch: "On the rectilinear congruence realizing the circular transformation of one plane into another," Annals of Mathematics, 2nd series, Vol. M 13, (1911-12), pp. 155-160.

III. A GENERALIZATION OF THE METHOD OF THE RIEMANN SPHERE

Let us choose the $\xi\eta$-plane as the common plane of the two complex variables. Project the variables

$$z=x+iy$$

upon a sphere S_1 of radius r_1 with its center at the origin, using as center of projection the point (o, o, r_1). Also project the second variable

$$w=u+iv$$

upon a concentric sphere S_2 with radius r_2 from the point (o, o, r_2). If the radii r_1 and r_2 are equal, the spheres will coincide and the situation will reduce to that considered by Wilczynski. A line of the congruence is obtained by joining a point P_z of S_1 to the points P_w on S_2 which correspond to it by means of

$$w=F(z).$$

The correspondence between the two spheres is of course conformal since the null lines on the two spheres correspond to each other. While this is not the most general extension which can be made of Wilczynski's method, it puts into evidence very clearly the difficulties which arise from any such generalization.

The coordinates of P_z and P_w can be found at once. The line joining the points (x, y, o) and (o, o, r_1) will cut the sphere

$$S_1 \qquad \xi^2+\eta^2+\zeta^2=r_1^2$$

in a point whose coordinates are

$$(26)\quad \xi_1=\frac{2r_1^2x}{x^2+y^2+r_1^2},\quad \eta_1=\frac{2r_1^2y}{x^2+y^2+r_1^2},\quad \zeta_1=\frac{r_1(x^2+y^2-r_1^2)}{x^2+y^2+r^2_1}.$$

Similarly, the coordinates of P_w, the intersection of S_2 with the line joining (u, v, o) and (o, o, r_2) are

$$(27)\quad \xi_2=\frac{2r_2^2u}{u^2+v^2+r_2^2},\quad \eta_2=\frac{2r_2^2v}{u^2+v^2+r_2^2},\quad \zeta_2=\frac{r_2(u^2+v^2-r_2^2)}{u^2+v^2+r_2^2}.$$

The homogeneous cartesian coordinates of the points may therefore be written in the form

$$(28)\qquad \begin{array}{ll} P_z: & P_w: \\ \left\{\begin{array}{l} \lambda_1=2r_1^2x, \\ \lambda_2=2r_1^2y, \\ \lambda_3=r_1(x^2+y^2-r_1^2), \\ \lambda_4=(x^2+y^2+r_1^2), \end{array}\right. & \begin{array}{l} \mu_1=2r_2^2u, \\ \mu_2=2r_2^2v, \\ \mu_3=r_2(u^2+v^2-r_2^2), \\ \mu_4=u^2+v^2+r_2^2. \end{array} \end{array}$$

or after introducing the complex variables of (1)

$$(29)\qquad \begin{array}{ll} P_z: & P_w: \\ \left\{\begin{array}{l} \lambda_1=r_1^2(z+z_0), \\ \lambda_2=-ir_1^2(z-z_0), \\ \lambda_3=r_1(zz_0-r_1^2), \\ \lambda_4=zz_0+r_1^2, \end{array}\right. & \begin{array}{l} \mu_1=r_2^2(w+w_0), \\ \mu_2=-ir_2^2(w-w_0), \\ \mu_3=r_2(ww_0-r_2^2), \\ \mu_4=ww_0+r_2^2. \end{array} \end{array}$$

The conditions (18) and (18′) that the two families of ruled surfaces $z=\text{const.}$, $z_0=\text{const.}$ shall be developables, reduces to

$$w'(r_1w-r_2z)=0$$

which can be interpreted immediately. If $w'=0$, or $w=\text{const.}$, the congruence becomes a bundle of lines through a point of S_2. If $r_1w-r_2z=0$ or

$$w=\frac{r_2z}{r_1},$$

the congruence reduces to the bundle of lines through the common center of the spheres.

For all other functions, the coordinates (29) will satisfy a system of differential equations of the form (4). In fact, the second order equations are identical with those obtained in section II. The coefficients of the first order equations have the following values:

$$
(30)\quad \left\{
\begin{aligned}
a_{11} &= \frac{(r_1-r_2)(r_1r_2+zw_0)^2}{(r_1+r_2)(r_1w_0-r_2z_0)^2},\\
a_{12} &= -\frac{r_1^2}{w'r_2^2}\cdot\\
&\left\{\frac{(r_1+r_2)(r_1w_0-r_2z_0)(r_1w-r_2z)+(r_1-r_2)(r_1r_2+zw_0)(r_1r_2+z_0w)}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\},\\
b_{11} &= -\left\{\frac{r_2(r_1+r_2)(r_1w_0-r_2z_0)+w_0(r_1-r_2)(r_1r_2+zw_0)}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\},\\
b_{12} &= \frac{r_1^2}{r_2^2}\left\{\frac{r_1(r_1+r_2)(r_1w_0-r_2z_0)+z_0(r_1-r_2)(r_1r_2+zw_0)}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\},\\
a_{21} &= -\frac{w_0'r_2^2}{r_1^2}\cdot\\
&\left\{\frac{(r_1+r_2)(r_1w_0-r_2z_0)(r_1w-r_2z)-(r_1-r_2)(r_1r_2+zw_0)(r_1r_2+z_0w)}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\},\\
a_{22} &= -\frac{w_0'}{w'}\left\{\frac{(r_1-r_2)(r_1r_2+z_0w)^2}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\},\\
b_{21} &= -\frac{w_0'r_2^2}{r_1^2}\left\{\frac{r_2(r_1+r_2)(r_1w_0-r_2z_0)+w_0(r_1-r_2)(r_1r_2+z_0w)}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\},\\
b_{22} &= -w_0'\left\{\frac{r_1(r_1+r_2)(r_1w_0-r_2z_0)+z_0(r_1-r_2)(r_1r_2+z_0w)}{(r_1+r_2)(r_1w_0-r_2z_0)^2}\right\}.
\end{aligned}
\right.
$$

The covariant (21) which determines the coordinates of the focal points of the lines of the congruence, becomes

$$
(31)\quad \left\{
\begin{aligned}
& w'w_0^{1}r_2^4\{(r_1+r_2)(r_1w_0-r_2z_0)(r_1w-r_2z)-(r_1-r_2)(r_1r_2+zw_0)\\
&\qquad (r_1r_2+z_0w)\}\lambda^2\\
& +r_1^2r_2^2(r_1-r_2)\{w'(r_1r_2+zw_0)^2+w_0'(r_1r_2+z_0w)^2\}\lambda\mu\\
& -r_1^4\{(r_1+r_2)(r_1w_0-r_2z_0)(r_1w-r_{2z})+(r_1-r_2)(r_1r_2+zw_0)\\
&\qquad (r_1r_2+z_0w)\}\mu^2
\end{aligned}
\right.
$$

whose coefficients are real and its discriminant is equal to

$$
(32)\ \frac{\Delta}{r_1^4r_2^4} = (r_1-r_2)^2[w'(r_1r_2+zw_0)^2-w_0'(r_1r_2+z_0w)^2]^2+4w'w_0'(r_1+r_2)^2 (r_1w_0-r_2z_0)^2(r_1w-r_2z)^2.
$$

If we exclude the case $r_1=r_2$, for which the preceding results all become simple, we see that the first term in Δ is always negative, while the second is always positive. The following example shows that for a given function, Δ can be changed in sign as z varies. Let

$$w = az,$$

where a is a complex constant. Then (32) gives the value

$$\begin{aligned}\frac{\Delta}{r_1^4 r_2^4} = {} & (r_1-r_2)^2(a-a_0)^2(aa_0z^2z_0^2)^2 \\ & +2(aa_0z^2z_0^2)[2(r_1+r_2)^2(r_1a_0-r_2)^2(r_1a-r_2)^2-r_1^2r_2^2(a-a_0)^2(r_1-r_2)^2] \\ & +r_1^4r_2^4(r_1-r_2)^2(a-a_0)^2\end{aligned}$$

which is a quadratic form

$$A(aa_0z^2z_0^2)^2+2B(aa_0z^2z_0^2)+C$$

with real coefficients $A<0$, $B>0$, $C<0$ and with

$$\begin{aligned}B^2-AC = {} & (r_1+r_2)^4(r_1a_0-r_2)^4(r_1a-r_2)^4 \\ & -(r_1^2-r_2^2)^2(r_1a_0-r_2)^2(r_1a-r_2)^2(a-a_0)^2\end{aligned}$$

which is always positive. Then there are two positive real values of $aa_0z^2z_0^2$ for which Δ will vanish. If we indicate them by ρ_1 and ρ_2,

$$\rho_1 \leqq (aa_0z^2z_0^2) \leqq \rho_2$$

represents the closed region for which the lines of the congruence have real focal points.

The situation here, then, is similar to that in section II. The reality of the focal sheets and developables of the congruence depends upon the special function under consideration. In the particular cases studied by Wilczynski, the reality conditions are independent of the particular functional relation. From the point of view of a general theory, then, there seem to be serious disadvantages connected with any attempt to generalize these two methods for representing a functional relation by means of a congruence.

VITA

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She studied at the University of Chicago for nine quarters during the years 1917-1920, holding a fellowship in the department of mathematics for the last two years of that period. She has had courses under Professors Moore, Dickson, Bliss, Wilczynski and Moulton, to all of whom she wishes to express her appreciation of their helpful interest. Her special thanks are due to Professor Wilczynski, whose teaching has been a constant inspiration in the writing of this thesis.