The University of Chicago

THE TRANSMISSION OF WAVES THROUGH A SYMMETRIC OPTICAL SYSTEM

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DEPARTMENT OF MATHEMATICS

BY

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THE TRANSMISSION OF WAVES THROUGH A SYMMETRIC OPTICAL INSTRUMENT.[1]

BY IRWIN ROMAN.

SYNOPSIS.

Gaussian Parameter Method of Studying the Transmission of a Wave Surface through a Centered System of Symmetric Lenses.—Since for each point of an incident wave surface there is a corresponding point on the refracting surface, the parameter β of the refracting surface may be expressed in terms of the parameter α of the wave surface, thus allowing the elimination of the parameter β. After expressing all quantities in terms of α, certain invariant combinations are found among the coefficients of the wave and lens surfaces, these furnishing the determination of the refracted surface. This gives a point by point correspondence between all surfaces of the system. As an application, the focal distance and longitudinal aberration are calculated and a numerical case given. The method is applicable to aspherical surfaces as simply as to spherical ones, and it is hoped that many problems in practical optics may be simplified by the present method, combined with proper choices of the parameter.

PART I. INTRODUCTION.

THE literature on the aberrations in an optical system contains a number of different methods of attack on the problems involved. Nearly all of these may be grouped into two classes, according as the chief emphasis is placed on the wave surface or the ray congruence. As a combination of the two points of view may be mentioned the methods using the characteristic function of Hamilton, which, in the hands of Bruns[2] and of Schwarzschild,[3] have yielded some valuable results. The characteristic function is essentially a relation between the coördinates of the points at the opposite ends of a particular ray segment, the results being studied without expressly finding the coördinates of one end in terms of those of the other end. The method is an implicit function method, and leads to fairly complicated analyses.

It was pointed out by Gullstrand[4] that the aberrations of an optical system could be studied more simply from the point of view of the wave surface than from that of the ray. He selected the equation of a particular wave in the form $z = f(x, y)$. So far as the present writer has been able to learn, Gullstrand has made no attempt to study the trans-

[1] Presented to The University of Chicago, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

[2] Abh. d. math.-phys. Kl. d. Kgl. Sächs. Ges., Bd. 21, s325 (1895).

[3] Abh. d. Kgl. Ges. d. Wiss. zu Göttingen, math.-phys. Kl. Neue Folge, IV (1905–1906).

[4] Ann. d. Phys. (4), Bd. 18, S. 941–973; also K. Svenska Vetenskapsakad. Handl., 41. No. 3 (1906–7).

formation of a wave surface, but restricts himself to the analysis of a particular surface, the aberrations being referred to that surface.

The purpose of the present investigation has been to replace the rectangular representation of the surface by a representation in terms of two Gaussian parameters such as are used in differential geometry. Let us consider a particular wave of a family incident on the surface of separation between two homogeneous and isotropic media. Suppose that the initial wave and the lens surface are determined in terms of separate parameters. At each regular point of the wave surface there is a unique normal, commonly called the ray. This ray, if effective in image formation, must pass through the lens surface, and the point of ntersection determines a correspondence between the two sets of parameters. By means of this correspondence, either set of parameters may be expressed in terms of the other set, thus allowing the elimination of all but a single set. The use of this single set of parameters throughout the system, makes it possible to determine corresponding points on all surfaces of the system, since all points on a particular ray are given by the same values of the parameters throughout the entire system. It should be noted, however, that if the parameters have a simple geometric significance for one surface, they will, usually, not have a simple significance for another surface. Having determined the various parameters in terms of a common parameter, we may proceed to the study of corresponding surfaces and to the study of the transformation of the surface as the wave is transmitted through the system. Knowing the initial and the final wave of the system, we may proceed to the study of the aberrations in accordance with any desired classification or analysis.

Besides the advantage of directness in the method, there are several other advantages. Since we have a single set of parameters, we have a definite interpretation for the "order" of a quantity. While a few attempts have been made to retain powers of small quantities beyond the second, such terms are usually neglected, probably because of the complexity involved in their retention, and possibly also because there is usually a certain vagueness in the analysis. While a quantity may be of one order in terms of one parameter, it may be of a different order in terms of another parameter. A second advantage lies in the facility with which questions involving aspherical surfaces may be treated. In the usual treatments, the surfaces are assumed to be spheres or to depart only slightly from spheres.

In the preceding paragraphs, the method has been sketched for a general system. Because of the complexity which arises in carrying out the method, the investigation in this paper has been restricted to the case in which all the surfaces are symmetrical with respect to a common axis of revolution. This allows the selection of one of the two Gaussian parameters as the azimuth around the common axis, reducing the problem to a plane problem. Besides assuming that each surface is a surface of revolution around a common axis, we shall assume that each surface and its representation is non-singular in the portions to be studied. This means that for the axis and for a certain region around the axis, there shall be no singular points of the surface or in its representation. The parameters will be selected so as to reduce to zero for the axial point. This choice of parameters is a restriction of small importance, since practically all discussions of the symmetrical system are made in terms of the optical height or of the paraxial angle. The restrictions here imposed still allow a considerable freedom in the choice of parameters to suit particular needs.

In following out the methods here discussed, we shall use power series expansion in the parameter selected, retaining all terms by means of recursion formulas. Except for the application to the first four orders, no attempt will be made to reduce these to explicit forms. The complexity increases rapidly with the order of the term, and in numerical calculations, the recursion formulas may even be more convenient than the explicit forms.

The method also makes use of invariants of two classes, parametric and optical. A parametric invariant is a quantity whose value is independent of the choice of parameter. An optical invariant is a quantity whose value is independent of which particular wave of the family is selected. The parametric invariants are geometric and are incidental rather than fundamental in the optical problem. The optical invariants, however, are fundamental, as they furnish the new surface in terms of the old surface and the interface. Unless specified to the contrary, we shall understand the single word "invariant" to refer to optical invariant.

In order to avoid possible confusion, we shall define several terms ordinarily used with varying meanings. By a refraction, we shall understand the transformation from a wave surface in one medium to that in the other, the two surfaces coinciding with the interface at the axis. By a propagation, we shall understand a transformation from one wave surface to another in the same medium. By a transmission, we shall understand a refraction between two propagations. Thus a transmission carries an arbitrary wave surface in one medium to a wave surface in the other medium at an arbitrary distance. When convenient, and where no confusion is likely to arise, the term transmission may be used for any combination of propagations and refractions, or either alone.

PART II. THE CORRESPONDENCE OF PARAMETERS.

As was pointed out in the introduction, there corresponds to each point on the initial wave a point on the lens surface uniquely determined as the point where a normal to the wave at the point in question cuts the lens surface. Since the system is one of revolution about a single axis, we may select that axis as the x-axis, and study the system by means of a typical meridian plane, which we may take as $z = 0$. Then the y-axis will be in the plane under consideration and normal to the axis of revolution. The symmetry assumed may be expressed in the equations of the wave surface by:

$$x = \sum_{i=0}^{\infty} A_{2i}\alpha^{2i} \qquad y = \sum_{i=0}^{\infty} B_{2i+1}\alpha^{2i+1}. \tag{1}$$

The assumption that the representation is regular at the axis is equivalent to the condition $B_1 \neq 0$. The other coefficients may be assumed to be all finite. Thus, $y = 0$ for $\alpha = 0$. Likewise, we may assume the lens surface to be given by

$$\xi = \sum_{i=0}^{\infty} C_{2i}\beta^{2i}, \qquad \eta = \sum_{i=0}^{\infty} D_{2i+1}\beta^{2i+1}, \tag{2}$$

where $D_1 \neq 0$ and all the coefficients are finite.

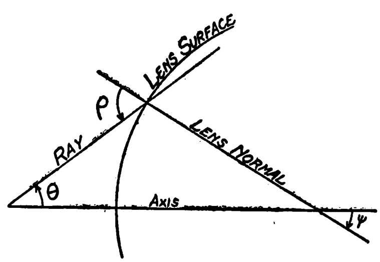

Fig. 1.

Let the normal to the lens surface make an angle ψ with the x-axis, measured as shown in the figure, positive angles being counterclockwise, as usual. Let the ray make an angle θ with the x-axis and an angle ρ with the lens surface normal. Then, since the ray is normal to the wave, it follows that

$$\tan\theta = -\frac{dx}{d\alpha}\Big/\frac{dy}{d\alpha} \qquad \tan\psi = -\frac{d\xi}{d\beta}\Big/\frac{d\eta}{d\beta}. \tag{3}$$

If the point (ξ, η) of the lens surface corresponds to the point (x, y) of the wave surface, then it follows that

$$(\eta - y)/(\xi - x) = \tan\theta = -\frac{dx}{d\alpha}\Big/\frac{dy}{d\alpha}$$

or

$$(\eta - y)\frac{dy}{d\alpha} + (\xi - x)\frac{dx}{d\alpha} = 0. \tag{4}$$

Equation (4) represents the correspondence between α and β and enables us to determine E_i where

$$\beta = \sum_{i=0}^{\infty} E_i\alpha^i. \tag{5}$$

From equation (1), we have

$$\left.\begin{aligned} \frac{dx}{d\alpha} &= \sum_{i=0}^{\infty}(2i+2)A_{2i+2}\alpha^{2i+1}, \\ \frac{dy}{d\alpha} &= \sum_{i=0}^{\infty}(2i+1)B_{2i+1}\alpha^{2i}. \end{aligned}\right\} \tag{6}$$

For $\alpha = 0$, $y = dx/d\alpha = 0$ and $dy/d\alpha = B_1 \neq 0$, so that equation (4) requires that $\eta = 0$ for $\alpha = 0$ or that the solution we want is the one for which $E_0 = 0$, giving

$$\beta = \sum_{i=1}^{\infty} E_i\alpha^i. \tag{7}$$

To express ξ and η in terms of α, we need the various powers of β in terms of α. For this purpose, let

$$\beta^k = \sum_{j=0}^{\infty} I_j^k\alpha^{j+k}. \tag{8}$$

Inspection of equations (7) and (8) shows that I_j^k has the following defining properties:

(a) I_j^k is the sum of terms each involving k of the E_i, one from each of the factors β.

(b) Since the order, or subscript, of each E_i denotes the power of α that it accompanies from β, the sum of the subscripts in each term of I_j^k is $(j + k)$.

(c) The numerical coefficient of each term in I_j^k is the number of distinct permutations possible among the subscripts of that term. If E_i occurs n_i times in a particular term, this number is

$$\frac{k!}{\Pi_i(n_i!)} \quad \text{where} \quad \sum_i n_i = k \quad \text{and} \quad \sum_i in_i = j + k.$$

Hence, we have

$$I_j^k = \sum_u \frac{k!}{\Pi_i(n_i!)}\prod\nolimits_i(E_i^{n_i}), \tag{9}$$

where $\sum_i n_i = k$ and $\sum_i in_i = j + k$ while Σ_u extends over all possible partitions of $(j + k)$ things into k parts.

In particular, I_j^k has the following useful properties:

(A) The highest order E_i among all the terms of I_j^k is in $E_1^{k-1}E_{j+1}$.

(B) If j is odd, each term of I_j^k must have at least one E_i of even order. This follows from the fact that if each E_i were of odd order, the sum would be even or odd with k, while $(j + k)$ is the opposite.

(C) $I_0^0 = 1$, $I_j^0 = 0$ for $j > 0$, $I_j^1 = E_{j+1}$.

Inserting equations (8) in equations (2), we get

$$\xi = \sum_{i,j=0}^{\infty} C_{2i} I_j^{2i} \alpha^{2i+j}, \qquad \eta = \sum_{i,j=0}^{\infty} D_{2i+1} I_j^{2i+1} \alpha^{2i+j+1}. \tag{10}$$

By means of equations (1), (6) and (10), equation (4) becomes

$$\sum_{i,j,k=0}^{\infty} \left\{ \begin{aligned} & C_{2i} I_j^{2i}(2k+2) A_{2k+2} \alpha^{2i+j+2k+1} \\ & \qquad - (2j+2) A_{2j+2} A_{2i} \alpha^{2i+2j+1} \\ & \qquad + D_{2i+1} I_j^{2i+1}(2k+1) B_{2k+1} \alpha^{2i+j+2k+1} \\ & \qquad - (2j+1) B_{2j+1} B_{2i+1} \alpha^{2i+2j+1} \end{aligned} \right\} = 0. \tag{11}$$

The coefficient of α^{2m} is

$$\sum_{k=0}^{m-1} \sum_{i=0}^{m-k-1} [(2k+2) A_{2k+2} C_{2i} I_{2m-2k-2i-1}^{2i} + (2k+1) B_{2k+1} D_{2i+1} I_{2m-2k-2i-1}^{2i+1}] = 0. \tag{12}$$

Since $j = 2m - 2k - 2i - 1$ is odd, each term in equation (12) contains an E_i of even order. Since $I_{2m-1}^0 = 0$, the highest order E_i occurs for $k = i = 0$ and is E_{2m}, the term being $B_1 D_1 I_{2m-1}^1 = B_1 D_1 E_{2m}$. Hence by (12), E_{2m} is the sum of terms each of which involves a factor E_i of even order below $2m$. In particular, for $m = 1$, we have, by equation (12), $k = i = 0$ so

$$2A_2 C_0 I_1^0 + B_1 D_1 L_1^1 = B_1 D_1 E_2 = 0. \tag{13}$$

Since B_1 and D_1 were selected different from zero, it follows that $E_2 = 0$. Further, we have

$$E_{2m} = 0 \qquad \text{and} \qquad I_{2j+1}^k = 0. \tag{14}$$

As a consequence of equations (14), we may write equations (8) and (9) as follows:

$$\left. \begin{aligned} \beta^k &= \sum_{j=0}^{\infty} I_{2j}^k \alpha^{2j+k}, \text{ where} \\ I_{2j}^k &= \sum_u \frac{k!}{\Pi_i(n_i!)} \prod_i E_{2i+1}^{n_i}, \\ & \sum_i n_i = k, \\ & \sum_i (2i+1) n_i = k + 2j \end{aligned} \right\} \tag{15}$$

and where $\sum_u$ extends over all partitions of $(k + 2j)$ into k parts.

By means of equations (14), equation (11) becomes

$$\sum_{i,j,k=0}^{\infty}\left\{\begin{array}{l}[(2k+2)A_{2k+2}C_{2i}I_{2j}{}^{2i} \\ \qquad + (2k+1)B_{2k+1}D_{2i+1}I_{2j}{}^{2i+1}]\alpha^{2i+2j+2k+1} \\ \qquad - [(2j+2)A_{2i}A_{2j+2} \\ \qquad + (2j+1)B_{2i+1}B_{2j+1}]\alpha^{2i+2j+1}\end{array}\right\} = 0. \quad (16)$$

The coefficient of α^{2m+1} is

$$\left.\begin{array}{l}\sum_{k=0}^{m}\sum_{i=0}^{m-k}[(2k+2)A_{2k+2}C_{2i}I_{2m-2k-2i}^{2i} \\ \qquad + (2k+1)B_{2k+1}D_{2i+1}I_{2m-2k-2i}^{2i+1}] \\ \quad - \sum_{i=0}^{m}[(2i+2)A_{2i+2}A_{2m-2i} + (2i+1)B_{2i+1}B_{2m-2i+1}]\end{array}\right\} = 0. \quad (17)$$

Since

$$\sum_{k=0}^{m}\sum_{i=0}^{m-k}\varphi(i,k) = \varphi(0,0) + \sum_{i=1}^{m}\varphi(i,0) + \sum_{k=1}^{m}\sum_{i=0}^{m-k}\varphi(i,k),$$

equation (17) becomes

$$\left\{\begin{array}{l}[2A_2C_0I_{2m}{}^0 + B_1D_1I_{2m}{}^1] \\ \quad + \sum_{i=1}^{m}[2A_2C_{2i}I_{2m-2i}^{2i} + B_1D_{2i+1}I_{2m-2i}^{2i+1}] \\ \quad + \sum_{k=1}^{m}\sum_{i=0}^{m-k}[(2k+2)A_{2k+2}C_{2i}I_{2m-2k-2i}^{2i} \\ \quad + (2k+1)B_{2k+1}D_{2i+1}I_{2m-2k-2i}^{2i+1}] \\ \quad - \sum_{i=0}^{m}[(2i+2)A_{2i+2}A_{2m-2i} + (2i+1)B_{2i+1}B_{2m-2i+1}]\end{array}\right\} = 0. \quad (18)$$

Since $I_{2m}{}^1 = E_{2m+1}$, this gives the recursion formula:

$$E_{2m+1} = \frac{1}{B_1D_1}\left\{\begin{array}{l}\sum_{i=0}^{m}[(2i+2)A_{2i+2}A_{2m-2i} + (2i+1)B_{2i+1}B_{2m-2i+1}] \\ - 2A_2C_0I_{2m}{}^0 - \sum_{i=1}^{m}[2A_2C_{2i}I_{2m-2i}^{2i} \\ \qquad\qquad + B_1D_{2i+1}I_{2m-2i}^{2i+1}] \\ - \sum_{k=1}^{m}\sum_{i=0}^{m-k}[(2k+2)A_{2k+2}C_{2i}I_{2m-2k-2i}^{2i} \\ \qquad\qquad + (2k+1)B_{2k+1}D_{2i+1}I_{2m-2k-2i}^{2i+1}]\end{array}\right\} \quad (19)$$

Equation (19) furnishes E_{2m+1}, after which equations (15) furnish $I_{2j}{}^k$ so that substitution in equation (10) gives:

$$\left.\begin{aligned}\xi &= \sum_{m=0}^{\infty} F_{2m}\alpha^{2m}\\ \eta &= \sum_{m=0}^{\infty} G_{2m+1}\alpha^{2m+1}\end{aligned}\right\}\text{ where }\left\{\begin{aligned}F_{2m} &= \sum_{i=0}^{m} C_{2i}I_{2m-2i}^{2i}\\ G_{2m+1} &= \sum_{i=0}^{m} D_{2i+1}I_{2m-2i}^{2i+1}\end{aligned}\right\}. \tag{20}$$

While equations (20) give the value of G_{2m+1}, it may be expressed in a more convenient form as follows. By equations (1), (4), (6), and (20), we have

$$\sum_{i,j=0}^{\infty}[(2i+1)(B_{2j+1}-G_{2j+1})B_{2i+1}+(2i+2)(A_{2j}-F_{2j})A_{2i+2}]\alpha^{2i+2j+1} = 0. \tag{21}$$

The coefficient of α^{2m+1} is

$$\sum_{j=0}^{m}[(2m-2j+1)(B_{2j+1}-G_{2j+1})B_{2m-2j+1} + (2m-2j+2)(A_{2j}-F_{2j})A_{2m-2j+2}] = 0. \tag{22}$$

Since $\sum_{j=0}^{m}\varphi(j) = \varphi(m)+\sum_{j=0}^{m-1}\varphi(j)$, we have from equation (22)

$$\left.\begin{aligned}G_{2m+1} = B_{2m+1} &+ \frac{1}{B_1}\sum_{j=0}^{m-1}(2m-2j+1)(B_{2j+1}-G_{2j+1})B_{2m-2j+1}\\ &+\frac{1}{B_1}\sum_{j=0}^{m}(2m-2j+2)(A_{2j}-F_{2j})A_{2m-2j+2}\end{aligned}\right\}. \tag{23}$$

We have thus expressed in β terms of α, and are able to eliminate the parameter β from the problem, expressing everything in terms of the parameter α. We have also found explicit values for the coördinates of each point of the lens surface in terms of the value of the parameter specifying the corresponding point on the incident wave surface. Since the coefficients F_{2m} and G_{2m+1} are for the lens surface in terms of α, it follows that they are independent of the choice of β. Thus, when F_{2m} and G_{2m+1} are expressed in terms of A_{2m}, B_{2m+1}, C_{2m} and D_{2m+1}, the coefficients of A_{2m} and B_{2m+1} will be combinations of the C_{2m} and D_{2m+1} which will be the same for all parameters β. Hence these give parametric invariants of the curve which describes the lens surface and hence of all curves with the assumed type of symmetry. The details of this will appear more clearly in the special cases treated in Part IV.

Part III. Transformation of the Wave Surface.

Having expressed the coördinates of the lens surface in terms of the parameter of the incident wave, we may now proceed at once to the study of how the wave is transformed in its passage through the system. We shall assume that the incident wave

$$x' = \sum_{i=0}^{\infty} A'_{2i}\alpha^{2i} \qquad y' = \sum_{i=0}^{\infty} B'_{2i+1}\alpha^{2i+1} \tag{1'}$$

is propagated to the lens surface

$$\xi = \sum_{i=0}^{\infty} F_{2i}\alpha^{2i} \qquad \eta = \sum_{i=0}^{\infty} G_{2i+1}\alpha^{2i+1}, \tag{20}$$

is there refracted in the usual way and is finally propagated into the wave

$$x'' = \sum_{i=0}^{\infty} A''_{2i}\alpha^{2i} \qquad y'' = \sum_{i=0}^{\infty} B''_{2i+1}\alpha^{2i+1}. \tag{1''}$$

We shall thus want to determine A''_{2i} and B''_{2i+1} in terms of A'_{2i}, B'_{2i+1}, F_{2i}, G_{2i+1} and A_0'', the latter quantity specifying which wave of the refracted family we are considering. The same value of α specifies corresponding points (x', y'), (ξ, η) and (x'', y'').

The law of refraction is equivalent to the optical invariance of $\mu \sin \rho$ where μ is the index of refraction of the medium and ρ is the angle between the normals to the lens surface and the wave surface, respectively. If we use primes to refer to the first medium and double primes to refer to the second medium, this may be stated in the form $\mu' \sin \rho' = \mu'' \sin \rho''$. From the figure, we have

$$\sin \rho = \sin (\theta - \psi) = \sin \theta \cos \psi - \cos \theta \sin \psi. \tag{24}$$

By means of equations (3) and (24) and setting

$$\mu \sin \rho = \sum_{i=0}^{\infty} L_{2i+1}\alpha^{2i+1} \tag{25}$$

we find that

$$\left.\begin{aligned} \sum_{i=0}^{\infty} L_{2i+1}\alpha^{2i+1} &= \frac{\mu \sum_{i=0}^{\infty} M_{2i+1}\alpha^{2i+1}}{\sum_{j=0}^{\infty} H_{2j}\alpha^{2j} \sum_{k=0}^{\infty} J_{2k}\alpha^{2k}} \text{ where} \\ \sum_{i=0}^{\infty} M_{2i+1}\alpha^{2i+1} &= \frac{d\xi}{d\alpha}\frac{dy}{d\alpha} - \frac{dx}{d\alpha}\frac{d\eta}{d\alpha} \\ \sum_{j=0}^{\infty} H_{2j}\alpha^{2j} &= \sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2} \\ \sum_{k=0}^{\infty} J_{2k}\alpha^{2k} &= \sqrt{\left(\frac{d\xi}{d\alpha}\right)^2 + \left(\frac{d\eta}{d\alpha}\right)^2} \end{aligned}\right\} \tag{26}$$

By means of equations (1), (20) and (26), after a few simple reductions, we find

$$\left.\begin{aligned}
&H_0 = B_1, \qquad J_0 = G_1, \qquad M_1 = 2(F_2B_1 - A_2G_1), \\
&\qquad\qquad L_1 = \frac{2\mu}{B_1G_1}(F_2B_1 - A_2G_1)
\end{aligned}\right.$$

while for $m > 0$, we get

$$\left.\begin{aligned}
H_{2m} &= \frac{1}{2B_1}\left\{\sum_{i=0}^{m-1}(2i+2)(2m-2i)A_{2m-2i}A_{2i+2} \right.\\
&\quad + \sum_{i=0}^{m}(2i+1)(2m-2i+1)B_{2i+1}B_{2m-2i+1} \\
&\quad \left. - \sum_{i=1}^{m-1} H_{2i}H_{2m-2i}\right\} \\
J_{2m} &= \frac{1}{2G_1}\left\{\sum_{i=0}^{m-1}(2i+2)(2m-2i)F_{2m-2i}F_{2i+2} \right.\\
&\quad + \sum_{i=0}^{m}(2i+1)(2m-2i+1)G_{2i+1}G_{2m-2i+1} \\
&\quad \left. - \sum_{i=1}^{m-1} J_{2i}J_{2m-2i}\right\} \\
M_{2m+1} &= \sum_{i=0}^{m}(2i+2)(2m-2i+1)(F_{2i+2}B_{2m-2i+1} \\
&\qquad\qquad - A_{2i+2}G_{2m-2i+1}) \\
L_{2m+1} &= \frac{1}{B_1G_1}\left\{\mu M_{2m+1} - \sum_{i=0}^{m-1} L_{2i+1}\sum_{j=0}^{m-i} H_{2j}J_{2m-2i-2j}\right\}
\end{aligned}\right\} \tag{27}$$

If we define

$$S_{2m+1} = \frac{1}{G_1}\sum_{i=0}^{m} L_{2i+1}J_{2m-2i} = L_{2m+1} + \frac{1}{G_1}\sum_{i=0}^{m-1} L_{2i+1}J_{2m-2i}, \tag{28}$$

we get

$$S_{2m+1} = \frac{1}{B_1G_1}\left\{\mu M_{2m+1} - \sum_{i=0}^{m-1} L_{2i+1}\sum_{j=1}^{m-i} H_{2j}J_{2m-2i-2j}\right\}. \tag{29}$$

The quantities F_{2m}, G_{2m+1}, J_{2m}, L_{2m+1} and S_{2m+1} are optical invariants and have the same value in the two sets of primed symbols, *i.e.*, in the two media. The invariants G_{2m+1} and S_{2m+1} involve B_{2m+1} and A_{2m+2} along with terms of lower order and F_{2m} and G_{2m+1}, which are invariant. Hence the equations

$$G'_{2m+1} = G''_{2m+1} \qquad \text{and} \qquad S'_{2m+1} = S''_{2m+1}, \tag{30}$$

which are linear in B_{2m+1} and A_{2m+2} if $m > 0$, may be solved for B''_{2m+1} and A''_{2m+2}. In the case $m = 0$, the equations are not linear, but, as shown in Part IV., the solution may be effected without difficulty. The invariant L_{2m+1} might be used instead of S_{2m+1}, but the latter is simpler. While S_{2m+1} involves G_{2m+1}, this may be eliminated by means of equation 23).

The solution for A''_{2m+2} and B''_{2m+1} completes the problem of finding the new wave in terms of the old wave and the refracting surface. The solution of equations (30) becomes increasingly complex as m increases.

Part IV. Terms of the First Four Orders.

To illustrate the use of the preceding formulas, we shall calculate the results for the cases $m = 0$ and $m = 1$. The case $m = 0$ is the elementary Gaussian theory, while the case $m = 1$ is essentially the theory of first order aberrations, so far as these aberrations have a direct interpretation in the case of a system of revolution. For the case of $m = 0$, we shall calculate the case of a general transmission, but for the case of $m = 1$, we shall calculate the cases of propagation and refraction separately, because of the complexity of the formulas. As a matter of notation, let

$$l' = F_0 - A_0', \quad l'' = F_0 - A_0'', \quad \text{and} \quad l = l' - l'' = A_0'' - A_0'. \tag{31}$$

Case I. Gaussian Theory. $m = 0$.—By substitution in the various formulas (23), (27), (29) and (30), we get

$$\left.\begin{aligned} G_1' &= B_1' - 2l'(A_2'/B_1') = G_1'' = B_1'' - 2l''(A_2''/B_1'') \\ S_1' &= L_1' = 2\mu'\left(\frac{F_2}{G_1} - \frac{A_2'}{B_1'}\right) = S_1'' = 2\mu''\left(\frac{F_2}{G_1} - \frac{A_2''}{B_1''}\right) \end{aligned}\right\}. \tag{32}$$

$$\left.\begin{aligned} B_1'' - 2l''(A_2''/B_1'') &= B_1' - 2l'(A_2'/B_1') \\ (A_2''/B_1'') &= \frac{\mu'}{\mu''}\left(\frac{A_2'}{B_1^1}\right) + \frac{F_2(\mu'' - \mu')}{G_1\mu''} \end{aligned}\right\}. \tag{33}$$

Solving for B_1'' and (A_2''/B_1''), we get

$$\left.\begin{aligned} B_1'' &= B_1' - \frac{2A_2'(l'\mu'' - l''\mu')}{B_1'\mu''} + \frac{2l''F_2(\mu'' - \mu')}{G_1\mu''} \\ A_2'' &= \left[\frac{\mu'A_2'}{\mu''B_1'} + \frac{F_2(\mu'' - \mu')}{G_1\mu''}\right]B_1'' \end{aligned}\right\}. \tag{34}$$

For refraction, $l' = l'' = 0$ so that

$$B_1'' = B_1' = G_1 \qquad \text{and} \qquad A_2'' = [\mu'A_2' + F_2(\mu'' - \mu')]/\mu''. \tag{34r}$$

For propagation, $\mu' = \mu'' = 1$ so that

$$B_1'' = B_1' - 2l(A_2'/B_1') \qquad A_2'' = A_2' - 2l(A_2'/B_1')^2 \tag{34p}$$

Case II. First Order Aberrations. $m = 1$.—In this case, we have, by equations (23), (27) and (29),

$$\left.\begin{aligned} G_3 &= B_3 + [3(B_1 - G_1)B_3 + 4(A_0 - F_0)A_4 + 2(A_2 - F_2)A_2]/B_1 \\ H_2 &= 3B_3 + (2A_2^2/B_1) \\ M_3 &= 6(F_2B_3 - A_2G_3) + 4(F_4B_1 - A_4G_1) \\ S_3 &= (\mu M_3/B_1G_1) - (L_1H_2/B_1) \end{aligned}\right\} \quad (35)$$

For refraction, $A_0'' = A_0' = F_0$, $B_1'' = B_1' = G_1$, so that

$$\left.\begin{aligned} G_3 &= B_3' + 2A_2'(A_2' - F_2)/B_1' = B_3'' + 2A_2''(A_2'' - F_2)/B_1'' \\ L_1 &= 2\mu(F_2 - A_2)/B_1 \\ M_3 &= (F_2 - A_2)(6B_1B_3 + 12A_2^2)/B_1 + 4B_1(F_4 - A_4) \\ S_3 &= \frac{4\mu'}{B_1'}\left[(F_4 - A_4') + \frac{2A_2'^2}{B_1'^2}(F_2 - A_2')\right] \\ &= \frac{4\mu''}{B_1''}\left[(F_4 - A_4'') + \frac{2A_2''^2}{B_1''^2}(F_2 - A_2'')\right] \end{aligned}\right\} \quad (36r)$$

$$\left.\begin{aligned} B_3'' &= B_3' + \frac{2(A_2' - F_2)}{B_1'}\left(A_2' - \frac{\mu'}{\mu''}A_2''\right) \\ A_4'' &= F_4 + \frac{\mu'}{\mu''}(A_4' - F_4) + \frac{2\mu'}{B_1'^2\mu''}(A_2' - F_2)(A_2'^2 - A_2''^2) \end{aligned}\right\} \quad (37r)$$

A_2'' is known by equations (34r). For propagation, we may select the initial wave as coinciding with the refracting surface, so that $A_{2k}' = F_{2k}$ and $B_{2k+1}' = G_{2k+1}$. Then, by equations (34p),

$$B_1' - B_1'' = G_1 - B_1'' = 2lA_2'/B_1', \qquad A_2' - A_2'' = 2lA_2'^2/B_1'^2,$$
$$A_2''/B_1'' = A_2'/B_1'.$$

Hence

$$\left.\begin{aligned} &\left(1 - \frac{6A_2'l}{B_1'B_1''}\right)B_3'' + \frac{4l}{B_1''}A_4'' = B_3' + 4l\left(\frac{A_2'}{B_1'}\right)^3, \\ &L_1'' = 2\mu''\left(\frac{A_2'}{B_1'} - \frac{A_2''}{B_1''}\right) = 0, \\ &M_3'' = 6(A_2'B_3'' - A_2''B_3') + 4(A_4'B_1'' - A_4''B_1'), \\ &S_3'' = \frac{\mu''}{B_1'B_1''}\{6(A_2'B_3'' - A_2''B_3') + 4(A_4'B_1'' - A_4''B_1')\} = 0, \\ &3A_2'B_3'' - 2B_1'A_4'' = 3A_2''B_3' - 2A_4'B_1''. \end{aligned}\right\} \quad (36p)$$

Solving the first and last of equations ($36p$), we get

$$\left.\begin{aligned} B_3'' &= B_3' + l\left[\frac{6A_2'B_3'}{B_1'^2} - \frac{4A_4'}{B_1'} + 4\left(\frac{A_2'}{B_1'}\right)^3\right], \\ A_4'' &= A_4' + l\left[6\left(\frac{A_2'}{B_1'}\right)^4 + 12\left(\frac{B_3'}{B_1'}\right)\left(\frac{A_2'}{B_1'}\right)^2 - 8\left(\frac{A_2'}{B_1'}\right)\left(\frac{A_4'}{B_1'}\right)\right]. \end{aligned}\right\} \quad (37p)$$

The calculations for F_0, F_2 and F_4 are made without special difficulty and the results are as follows:

$$\left.\begin{aligned} F_0 &= C_0 I_0^0 = C_0 \qquad F_2 = C_0 I_2^0 + C_2 I_0^2 = C_2 E_1^2, \\ F_4 &= C_0 I_4^0 + C_2 I_2^2 + C_4 I_0^4 = 2C_2 E_1 E_3 + C_4 E_1^4, \\ E_1 &= \frac{B_1}{D_1} + \frac{2A_2(A_0 - C_0)}{B_1 D_1}, \\ E_3 &= \frac{2A_2^2}{B_1 D_1} + \frac{4B_3}{D_1} - \frac{2A_2 C_2 E_1^2}{B_1 D_1} - \frac{D_3 E_1^3}{D_1} + \frac{4A_4(A_0 - C_0)}{B_1 D_1} \\ &\qquad - \frac{3B_3 E_1}{B_1}. \end{aligned}\right\} \quad (38)$$

For pure refraction, $C_0 = A_0$ so that equations (38) become

$$\left.\begin{aligned} E_1 &= \frac{B_1}{D_1}, \qquad E_3 = \frac{2A_2^2}{B_1 D_1} + \frac{B_3}{D_1} - \frac{2A_2 C_2 B_1}{D_1^3} - \frac{D_3 B_1^3}{D_1^4}, \\ F_0 &= A_0, \qquad F_2 = \frac{C_2 B_1^2}{D_1^2}, \\ F_4 &= \left(\frac{C_2}{D_1^2}\right)(4A_2^2 + 2B_1 B_3) - 4A_2 B_1^2\left(\frac{C_2}{D_1^2}\right)^2 \\ &\qquad - B_1^4\left(\frac{2C_2 D_3 - C_4 D_1}{D_1^5}\right). \end{aligned}\right\} \quad (38r)$$

As was noted above (see end of part III.), the coefficients of the wave surface symbols are combinations of the lens surface coefficients which are invariant under a transformation of the parameter β of the type assumed. Hence we have two parametric invariants $P_2 = C_2/D_1^2$ and $P_4 = (C_4 D_1 - 2C_2 D_3)/D_1^5$. That these are actually parametric invariants may be verified by direct calculation, assuming that $\beta = a_1\gamma + a_3\gamma^3$. While it may appear that the assumption of pure refraction is a loss of generality, such is not the case. The values for F_2 and F_4 using the general values in equations (38) are quite complicated and lead to the same values of P_2 and P_4 as given above.

Part V. Concluding Remarks.

While no exhaustive analysis of the preceding results has been made, several conclusions may be mentioned.

The invariant S_{2k+1} may be considered as a generalization of the so-called "zero invariant." Since the axial curvature of the wave is $K = 2A_2/B_1^2$ and of the lens is $k = 2F_2/G_1^2$, the invariant S_1 is

$$S_1 = \mu\left(\frac{2F_2}{G_1} - \frac{2A_2}{B_1}\right) = \mu(kG_1 - KB_1).$$

For pure refraction, $G_1 = B_1$, so that we get the usual zero invariant

$$Q = S_1/G_1 = \mu(k - K). \tag{39}$$

While the assumption that the entire system is one of revolution makes a full analysis of the image formation unsatisfactory, we may nevertheless calculate the longitudinal aberration. Let the normal to the wave at the point (x, y) cut the axis at the point $(X, 0)$. Then X will vary with α, unless the surface is spherical. If we set

$$X = \sum_{m=0}^{\infty} \lambda_{2m}\alpha^{2m}, \tag{40}$$

λ_{2m} will be the longitudinal aberration of order $2m$. It may be shown without difficulty, by the preceding methods, that

$$\lambda_{2m} = \frac{1}{A_2}\left[\frac{1}{2}\sum_{i=0}^{m}(2i + 1)B_{2i+1}B_{2m-2i+1} + \sum_{i=0}^{m}(i + 1)A_{2i+2}A_{2m-2i} - \sum_{i=1}^{m}(i + 1)A_{2i+2}\lambda_{2m-2i}\right]. \tag{41}$$

For $m = 0$, we have $\lambda_0 = A_0 + B_1^2/2A_2$. Since the value of X reduces to λ_0 for $\alpha = 0$, and since the wave surface cuts the axis at A_0, the quantity $B_1^2/2A_2$ represents the axial distance from the wave to the cusp of the caustic. If the initial wave is plane, this distance reduces to the ordinary focal length. The ordinary value of the longitudinal aberration (the so-called spherical aberration) is

$$\lambda_2 = A_2 + 2B_1B_3/A_2 - A_4(B_1/A_2)^2.$$

The results calculated by this method agree very satisfactorily with the results obtained by the usual methods. It may be mentioned that in this method the emphasis is placed on the wave and not on the aberrations. In order to study the aberrations and the effects of the transmission on them, we have merely to calculate the aberrations for the initial and for the final surfaces of the wave family and to subtract the one from the other.

As an example of the agreement between the present method and the usual one, let us consider a particular telescopic system as follows. Let the front lens have a front curvature of 2.0538 and a rear curvature of

$-$ 3.0538. Let the rear lens have a front curvature of $-$ 3.0538 and a rear curvature of $-$ 0.4394. Let the indices of refraction of the front and rear lenses be 1.51115 and 1.61624, respectively, with respect to air, the system being used in air. Let the lenses have axial thicknesses of 0.0065 and 0.0050, respectively, the axial separation being 0.0005. Let the aperture be 1/30. Then the present method gives the following results, for a plane incident wave:

	B_1	A_2	B_3	A_4
Approaching first surface	1.	0.	0.	0.
Leaving " "	1.	0.34735	0.47208	0.53027
Approaching second "	0.99548	0.34578	0.46578	0.52570
Leaving " "	0.99548	1.29596	−5.55684	−3.53208
Approaching third "	0.99418	1.29427	−5.56713	−3.56175
Leaving " "	0.99418	0.22537	0.94578	0.16635
Approaching fourth "	0.99191	0.22486	0.94914	0.16785
Leaving " "	0.99191	0.49663	0.43533	0.30213

The focal point is 0.99057 beyond the axial point of the last surface. The usual theory gives this value as 0.99059. The longitudinal aberration here is 1.03034 per unit aperture or 0.0011448, the usual theory giving 0.0011446.

While some of the formulas have been given for a refraction between two propagations, it is usually more convenient to treat the cases of refraction and of propagation separately, especially in numerical calculations. The formulas so divided are readily adapted to the use of a calculating machine, although not to the use of logarithms.

As a particular case,[1] let us consider the case of a plane wave refracted through a single spherical surface of curvature k, and let us write μ for μ'/μ''. Let us take the optical height as parametric. Then

$$\left.\begin{aligned} x &= a, \\ y &= \alpha, \\ \xi &= a + \tfrac{1}{2}k\alpha^2 + \tfrac{1}{8}k^3\alpha^4, \\ \eta &= \alpha, \end{aligned}\right\} \qquad \left\{\begin{aligned} A_0 &= C_0 = a, \\ A_2 &= A_4 = B_3 = D_3 = 0, \\ B_1 &= D_1 = 1, \\ C_2 &= \tfrac{1}{2}k, \\ C_4 &= \tfrac{1}{8}k^3, \end{aligned}\right.$$

$$\left.\begin{aligned} &\frac{C_2}{D_1^2} = \frac{1}{2}k, \\ &\frac{C^4}{D_1^4} - \frac{2C_2D_3}{D_1^5} = \frac{1}{8}k^3, \end{aligned}\right\} \qquad \left\{\begin{aligned} F_0 &= C_0, \\ F_2 &= \tfrac{1}{2}k, \\ F_4 &= \tfrac{1}{8}k^3. \end{aligned}\right.$$

[1] See, e.g., Preston, Theory of Light, § 73.

The refracted wave has

$$B_1 = 1,$$
$$A_2 = \tfrac{1}{2}k(1 - \mu),$$
$$B_3 = \tfrac{1}{2}k^2\mu(1 - \mu),$$
$$A_4 = \tfrac{1}{8}k^3(1 - \mu)(1 + 2\mu - 2\mu^2),$$

The second order longitudinal aberration is

$$\lambda_2 = \frac{k\mu^2}{2(\mu - 1)},$$

This shows that if the optically denser medium is on the concave side of the refracting surface, the surface is over-corrected for aberration, for λ_2 is positive which means that the cusp of the caustic is towards the lens surface. The focal length of the surface is $B_1^2/2A_2 = 1/k(1 - \mu)$.

While no attempt has been made to apply the foregoing results to the problems of practical optics, it is to be hoped that these problems may be materially simplified by the proper choice of parameters to meet the demands of each particular case. Since the geometrical significance of the parameter is of no essential value, there is more freedom in the choice of parameters than is customary in most treatments of geometrical optics. To review what has been said, the use of a single parameter, besides the azimuth, clarifies the interpretation of orders of small quantities. The recursion formulas, furnish terms of all orders, to be used when occasion demands. Since the sphericity of the lens surfaces has no essential part in the method, it follows that the method is directly applicable to the cases of aspherical lenses. Besides furnishing the final surface as a whole, the present method gives a point by point correspondence between all the surfaces of the system. While many of the quantities involved become indeterminate at a focal point, this furnishes no special difficulty in tracing a surface through a focal point.

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VITA

Irwin Roman was born at St. Louis, Mo., on May 2, 1893. After the customary course in the public schools, he matriculated in Washington University in February, 1910. In June, 1913, he was graduated with the degree of Bachelor of Arts, and the following year he served as Assistant in Physics, spending part time in graduate work in Mathematics. From autumn, 1914, to summer, 1916, he was in residence at The University of Chicago, as also during the summers of 1917, 1919, and 1920. In December, 1915, The University of Chicago granted him the degree of Master of Arts. Since the academic year 1916–17, he has been Instructor in Mathematics at Northwestern University.

While at Washington University, his principal instructors were Professors W. H. Roever and G. O. James. At The University of Chicago, his principal instructor was Professor A. C. Lunn. In the department of Mathematics, his other instructors were Professors G. D. Birkhoff, G. A. Bliss, L. E. Dickson, E. H. Moore, and E. J. Wilczynski. In the Department of Physics, his instructors were Professors C. Kinsley, A. A. Michelson, R. A. Millikan, and A. G. Webster. To all of his instructors he expresses his appreciation of their enthusiasm for research and for the dissemination of knowledge.