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THE CLASS NUMBER OF BINARY QUADRATIC FORMS

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CHAPTER VI.

NUMBER OF CLASSES OF BINARY QUADRATIC FORMS WITH INTEGRAL COEFFICIENTS.

INTRODUCTION.

Particular interest in the mere number of the classes of binary quadratic forms of a given determinant dates from the establishment by C. F. Gauss of the relation between the number h of properly primitive classes of the negative determinant $-D$ and the number of proper representations of D as the sum of three squares. Gauss himself found various expressions for h. G. L. Dirichlet elaborated Gauss' method exhaustively and rigorously.

L. Kronecker, by a study of elliptic modular equations, deduced recurrence formulas for class-number which have come to be called *class-number relations*. C. Hermite obtained many relations of the same general type by equating certain coefficients in two different expansions of pseudo-doubly periodic functions. Hermite's method was extended by K. Petr and G. Humbert to deduce all of Kronecker's relations as well as new and independent ones of the same general type. The method of Hermite was translated by J. Liouville into a purely arithmetical deduction of Kronecker's relations.

The modular function of F. Klein, which is invariant only under a certain congruencial sub-group of the group of unitary substitutions, was employed by A. Hurwitz and J. Gierster just as elliptic moduli had been employed by Kronecker and so the range of class-number relations was vastly extended.

Taking the suggestion from R. Dedekind in his investigation of the classes of ideals of the quadratic field of discriminant D, Kronecker departed from the tradition of Gauss and chose the representative form $ax^2+bxy+cy^2$, where b is indifferently odd or even, and regarded as primitive only forms in which the coefficients have no common divisor. Kronecker thus simplified Dirichlet's results and at the same time set up a relation in terms of elliptic theta functions between the class-number of two discriminants; so he referred the problem of the class-number of a positive discriminant to that of a negative discriminant.

By a study of quadratic residues, M. Lerch and others have curtailed the computation of the class-number. A. Hurwitz has accomplished the same object by approximating $h(p)$, p a prime, by a rapidly converging series and then applying a congruencial condition which selects the exact value of $h(p)$.

Reports are made on several independent methods of obtaining the asymptotic expression for the class-number, and also methods of obtaining the ratio between the number of classes of different orders of the same determinant. The chief advances that have been made in recent years have been made by extending the method of Hermite.

We shall frequently avoid the explanation of an author's peculiar symbols by using the more current notation. Where there is no local indication to the contrary, $h(D)$ denotes the number of properly primitive, and $h'(D)$ the number of improperly primitive, classes of Gauss forms (a, b, c) of determinant $D=b^2-ac$. Referring to Gauss' forms, $F(D)$, $G(D)$, $E(D)$, though printed in italics, will have the meaning which L. Kronecker (p. 109) assigned to them when printed in Roman type. The class-number symbol $H(D)$ is defined as $G(D)-F(D)$. By $K(D)$ or ClD, we denote the number of classes of primitive Kronecker forms of discriminant $D=b^2-4ac$. A determinant is *fundamental* if it is of the form P or $2P$; a discriminant is *fundamental* if it is of the form P, $4P=4(4n-1)$ or $8P$, where P is an odd number without a square divisor other than 1. The context will usually be depended on to show to what extent the Legendre symbol (P/Q) is generalized.

Reduced form and *equivalence* will have the meanings assigned by Gauss (cf. Ch. I). Among definite forms, only positive forms will be considered; and the leading coefficient of representative indefinite forms will be understood to be positive. Ordinarily, τ will be used to denote the number of automorphs for a form under consideration; but when $D>0$, $\tau=1$.

Some account will be given of the modular equations which lead to class-number relations. In reports of papers involving elliptic theta functions, the notations of the original authors will be adopted without giving definitions of the symbols. For the definitions and a comparison of the systems of theta-function notation, the reader is referred to the accompanying table. The different functions of the divisors of a number will be denoted by the symbols of Kronecker,[54] and without repeating the definition. A Gauss form will be called *odd* if it has at least one odd outer coefficient; otherwise it is an *even* form. These terms are not applied to Kronecker forms.

TABLES OF THETA-FUNCTIONS.

Jacobi Hermite	Kronecker	Klein-Fricke	Smith	Hermite Weber Mordell	Petr	Humbert Chapelon	Bell (Tannery)	Fricke
$\Theta_1(z)$ or Θ_1	$=\vartheta_3(x)$	$=\vartheta_3(x, r)$	$=\theta_{00}(z)$	$=\theta_{00}(v)$	$=\Theta_3(v)$	$=\Theta_1(x)$ or Θ_1	$=\vartheta_3(x)$	$=\vartheta_3(v, q)$
$\Theta(z)$ or Θ	$=\vartheta_0(x)$	$=\vartheta_0(x, r)$	$=\theta_{01}(z)$	$=\theta_{01}(v)$	$=\Theta_2(v)$	$=\Theta(x)$ or Θ	$=\vartheta_4(x)$	$=\vartheta_0(v, q)$
$H_1(z)$ or H_1	$=\vartheta_2(x)$	$=\vartheta_2(x, r)$	$=\theta_{10}(z)$	$=\theta_{10}(v)$	$=H_1(v)$	$=H_1(x)$ or H_1	$=\vartheta_2(x)$	$=\vartheta_2(v, q)$
$H(z)$ or H	$=\vartheta_1(x)$	$=\vartheta_1(x, r)$	$=\theta_{11}(z)$	$=\theta_{11}(v)$	$=H(v)$	$=H(x)$ or H	$=\vartheta_1(x)$	$=\vartheta_1(v, q)$

Here, $r=q^2$, $z=2Kx/\pi$, $v=x/\pi$, n is any, m is any odd, integer; and, according to Humbert,

$$\Theta_1(x)=\sum_{n=-\infty}^{\infty} q^{n^2}\cos 2nx,\quad \Theta(x)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\cos 2nx,$$

$$H_1(x)=\sum_{n=-\infty}^{\infty} q^{m^2/4}\cos mx,\ H(x)=\sum_{n=-\infty}^{\infty}(-1)^{\frac{1}{2}(m-1)} q^{m^2/4}\sin mx.$$

For $x=0$, the following systems of special symbols are represented in this chapter.

Humbert	Kronecker, Hurwitz, Hermite, Stieltjes	Weber, Mordell	Hermite, Smith	Humbert, Chapelon	Petr	Bell	Fricke
$\Theta_1(0)=$	$\vartheta_3(q)$ or $\theta_3(q)=$	$\theta_{00}=$	$\theta_1=$	$\theta_1=$	$\Theta_3=$	$\vartheta_3=$	ϑ_3
$\Theta(0)=$	$\vartheta_0(q)$ or $\theta(q)=$	$\theta_{01}=$	$\theta=$	$\theta=$	$\Theta_2=$	$\vartheta_4=$	ϑ_0
$H_1(0)=$	$\vartheta_2(q)$ or $\theta_2(q)=$	$\theta_{10}=$	$\eta=$	$\eta_1=$	$\Theta_1=$	$\vartheta_2=$	ϑ_2
$H'(0)=$	$\vartheta_1'(q)$ or $\theta_1'(q)=$	$\pi\theta_{11}'=$	η'		$=$	ϑ_1'	

In connection with these tables, the following relations will need to be recalled:

$$q=e^{i\pi\omega},\quad \omega=\tau=iK'/K;$$
$$\sqrt{\kappa}=\theta_2(q)/\theta_3(q),\quad \sqrt{\kappa'}=\theta(q)/\theta_3(q),\quad \theta_1'(q)=\theta(q)\theta_2(q)\theta_3(q);$$
$$\sqrt{2K/\pi}=\theta_3(q),\quad \sqrt{2\kappa'K/\pi}=\theta(q),\quad \sqrt{2\kappa K/\pi}=\theta_2(q);$$
$$k=\kappa=\phi^4(\omega),\ k'=\kappa'=\psi^4(\omega).$$

A. M. Legendre[1] excluded every reduced form ("quadratic divisor") whose determinant has a square divisor. Each reduced form $py^2+2qyz+2mz^2$ of determinant $-a=-(4n+1)$ has a *conjugate* reduced form $2py^2+2qyz+mz^2$; here p, q, m are all odd.

If a is of the form $8n+5$, one of p, m is of the form $4n+1$ and the other of the form $4n-1$. Hence the odd numbers represented by one of the quadratic forms are all of the form $4n+1$ and those represented by the conjugate form are of the form $4n+3$. Thus a form and its conjugate are not equivalent and the total number of reduced forms is even.

If $a=8n+1$, the number of reduced forms may be even or odd,[1] but is odd[2] if $a=8n+1$ is prime.

Legendre[3] counted (r, s, t) and $(r, -s, t)$ as the same form. Hence for $a=4n+1$, his number of forms is $\frac{1}{2}\{h(-a)-A\}$, where $h(-a)$, in the terminology of Gauss[4] (Art. 172), is the number of properly primitive classes and A is the number of ambiguous properly primitive classes plus the number of classes represented by forms of the type (r, s, r).

C. F. Gauss,[4] by the composition of classes, proved (Art. 252) that the different genera of the same order have the same number of classes (cf. Ch. IV). He[5] then set for himself the problem of finding an expression in terms of D for the number of classes in the principal genus of determinant D. He succeeded later[8] in finding an expression for the total number of primitive classes of the determinant and thus solved his former problem only incidentally.

[1] Theorie des nombres, Paris, 1798, 267–8; ed. 2, 1808, 245–6; ed. 3, 1830, Vol. I, Part II, § XI (No. 217), pp. 287–8; German transl. by H. Maser, Zahlentheorie, I, 283.
[2] *Ibid.*, Part IV, Prop. VIII, 1798, 449; ed. 2, 1808, 385; ed. 3, II, 1830, 55; Zahlentheorie, II, 56.
[3] *Ibid.*, 1798, No. 48, p. 74; ed. 2, 1808, p. 65; ed. 3, I, p. 77; Zahlentheorie, I, p. 79.
[4] Disquisitiones Arithmeticae, 1801; Werke, I, 1876; German transl. by H. Maser, Untersuchungen ueber Höhere Arithmetik, 1889; French transl. by A. C. M. Poullet-Delisle, Reserches Arithmetiques, 1807, 1910.
[5] Werke, I, 466; Maser, 450; Supplement X to Art. 306. Cf. opening of Gauss'[8, 9] memoirs of 1834, 1837.

If (Art. 253) Q denotes the number of classes of the (positive) order O of determinant D, and if r denotes the number of properly primitive classes of determinant D which, being compounded with an arbitrary class K of the order O, produce a given arbitrary class L of the order O, then the number of properly primitive (positive) classes is rQ. We take both K and L to be the simplest form (Art. 250). It is proved (Arts. 254–6) by the composition of forms that the above r classes are included among certain r' primitive forms, r' being given by

$$r'=\tfrac{1}{2}An\prod_a\left[1-\left(\frac{D'}{a}\right)\frac{1}{a}\right],$$

in which (A, B, C) is the simplest form of order O, $D'=4D/A^2$, and a ranges over the distinct odd divisors of A, while $n=2$ if D/A^2 is an integer, $n=1$ if $4D/A^2\equiv 1 \pmod 8$, $n=3$ if $4D/A^2\equiv 5 \pmod 8$.

Now $r=r'$ if D is a positive square or a negative number except in the cases $D=-A^2$ and $-\frac{3}{4}A^2$, in which cases $r=r'/2$ and $r'/3$ respectively. No general relation (Art. 256, IV, V) is found between r and r' for D positive and not a square.

The problem of finding the ratio of the number of classes of different orders of a determinant will be hereafter referred to as the Gauss Problem. It was solved completely by Dirichlet,[20, 98] Lipschitz,[41] Dedekind,[115] Pepin,[120, 187] Dedekind,[127a] Kronecker,[171] Weber,[220] Mertens,[287] Lerch,[277] Chatelain,[316] and de Séguier.[226]

If O is the improperly primitive order, the same method gives the following result (Art. 256, VI):

If $D\equiv 1 \pmod 8$, $r=1$; if $D<0$ and $\equiv 5 \pmod 8$, $r=3$ (except when $D=-3$ and then $r=1$); if $D>0$ and $\equiv 5 \pmod 8$, $r=1$ or 3, according as the three properly primitive forms

$$(1, 0, -D),\qquad (4, 1, \tfrac{1}{4}(1-D)),\qquad (4, 3, \tfrac{1}{4}(9-D))$$

belong to one or three different classes.

Gauss (Art. 302) gave the following expression for the asymptotic median number of the properly primitive classes of a negative determinant $-D$:

$$M(D)=m\sqrt{D}-\frac{2}{\pi^2},\qquad m=\frac{2\pi}{7(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\dots)}.$$

He later corrected[6] this formula to $m\sqrt{D}$.

His tables of genera and classes led him (Art. 303) to the conjecture[304] that the number of negative determinants which have a given class-number h is finite for every h (cf. Joubert,[60] Landau,[260] Lerch,[262] Dickson,[327] Rabinovitch[336a] and Nagel[336b]).

The asymptotic median value of $h(k^2)$ is $8k/\pi^2$ (Art. 304). He conjectured that the number of positive determinants which have genera of a single class is infinite. Dirichlet[40] proved that this is true. He stated (Art. 304) that, for a positive determinant D, the asymptotic median value of $h(D)\log(T+U\sqrt{D})$ is $m\sqrt{D}-n$, where T, U give the fundamental solution of $t^2-Du^2=1$ and[7] for m as above, while n is a constant as yet not evaluated (cf. Lipschitz[102]).

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[6] Werke, II, 1876, 284; Maser's transl., 670. Cf. Lipschitz.[102]

[7] On the value of m, see Supplement referring to Art. 306 (X). Maser's transl., p. 450; Werke, 1, 1863, 466.

C. F. Gauss[8] considered the lattice points within or on the boundary of an ellipse $ax^2+2bxy+cy^2=A$, where A is a positive integer. The area is $\pi A/\sqrt{D}$, where $-D=b^2-ac$. Hence as A increases indefinitely, the number of representations of all positive numbers $\leqq A$ by means of the definite form (a, b, c) bears to A a ratio which approaches $\pi/\sqrt{D}$ as a limit.

Hereafter[9] the determinant $-D$ has no square divisors, and the asymptotic number of representations of odd numbers $\leqq M$ by the complex C of representative properly primitive forms of determinant $-D$ is

$$\frac{\pi M}{2\sqrt{D}}h(-D).$$

To evaluate $h(-D)$, a second expression for this number of representations is found; but Gauss gives the deduction only in fragments. Thus if (n) denotes the number of representations of n by C and p is an odd prime, then[10].

1. $(pn)=(n)$, if p is a divisor of D;

2. $(pn)=(n)+(h)$, if $\left(\frac{-D}{p}\right)=1$;

3. $(pn)=-(n)+h$, if $\left(\frac{-D}{p}\right)=-1$,

where $n=hp^\mu$, μ arbitrary, h prime to p.

This implies in the three cases

1. $(h)=(ph)=(p^2h)=(p^3h)=\dots$;
2. $(ph)=2(h),\ (p^2h)=3(h),\ (p^3h)=4(h),\ \dots$;
3. $(ph)=0,\ (p^2h)=(h),\ (p^3h)=0,\ (p^4h)=(h),\ \dots$.

Hence the ratio of the mean number of representations by C of all odd numbers $\leqq M$ to the mean number of representations of those numbers after the highest possible power of p has been removed from each as a factor is, in each of the three cases,[11]

$$1/\left\{1-\left(\frac{-D}{p}\right)\frac{1}{p}\right\}.$$

A second odd prime divisor p' is similarly eliminated from the odd numbers $\leqq M$; and so on. Eventually the number of representations of the numbers is asymptotically $\frac{1}{2}\tau M$. Gauss, supposing $-D<-1$, takes the number τ of automorphs to be 2. (See Disq. Arith., Art. 179; Gauss[85] of Ch. I.) Hence the original number of representations is asymptotically[12]

$$M/\Pi\left(1-\left(\frac{-D}{p}\right)\frac{1}{p}\right).$$

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[8] Posthumous paper presented to König. Gesells. der Wiss. Göttingen, 1834; Werke, II, 1876, 269–276; Untersuchungen über Höhere Arith., 1889, 655–661.

[9] Posthumous fragmentary paper presented to König. Gesells. der Wiss. Göthingen, 1837; Werke, II, 1876, 276–291; Untersuchungen über Höhere Arith., 1889, 662–677.

[10] Cf. remarks by R. Dedekind, Werke of Gauss, 1876, II, 293–294; Untersuchungen über Höhere Arith., 1889, 686.

[11] Cf. R. Dedekind, Werke of Gauss, II, 1876, 295–296; Untersuchungen, 1889, 687.

[12] Cf. remarks of R. Dedekind, Werke of Gauss, II, 1876, 296; Untersuchungen, 688.

And hence (Untersuchungen, 670, III; cf. Dirichlet,[19] (1))

$$h(-D)=\frac{2\sqrt{D}}{\pi}\Pi\left\{1-\left(\frac{-D}{p}\right)\frac{1}{p}\right\}^{-1}.$$

Gauss gives without proof five further forms of $h(-D)$ including

$$h(-D)=\Sigma\frac{\sqrt{D}}{N}\left(\frac{-D}{n}\right)\cot n\theta,$$

where $\theta=\pi/N$, $N=D$ *or* $4D$, n is odd >0 and $<D$. Cf. Lebesgue,[36] (1).

By considering the number of lattice points in a certain hyperbolic sector,[13] $h(D)$ is found to be, for $D>0$,

$$\frac{2\sqrt{D}(1\pm\frac{1}{3}\pm\frac{1}{5}\pm\ldots)}{\log(T+U\sqrt{D})}=\frac{\log\sin\frac{\theta}{2}\pm\log\sin\frac{3\theta}{2}\pm\log\sin\frac{5\theta}{2}\pm\ldots}{\log(T+U\sqrt{D})},$$

where the coefficient ± 1 of $1/m$ and of $\log\sin m\theta/2$ is (D/m). Cf. Dirichlet,[28] (7), (8).

For a negative prime determinant, $-D=-(4n+1)$, $h(-D)$ is stated incorrectly to be $\alpha-\beta$, where α and β are respectively the number of quadratic residues and non-residues of D in the first quadrant of D. [This should be $2(\alpha-\beta)$; see Dirichlet,[28] formula (5).]

Extensive tables lead by induction to laws which state, in terms of the class-number of a prime determinant p, the distribution of quadratic residues of p in its octants and 12th intervals.

G. L. Dirichlet[14] obtained $h(-q)$, where q is a positive prime $=4n+3>3$. By replacing infinite sums by infinite products he obtained the lemma:

$$\sum\frac{2^{\mu}}{m^s}=\sum\frac{1}{n^s}\cdot\sum\left(\frac{n}{q}\right)\frac{1}{n^s}\Big/\sum\frac{1}{n^{2s}},$$

where n ranges in order over all positive odd integers prime to q, and m ranges over all positive numbers which have only prime divisors f such that $(f/q)=1$; while μ is the number of such distinct divisors of m; and s is arbitrary >1. Now

$$ax^2+2bxy+cy^2,\qquad a'x^2+2b'xy+c'y^2,\ \ldots$$

denotes a complete set of representative properly primitive (positive) forms of determinant $-q$. Then, by the lemma, since the number of representations of m by the forms is $2^{\mu+1}$ (cf. Dirichlet, Zahlentheorie, § 87), we have

$$(1)\quad 2\Sigma\frac{1}{n^s}\cdot\Sigma\left(\frac{n}{q}\right)\frac{1}{n^s}=\Sigma\frac{1}{(ax^2+2bxy+cy^2)^s}+\Sigma\frac{1}{(a'x^2+2b'xy+c'y^2)^s}+\ldots,$$

where x, y take every pair of values for which the values of the quadratic forms are

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[13] Remarks of R. Dedekind, Gauss' Werke, II, 1876, 299; Maser's translation, 691. Cf. G. L. Dirichlet, Zahlentheorie, §98.

[14] Jour. für Math., 18, 1838, 259–274; Werke, I, 1889, 357–370.

prime[15] to $2q$. We let $s=1+\rho$, and let $\rho>0$ approach zero. The limit of the ratio of each double sum in the right member to

$$\frac{q-1}{2q\sqrt{q}}\frac{\pi}{\rho}.$$

is found from the lattice points of an ellipse to be 1. But

$$\lim \Sigma \frac{1}{n^s} : \frac{q-1}{2q} \cdot \frac{1}{\rho} = 1.$$

Hence (cf. C. F. Gauss,[9] Werke, II, 1876, 285),

$$h(-q) = \frac{2\sqrt{q}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{q}\right)\frac{1}{n} \equiv \frac{2\sqrt{q}}{\pi} S. \tag{2}$$

To evaluate S, we consider

$$S/\left\{1-\left(\frac{2}{q}\right)\frac{1}{2}\right\} = \Sigma\left(\frac{n}{q}\right)\frac{1}{n},$$

where n now ranges in order over all integers $\overline{\gtrless} 1$. In the cyclotomic theory,[16]

$$\Sigma \sin\frac{2an\pi}{q} - \Sigma \sin\frac{2bn\pi}{q} = \left(\frac{n}{q}\right)\sqrt{q}.$$

Hence

$$\frac{\sqrt{q}}{1-\left(\frac{2}{q}\right)\frac{1}{2}} \cdot S = \sum_a \sum_n \frac{1}{n} \sin\frac{2an\pi}{q} - \sum_b \sum_n \frac{1}{n} \sin\frac{2bn\pi}{q},$$

where $(a/q)=1$, $(b/q)=-1$, and a, b are >0 and $<q$. Since (cf. W. E. Byerly, Fourier's Series, 1893, 39) $z=2b\pi/q$ is between 0 and 2π,

$$\frac{\pi-z}{2} = \Sigma \frac{1}{n} \sin nz;$$

and so[17]

$$h = 2\left[1-\left(\frac{2}{q}\right)\frac{1}{2}\right]\frac{\Sigma b - \Sigma a}{q}. \tag{3}$$

Evaluating S itself by cyclotomic considerations, Dirichlet gives the result[18]

$$h = A - B = 2A - \tfrac{1}{2}(q-1), \tag{4}$$

where A and B are respectively the number of quadratic residues and non-residues of q which are $<\frac{1}{2}q$. For $p=4n+1$, Dirichlet obtained

$$h(-p) = 2(A-B) = 4A - \tfrac{1}{2}(p-1), \tag{5}$$

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[15] This restriction is removed by G. Humbert, Comptes Rendus, Paris, 169, 1919, 360–361.
[16] Cf. C. F. Gauss, Werke, II, 1876, 12. G. L. Dirichlet, Zahlentheorie, §116.
[17] Stated empirically by C. G. J. Jacobi, Jour. für Math., 9, 1832, 189–192; detailed report in this History, Vol. I, 275–6; J. V. Pexider,[220] Archiv Math. Phys., (3), 14, 1909, 84–88, combined (3) with the known relation $\Sigma b + \Sigma a = \frac{1}{2}q\,(q-1)$ to express h in terms of Σa alone or Σb alone.
[18] G. B. Mathews, Proc. London Math. Soc., 31, 1899, 355–8, expressed $A-B$ in terms of the greatest integer function.

where A and B are the number of a's and b's respectively between 0 and $\frac{1}{2}p$; and without proof, he stated that

$$h(-pq)=(\Sigma b-\Sigma a)/pq \text{ or } 3(\Sigma b-\Sigma a)/pq,$$

according as pq is $\equiv 7$ or 3 (mod 8), where a, b are positive integers $<pq$, and $(a/p)=(a/q)$, $(b/p)=-(b/q)$. For $h(q)$, the factor π in (2) must be replaced by $\log(T+U\sqrt{D})$; as the lattice points involved must now lie in a certain hyperbolic sector rather than an ellipse (cf. Gauss,[9] Dirichlet[19]).

G. L. Dirichlet[19] considered the four cases of a determinant: $D=P\cdot S^2$, $P\equiv 1$ and 3 (mod 4); $D=2P\cdot S^2$, $P\equiv 1$ and 3 (mod 4), where S^2 is the greatest square divisor of D. He defined δ and ϵ in the four cases as follows:

$$\delta=\epsilon=1,\qquad \delta=-1,\ \epsilon=1,\qquad \delta=1,\ \epsilon=-1,\qquad \delta=\epsilon=-1.$$

Employing the notation of his former memoir,[14] he found for all four cases, if m is representable,

$$\delta^{\frac{1}{2}(f-1)}\epsilon^{\frac{1}{8}(f^2-1)}\left(\frac{f}{P}\right)=1.$$

Consequently the generalization of (1) of the preceding memoir[14] is, for $D=-D_1<0$,

$$\Sigma\frac{1}{(ax^2+2bxy+cy^2)^s}+\Sigma\frac{1}{(a'x^2+2b'xy+c'y^2)^s}+\ldots=2\Sigma\frac{1}{n^s}\Sigma\delta^{\frac{1}{2}(n-1)}\epsilon^{\frac{1}{8}(n^2-1)}\left(\frac{n}{P}\right)\frac{1}{n^s},$$

where the restrictions on s, x, y, n are the same as for (1) in the preceding memoir.

A lemma shows that

$$\Sigma\frac{1}{n^s}=\frac{\phi(D_1)}{2D_1}\cdot\frac{1}{\rho}\text{ or }\frac{\phi(D_1)}{D_1}\cdot\frac{1}{\rho},$$

according as D is odd or even, where $s=1+\rho$ and ρ is indefinitely small, and ϕ is the Euler symbol. The study of lattice points in the ellipse $ax^2+2bxy+cy^2=N$ for very great N leads to

$$\frac{\pi}{2}\frac{\phi(D_1)}{\sqrt{D_1^3}}\cdot\frac{1}{\rho}\text{ or }\frac{\pi\phi(D_1)}{\sqrt{D_1^3}}\cdot\frac{1}{\rho}$$

as the asymptotic value of each of the h sums in the first member, according as D is odd or even. Hence for $D=-D_1<0$,

$$h=\frac{2}{\pi}\sqrt{D_1}\Sigma\delta^{\frac{1}{2}(n-1)}\epsilon^{\frac{1}{8}(n^2-1)}\left(\frac{n}{P}\right)\frac{1}{n}. \tag{1}$$

Dirichlet obtained independently an analogous formula for the number h' of improperly primitive classes of determinant $D=-D_1<0$. For $D>0$, results analogous to all those for $D<0$, are obtained by considering all the representations of positive numbers $\leqq N$ by $ax^2+2bxy+cy^2$, where a is >0 and (x, y) are lattice points in the hyperbolic sector having $y>0$ and bounded by $y=0$, $U(ax+by)=Ty$, and $ax^2+2bxy+cy^2=N$. For $D>0$, these restrictions on a, x, y are hereafter understood in this chapter of the History.

[19] Jour. für Math., 19, 1839, 324–369; 21, 1840, 1–12, 134–155; Werke, I, 1889, 411–496; Ostwald's Klassiker der exakten Wissenschaften, No. 91, 1897, with explanatory notes by R. Haussner.

Incidentally Dirichlet[20] stated for $D<-3$ the "fundamental equation of Dirichlet" (see Zahlentheorie, § 92, for the general statement):

$$(2)\quad \Sigma'\phi(ax^2+2bxy+cy^2)+\Sigma'\phi(a'x^2+2b'xy+c'y\ )+\dots$$
$$=2\Sigma\delta^{\frac{1}{2}(n-1)}\epsilon^{\frac{1}{8}(n^2-1)}\left(\frac{n}{P}\right)\phi(nn'),$$

where ϕ is an arbitrary function which gives absolute convergence in both members; the forms are a representative primitive system; x and y take all pairs of integral values (excepting $x=y=0$) in each form for which the value of the form is prime[21] to $2D$ if the form is properly primitive, but half of the value of the form is prime to $2D$ if the form is improperly primitive; the second member is a double sum as to n and n'. Kronecker[171] and Lerch[277] (Chapter I of his Prize Essay) used this identity to obtain a class-number formula.

Dirichlet noted from the results in his[19] former memoir that for $D<0$, $h=h'$ or $3h'$, according as $D\equiv 1$ or 5 (mod 8), except that $h=h'$ for $D=-3$. For $D>0$, if $D=8n+1$, $h=h'$; but, if $D=8n+5$, $h=h'$ or $3h'$, according as the fundamental solutions of $t^2-Du^2=4$ are odd or even. (Cf. Gauss,[4] Disq. Arith., Art. 256.)

Since the series in (1) may be written as

$$\Pi\left\{1-\delta^{\frac{1}{2}(n-1)}\epsilon^{\frac{1}{8}(n^2-1)}\left(\frac{n}{P}\right)\frac{1}{n}\right\}^{-1},$$

where n is a positive odd prime, and prime to D, it follows that if h and h' denote respectively the number of properly primitive classes of the two negative determinants D and $D'=D\cdot S^2$, D having no square divisor, then

$$\frac{h'}{h}=S\prod_r\left[1-\delta^{\frac{r-1}{2}}\epsilon^{\frac{r^2-1}{8}}\left(\frac{r}{P}\right)\frac{1}{r}\right],$$

where[22] r ranges over the odd prime positive divisors of S (except if $D=-1$, the ratio thus given should be divided by 2). The corresponding ratio is found for $D'>0$.

Dirichlet[23] hereafter took $S=1$ and, representing the series in (1) by V, found that for $D=\pm P\equiv 1$ (mod 4), for example,

$$\frac{V}{1-\left(\frac{2}{P}\right)\frac{1}{2}}=-\int_0^1\frac{\Sigma\left(\frac{n}{P}\right)x^{n-1}}{x^P-1}\,dx=-\frac{1}{P}\sum_{m}\sum_{n}\left(\frac{n}{P}\right)e^{\frac{n}{P}\cdot 2m\pi i}\int_0^1\frac{dx}{x-e^{2m\pi i/P}},$$

$n,\ m=0,1,2,\dots,P-1;\ P=|D|$.

[20] Jour. für Math., 21, 1840, 7; Werke, I, 1889, 467. The text is a report of Jour. für Math. 21, 1840, 1-12; Werke, I, 1889, 461-72.

[21] This restriction is removed by G. Humbert, Comptes Rendus, Paris, 169, 1919, 360-361.

[22] Cf. Disq. Arith., Art. 256, V; R. Lipschitz,[41] Jour. für Math., 53, 1857, 238.

[23] From this point, Jour für Math., 21, 1840, 134-155; Werke, I, 1889, 479-496. Cf. Zahlentheorie, §§-103-105.

The identity, in Gauss sums,

$$(3)\qquad \Sigma\left(\frac{n}{P}\right)e^{\frac{n}{P}\cdot 2m\pi i}=i^{\left(\frac{P-1}{2}\right)^2}\left(\frac{m}{P}\right)\sqrt{P}$$

now gives

$$\frac{V}{1-\left(\frac{2}{P}\right)\frac{1}{2}}=-\frac{i^{\left(\frac{P-1}{2}\right)^2}}{\sqrt{P}}\sum_{m=1}^{P}\left(\frac{m}{P}\right)\left(\log\sin\frac{m\pi}{P}-\frac{m\pi i}{P}\right);$$

whence

$$(4)\qquad \begin{cases} V=-\dfrac{1}{\sqrt{P}}\left(1-\left(\dfrac{2}{P}\right)\dfrac{1}{2}\right)\Sigma\left(\dfrac{m}{P}\right)\log\sin\dfrac{m\pi}{P}, \text{ if } P=4\mu+1,\\[2ex] V=\dfrac{-\pi}{\sqrt{P^3}}\left(1-\left(\dfrac{2}{P}\right)\dfrac{1}{2}\right)\Sigma\left(\dfrac{m}{P}\right)m, \text{ if } P=4\mu-1.\end{cases}$$

For $D=-P$, $P=4\mu-1$, the comparison of (1) and (3) gives

$$h(D)=\frac{2}{\pi}\Sigma\left(\frac{m}{P}\right)\Sigma\frac{1}{n}\sin n\frac{2m\pi}{P};$$

whence[24] finally by grouping quadratic residues and non-residues, we have:

$$h(D)=\Sigma\left(\frac{m'}{P}\right),\quad 0<m'<\frac{P}{2}.$$

So Dirichlet[25] obtained his classic formulas for $D<0$:

$$(5)\qquad \begin{cases} D=-P,\quad P=4\mu+3,\ h(D)=\displaystyle\sum_{0}^{\frac{1}{2}P}\left(\frac{S}{P}\right);\\[2ex] D=-P,\quad P=4\mu+1,\ h(D)=2\displaystyle\sum_{0}^{\frac{1}{4}P}\left(\frac{S}{P}\right);\\[2ex] D=-2P,\ P=4\mu+3,\ h(D)=2\displaystyle\sum_{\frac{1}{8}P}^{\frac{3}{8}P}\left(\frac{S}{P}\right);\\[2ex] D=-2P,\ P=4\mu+1,\ h(D)=2\displaystyle\sum_{0}^{\frac{1}{8}P}\left(\frac{S}{P}\right)-2\sum_{\frac{3}{8}P}^{\frac{1}{2}P}\left(\frac{S}{P}\right).\end{cases}$$

From (1) and (4) and their analogues, he wrote also in the four cases of $D<0$:

$$(6)\quad -h(D)=\frac{1}{P}\left(2-\left(\frac{2}{P}\right)\right)\Sigma\left(\frac{S}{P}\right)\epsilon_1 S,\ \frac{1}{4P}\Sigma\left(\frac{S}{P}\right)\epsilon_2 S,\ \frac{1}{8P}\Sigma\left(\frac{S}{P}\right)\epsilon_3 S,\ \frac{1}{8P}\Sigma\left(\frac{S}{P}\right)\epsilon_4 S,$$

where $S=m$ ranges from 0 to P, $4P$, $8P$, $8P$ in the four respective cases, and

$$\epsilon_1=1,\ \epsilon_2=(-1)^{\frac{1}{2}(m-1)},\ \epsilon_3=(-1)^{\frac{1}{8}(m^2-1)},\ \epsilon_4=(-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)}.$$

For $D>0$, the analogue of (1) is

$$(7)\qquad h(D)=\frac{2\sqrt{D}}{\log(T+U\sqrt{D})}\Sigma\delta^{\frac{1}{2}(n-1)}\epsilon^{\frac{1}{8}(n^2-1)}\left(\frac{n}{P}\right)\frac{1}{n},$$

[24] Fourier-Freeman, Theory of Heat, Cambridge, 1878, 243.
[25] Jour. für Math., 21, 1840, 152; Werke, I, 492–3; Zahlentheorie, §106.

where T, U are the fundamental solution of $t^2-Du^2=1$. Hence[26] from equations like (4), we obtain:

$$(8_1)\qquad D=P,\qquad P=4\mu+1,\qquad h(D)=\frac{2-\left(\frac{2}{P}\right)}{\log(T+U\sqrt{P})}\log\frac{\Pi\sin b\pi/P}{\Pi\sin a\pi/P},$$

where a and b range over the integers $<P$ and prime to P for which

$$(a/P)=+1,\qquad (b/P)=-1;$$

$$(8_2)\qquad D=P,\qquad P=4\mu+3,\qquad h(D)=\frac{1}{\log(T+U\sqrt{P})}\log\frac{\Pi\sin\frac{1}{4}b\pi/P}{\Pi\sin\frac{1}{4}a\pi/P},$$

where a and b range over the integers $m<4P$ and prime to $4P$ for which

$$(-1)^{\frac{1}{2}(m-1)}\left(\frac{m}{P}\right)=+1 \text{ or } -1, \text{ according as } m=a \text{ or } b;$$

$$(8_3)\qquad D=2P,\qquad h(D)=\frac{1}{\log(T+U\sqrt{2P})}\log\frac{\Pi\sin\frac{1}{8}b\pi/P}{\Pi\sin\frac{1}{8}a\pi/P},$$

where a and b range over the integers $m<8P$ and prime to $8P$, for which

if $P\equiv 1 \pmod 4$, $(-1)^{\frac{1}{8}(m^2-1)}\left(\frac{m}{P}\right)=+1$ or -1, according as $m=a$ or b;

if $P\equiv 3 \pmod 4$, $(-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)}\left(\frac{m}{P}\right)=+1$ or -1, according as $m=a$ or b.

If $D=P=4\mu+1>0$, (4) and (7) with cyclotomic considerations give[27]

$$(9)\qquad h(D)=\left[4-2\left(\frac{2}{P}\right)\right]\frac{\log\frac{1}{2}[Y(1)+Z(1)\sqrt{P}]}{\log(T+U\sqrt{P})},$$

where $\frac{1}{2}[Y(x)+Z(x)\sqrt{P}]\equiv\Pi(x-e^{2\pi bi/P})$.

Arndt[58] supplied formulas for the other three cases.

A. L. Cauchy[28] proved that if p is a prime of the form $4l+3$,

$$\frac{A-B}{2}\equiv -3B_{(p+1)/4} \text{ or } B_{(p+1)/4} \pmod p,$$

according as $p=8l+3$ or $8l+7$, where A is the number of quadratic residues and B that of the non-residues of p which are >0 and $<\frac{1}{2}p$, and B_k is the kth Bernoullian number. This implies, by G. L. Dirichlet,[28] (5), that

$$(1)\qquad h(-p)\equiv 2B_{(p+1)/4} \text{ or } -6B_{(p+1)/4} \pmod p,$$

according as $p=8l+7$ or $8l+3$ [cf. Friedmann and Tamarkine[321]].

Cauchy[29] obtained also the equivalent of the following for n free from square factors, and of the form $4x+3$:

$$(2)\qquad A-B=\left[2-\left(\frac{2}{n}\right)\right]\frac{\Sigma b-\Sigma a}{n}=\left[2-\left(\frac{2}{n}\right)\right]\frac{\Sigma b^2-\Sigma a^2}{n^2},$$

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26 Jour. für Math., 21, 1840, 151; Werke, I, 492.

27 See this History, Vol. II, Ch. XII, 372 [117]; Cf. Dirichlet, Zahlentheorie, 1894, 279, § 107.

28 Mém. Institut de France, 17, 1840, 445; Oeuvres, (1), III, 172. Bull. Sc. Math., Phys., Chim. (ed., Férussac), 1831.

29 Mém. Institut de France, 17, 1840, 697; Oeuvres, (1), III, 388. Comptes Rendus, Paris, 10, 1840, 451.

where A, B are the number of quadratic residues and non-residues of n, which are $<\frac{1}{2}n$, while a, b are >0 and $<n$, $(a/n)=1$, $(b/n)=-1$: and similar formulas for $n=4x+1$. Hence, for $n=4x+3$,

$$h(-n)=\left[2-\left(\frac{2}{n}\right)\right]\frac{\Sigma b^2-\Sigma a^2}{n^2}$$

is called Cauchy's class-number formula.[30]

M. A. Stern[31] found that when P is a prime $8m+7$, or $8m+3$ respectively,

$$\prod_a \cot\frac{2\pi a}{P}=\pm(-1)^N\frac{1}{\sqrt{P}},$$

where a ranges over all positive integers $<P$ prime to P such that $(a/P)=1$, and N denotes the number of quadratic divisors of determinant $-P$. This formula has been made to include the case $P=4m+1$ by Lerch.[328]

G. Eisenstein[32] proposed the problem: If $D>0$ is $\equiv 5 \pmod 8$, to determine *a priori* whether $p^2-Dq^2=4$ can be solved in odd or even integers p, q; that is[33] to furnish a criterion to determine whether the number of properly primitive classes of determinant D is 1 or 3 times the number of improperly primitive classes of the same determinant. He also proposed the problem[34]: To find a criterion to determine whether the number of properly primitive classes of a determinant D is divisible by 3; and if this is the case, a criterion to determine those classes which can be obtained by triplication[35] of other classes.

V. A. Lebesgue[36] employed the notation of Dirichlet[23] and, in his four cases, set $p=P$, $4P$, $8P$, $8P$, and $f(x)=\Sigma\epsilon_i(a/p)x^a$, summed over all the positive integers $a<p$, for $i=1, 2, 3, 4$. Then

$$V'=\int_0^1\frac{f(x)\,dx}{x(1-x^p)}$$

is the sum of integrals (for the various values of a), with proper signs prefixed,

$$\int_0^1\frac{x^{a-1}dx}{1-x^p}=-\frac{1}{p}\sum_{m=1}^{p}\cos m\,\frac{2a\pi}{p}\log\sin\frac{m\pi}{p}+\frac{\pi}{2p}\cot\frac{a\pi}{p}.$$

For a negative determinant, the terms involving the logarithm cancel each other and then, by the theory of Gauss[37] sums, V' reduces to[38]

$$V'=\frac{\pi}{p}\Sigma\cot\frac{A\pi}{p},\quad \epsilon_i\left(\frac{A}{p}\right)=1,\quad 1\leqq A<p. \tag{1}$$

H. W. Erler[39] developed a hint by Gauss (Disq. Arith., Art. 256, § V, third case) that there is a remarkable relation between the totality B of properly primitive forms

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30 Cf. T. Pepin,[120] Annales sc. de l'Ecole Norm. Sup., (2), 3, 1874, 205; M. Lerch,[277] Acta Math., 29, 1905, 381.
31 Jour. de Math., (1), 5, 1840, 216–7. This is proved by means of C. G. J. Jacobi's result in this History, Vol. I, 275–6.
32 Jour. für Math., 27, 1844, 86.
33 Cf. G. L. Dirichlet, Zahlentheorie, 1894, Art. 99; Dirichlet.[23]
34 Jour. für Math., 27, 1844, 87.
35 Cf. C. F. Gauss, Disq. Arith., Art. 249; Maser's translation, 1889, 261; Werke, I, 1876, 272.
36 Jour. de Math., 15, 1850, 227–232.
37 Disq. Arith., Art. 356.
38 Cf. C. F. Gauss, memoir of 1837, Werke, II, 1876, 286; Untersuchungen, 1889, 671.
39 Eine Zahlentheoretische Abhandlung, Progr. Zullichau, 1855, p. 18.

of determinant D which represent A^2 and the least solution t_1, u_1 of $t^2-Du^2=A^2$. Erler considered the case in which A^2 divides D, whence A divides t_1. Write $\tau_1=t/A$, $D'=D/A^2$, whence $t_1^2-D'u_1^2=1$. Find the period of the solution of the latter for modulus A. From each pair of simultaneous values of τ_1, u_1, we can derive one and only one from the set B which is equivalent to the principal form. The terms of every later period give the same forms in the same sequence as those of the first period. In case bisection of the period is possible, the terms of the second half are the same as in the first half. The forms obtained from the terms of the first half (or from the entire first period, if bisection is impossible) are distinct.

G. L. Dirichlet[40] recalled (see Dirichlet,[23] (8)) that for a positive determinant $D'=D\cdot S^2$,

$$h(D')=h(D)\,\frac{\log(T+U\sqrt{D})}{\log(T'+U'\sqrt{D'})}\cdot S\cdot R\equiv\frac{h(D)\cdot S\cdot R}{N},$$

in which R is independent of $a_1, a_2, \ldots, a_k$ in $S=p_1^{a_1}\cdot p_2^{a_2}\ldots p_k^{a_k}$, where the p's are distinct primes. By the theory of the Pell equation it is found (see this History, Vol. II, p. 377, Dirichlet[184]) that if each a increases indefinitely, S/N is eventually a constant. Hence for every D, there is an infinitude of determinants $D'=DS^2$ for which $h(D')=h(D)$. And a proper choice of D and the primes $p_1, p_2, \ldots, p_k$ leads to an infinite sequence of determinants D' for which the number of genera coincides with the value of $h(D')$. This establishes the conjecture of C. F. Gauss (Disq. Arith.,[4] Art. 304) that there is an infinitude of determinants which have genera of a single class.

R. Lipschitz[41] called the linear substitutions

$$\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix},\quad \begin{pmatrix}\mathrm{A} & \mathrm{B}\\ \Gamma & \Delta\end{pmatrix}$$

equivalent if $\alpha, \ldots, \Delta$ are integers and if integers $\alpha', \beta', \gamma', \delta'$ exist such that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}=\begin{pmatrix}\mathrm{A} & \mathrm{B}\\ \Gamma & \Delta\end{pmatrix},\quad \alpha'\delta'-\beta'\gamma'=1.$$

Every substitution of odd prime order p is equivalent to one of the $p+1$ non-equivalent substitutions:

$$(1)\qquad \begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix},\ \begin{pmatrix}0 & -p\\ 1 & 0\end{pmatrix},\ \begin{pmatrix}1 & -p\\ 1 & 0\end{pmatrix},\ \begin{pmatrix}2 & -p\\ 1 & 0\end{pmatrix},\ldots,\ \begin{pmatrix}p-1 & -p\\ 1 & 0\end{pmatrix}.$$

Let (a, b, c), *a properly primitive form of determinant D, be transformed by* (1) into $p+1$ forms (a', b', c'). Then $D'=D\cdot p^2$. The coefficients of every form (a', b', c') satisfy the system of equations

$$\begin{aligned}ap^2&=a'\delta^2-2b'\delta\gamma+c'\gamma^2,\\ bp^2&=-a'\delta\beta+b'(\alpha\delta+\beta\gamma)-c'\gamma\alpha,\\ cp^2&=a'\beta^2-2b'\beta\alpha+c'\alpha^2.\end{aligned}$$

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[40] Bericht. Acad. Berlin, 1855, 493–495; Jour. de Math., (2), 1, 1856, 76–79; Jour. für Math., 53, 1857, 127–129; Werke, II, 191–194.

[41] Jour. für Math., 53, 1857, 238–259. See H. J. S. Smith, Report Brit. Assoc., 1862, § 113; Coll. Math. Papers, I, 246–9; also G. B. Mathews, Theory of Numbers, 1892, 159–170.

Hence (a', b', c') has no other divisor than p, and the condition that p be a divisor is

$$aa' = a(a\alpha^2 + 2b\alpha\gamma + c\gamma^2) = (a\alpha + b\gamma)^2 - D\gamma^2 \equiv 0 \pmod{p}.$$

Now, a may be assumed relatively prime to p. The number of solutions of this congruence is the number of substitutions in (1) which do not lead to properly primitive forms (a', b', c'). This number is 2, 0, 1 according as $(D/p) = 1, -1, 0$. Hence the number of properly primitive forms (a', b', c') is $p - (D/p)$.

If $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ be one substitution of (1) which carries (a, b, c) over into a particular (a', b', c'), then all the substitutions in (1) which effect the transformation are $\left(\begin{smallmatrix}\mathrm{A} & \mathrm{B} \\ \Gamma & \Delta\end{smallmatrix}\right)$, in which (Gauss, Disq. Arith., Art. 162; report in Ch. I)

$$\begin{aligned}\mathrm{A} &= \alpha t - (\alpha b + \gamma c)u, & \Gamma &= \gamma t + (a\alpha + \gamma b)u,\\ \mathrm{B} &= \beta t - (\beta b + \delta c)u, & \Delta &= \delta t + (\beta a + \delta b)u,\end{aligned}$$

t, u ranging over σ pairs of integers which satisfy $t^2 - Du^2 = 1$, where σ is the smallest value of i for which u_i is a multiple of p in

$$(T + U\sqrt{D})^i = t_i + u_i\sqrt{D}.$$

If $u_\sigma = pu'$, $t_\sigma = t'$, then

$$(T + U\sqrt{D})^\sigma = t' + u'\sqrt{p^2 \cdot D}, \qquad \sigma = \frac{\log(T' + U'\sqrt{p^2D})}{\log(T + U\sqrt{D})},$$

where T, U is the fundamental solution of $t^2 - Du^2 = 1$, and T', U', of $t^2 - Dp^2u^2 = 1$. Since only one form (a, b, c) can be carried over into a particular form (a', b', c') by (1), Dirichlet's[42] ratio $h(S^2D)/h(D)$ follows at once.[43] Similarly, Lipschitz obtained the ratio of the number of improperly primitive classes to the number of properly primitive classes for the same determinant.[44]

L. Kronecker[45] stated that if n denotes a positive odd number > 3 and κ denotes the modulus of an elliptic function, then the number of different values of κ^2 which admit of *complex multiplication* by $\sqrt{-n}$ [i. e., for which $sn^2(u\sqrt{-n}, \kappa)$ is rationally expressible in terms of $sn^2(u, \kappa)$ and κ] is six times[46] the number of classes of quadratic forms of determinant $-n$. These values of κ^2 are the sole roots of an algebraic equation with integral coefficients, which splits into as many integral factors as there are orders of binary quadratic forms of determinant $-n$. To each order corresponds one factor whose degree is six times the number of classes belonging to that order. The two following recursion formulas[47] and one immediately deducible from them are given. Let $n \equiv 3 \pmod 4$; let $F(m)$ be the number of properly primitive classes of $-m$ plus the number of classes derived from them;

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[42] Jour. für Math., 21, 1840, 12. See Dirichlet.[20]

[43] For details, see G. B. Mathews,[218] Theory of Numbers, Cambridge, 1892, 159–166; also H. J. S. Smith's Report.[79]

[44] For details see G. B. Mathews,[218] Theory of Numbers, 1892, 166–169.

[45] Monatsber. Akad. Wiss. Berlin, Oct., 1857, 455–460. French trans., Jour. de Math., (2), 3, 1858, 265–270.

[46] Cf. H. J. S. Smith, Report Brit. Assoc., 35, 1865, top of p. 335; Coll. Math. Papers, I, 305.

[47] Cf. L. Kronecker, Jour. für Math., 57, 1860, 249.

$\phi(n)$ be the sum of the divisors of n which are $>\sqrt{n}$; $\psi(n)$ be the sum of the other divisors. Then

$$\text{(I)} \qquad 2F(n)+4F(n-2^2)+4F(n-4^2)+\ldots=\phi(n)-\psi(n),$$

$$\text{(II)} \qquad 4F(n-1^2)+4F(n-3^2)+4F(n-5^2)+\ldots=\phi(n)+\psi(n),$$

where, in the left members, $n-i^2>0$.

Using the *absolute invariant* j instead of κ^2, H. Weber[48] has deduced in detail a similar relation which these two imply.[214]

C. Hermite[49] set $u=\phi(\omega)=\kappa^{\frac{1}{4}}$, κ being the ordinary modulus in elliptic functions, and found that the algebraic discriminant of the standard modular equation for transformations of prime order n,

$$\Pi\left[\phi^8(\omega)-\phi^8\left(\frac{\delta_1\omega+16m}{\delta_2}\right)\right]=0,\ \delta_1\delta_2=n,\ m=0,\ 1,\ 2,\ \ldots,\ \delta_2-1,$$

is of the form

$$u^{n+1}(1-u^8)^{n+(2/n)}\theta^2(u),$$

where $\theta(u)=a_0+a_1u^8+a_2u^{16}+\ldots+a_\nu u^{8\nu}$ is a reciprocal polynomial with no multiple roots and $\theta(u)$ is relatively prime to u and $1-u^8$; moreover,

$$\nu=\frac{n^2-1}{8}-\frac{1}{2}\left[n+\left(\frac{2}{n}\right)\right].$$

By means of the condition for equality of two roots[50] of the modular equation, he set up a correspondence between these equal roots and the roots of certain quadratic equations of determinant $-\Delta$ and so proved the following theorem.[51] Let

$$\Delta'=(8\delta-3n)(n-2\delta)>0,\ \Delta''=8\delta(n-8\delta)>0,\ \Delta'''=\delta(n-16\delta)>0.$$

Then

$$\nu=2\Sigma F(\Delta')+2\Sigma F(\Delta'')+6\Sigma G(\Delta''').$$

(Cf. H. J. S. Smith, Report Brit. Assoc., 1865; Coll. Math. Papers, I, 344–5.)

Those roots $x=\phi(\omega)$ of $\theta(u)=0$ are now segregated which correspond to the roots ω of a representative system of properly primitive forms of a given negative determinant $-\Delta$; similarly for a system of improperly primitive forms. If the representative form (A, B, C) of each properly primitive class is chosen with C even, A uneven, then to the roots ω of the equations $A\omega^2+2B\omega+C=0$ correspond values of $u^8=\phi^8(\omega)$ which are the principal roots of a reciprocal equation $F\ (x,\ \Delta)=0$ with integral

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[48] Elliptische Functionen und Algebraische Zahlen, Braunschweig, 1891, 393–401; Algebra, III, Braunschweig, 1908, 423–426. For the same theory see also Klein-Fricke, Elliptischen Modulfunctionen, Leipzig, 1892, II, 160–184.

[49] Comptes Rendus, Paris, 48, 1859, 940–948, 1079–1084, 1096–1105; 49, 1859, 16–24, 110–118, 141–144. Oeuvres, II, 1908, 38–82. Reprint, Paris, 1859, Sur la théorie des équations modulaires et la résolution de l'équation du cinquième degré, 29–68.

[50] Cf. C. Hermite, Sur la théorie des équations modulaires, 1859, 4; Comptes Rendus, Paris, 46, 1859, 511; Oeuvres, II, 1908, 8. Cf. also H. J. S. Smith, Report Brit. Assoc., 1865, 330; Coll. Math. Papers, I, 299. For properties of the discriminant of the modular equation, see L. Koenigsberger, Vorlesungen über die Theorie der Elliptischen Functionen, Leipzig, 1874, Part II, 154–6.

[51] For an equivalent result see Kronecker.[124]

coefficients and of degree the double of the number of those classes. Moreover, $F(x, \Delta)$ can be decomposed into factors of the form

$$(x+1)^4+a_1x(x-1)^2, \text{ if } \Delta=\begin{cases}2n-1,\ 2n-9,\ 2n-25, \ldots; \\ 2n,\ 2n-4,\ 2n-16, \ldots.\end{cases}$$

This illustrates the rule that, excepting $\Delta=1$, 2, the number of properly primitive classes of $-\Delta$ is even if Δ is $\equiv 1$ or 2 (mod 4).

In a theorem analogous to the preceding and concerning improperly primitive classes, $\mathcal{F}(x, \Delta)=0$ is a reciprocal equation with integral coefficients and of degree 2 or 6 times the number of those classes, according as $\Delta\equiv -1$ or 3 (mod 8); and $\mathcal{F}(x,\Delta)$ can be decomposed into factors of the form $(x^2-x+1)^3+a(x^2-x)^2$, if $\Delta=4n-1,\ 4n-9,\ 4n-25, \ldots$.

For a few small determinants the class-number is exhibited as by the following example. After Jacobi, the modular equations of orders 3 and 5 respectively are

$$(q-l)^4=64(1-q^2)(1-l^2)(3+ql),$$
$$(q-l)^6=256(1-q^2)(1-l^2)[16ql(9-ql)^2+9(45-ql)(q-l)^2]$$

where $q=1-2\kappa^2$, $l=1-2\lambda^2$. These equations combined with

$$u^8=\frac{1}{1-v^8}=x, \text{ where } u^8=\kappa^2,\ v^8=\lambda^2,$$

give, respectively,

$$[x^2-x+1][(x^2-x+1)^3+2^7(x^2-x)^2]\equiv\mathcal{F}(x,\ 3)\cdot\mathcal{F}(x,\ 11)=0,$$
$$[(x^2-x+1)^3+2^7(x^2-x)^2][(x^2-x+1)^3+2^7\cdot 3^3(x^2-x)^2]$$
$$\equiv\mathcal{F}(x,\ 11)\cdot\mathcal{F}(x,\ 19)=0.$$

The common factor of the two left members must be identical with $\mathcal{F}(x, 11)$. Then the numbers of improperly primitive classes of determinants -3, -11, -19 are one-sixth of the degrees of the expressions in brackets in the left members of the last two equations. P. Joubert's[52] modification of this method is given for determinants -15, -23, -31.

F. Arndt[53] wrote

$$\phi(x)=\prod_a(x-e^{2a\pi i/P}),\qquad \psi(x)=\prod_b(x-e^{2b\pi i/P})$$

and, in the three cases which Dirichlet had omitted (see Dirichlet,[28] (9)), obtained the following:

(II) $\quad D=P,\ P=4\mu+3,\ (T+U\sqrt{P})^h=\mp\psi(i)^4,$

where $\mp$ means $-$ or $+$ according to P is or is not prime;

$$\left.\begin{array}{ll}\text{(III)} & D=2P,\ P=4\mu+1,\ (T+U\sqrt{2P})^h=\psi(x)^4\cdot\phi(-x)^4 \\ \text{(IV)} & D=2P,\ P=4\mu+3,\ (T+U\sqrt{2P})^h=\psi(x)^4\cdot\psi(x^3)^4\end{array}\right\},\ x=\sqrt{\tfrac{1}{2}}(1+i).$$

[52] Cf. Joubert,[62] Comptes Rendus, Paris, 50, 1860, 911.
[53] Jour. für Math., 56, 1859, 100.

L. Kronecker[54] published without demonstration eight class-number recursion formulas derived from singular moduli in the theory of elliptic functions.[55] They are algebraically-arithmetically independent of each other; and any other formula of this type derived from an elliptic modular equation[49] is a linear combination of Kronecker's eight. He employed the following permanent[56] notations.

n is any positive integer; m any positive uneven integer; r any positive integer $8k-1$; $s=8k+1>0$.

$G(n)$ is the number of classes of determinant $-n$; $F(n)$ is the number of uneven classes.

$X(n)$ is the sum of the odd divisors of n; $\Phi(n)$ is the sum of all divisors.

$\Psi(n)$ is the sum of the divisors of n which are $>\sqrt{n}$ minus the sum of those which are $<\sqrt{n}$.

$\Phi'(n)$ is the sum of the divisors of the form $8k\pm1$ minus the sum of the divisors of the form $8k\pm3$.

$\Psi'(n)$ is the sum both of the divisors of the form $8k\pm1$ which are $>\sqrt{n}$ and of the divisors of the form $8k\pm3$ which are $<\sqrt{n}$ minus the sum both of the divisors of the form $8k\pm1$ which are $<\sqrt{n}$ and of the divisors of the form $8k\pm3$ which are $>\sqrt{n}$.

$\phi(n)$ is the number of divisors of n which are of the form $4k+1$ minus the number of those of the form $4k-1$.

$\psi(n)$ is the number of divisors of n which are of the form $3k+1$ minus the number of those of the form $3k-1$.

$\phi'(n)$ is half the number of solutions of $n=x^2+64y^2$; and $\psi'(n)$ is half the number of solutions of $n=x^2+3\cdot64y^2$, in which positive, negative, and zero values of x and y are counted for both equations.

$$\text{(I)}\quad F(4n)+2F(4n-1^2)+2F(4n-2^2)+2F(4n-3^2)+\dots =2X(n)+\Phi(n)+\Psi(n),$$

$$\text{(II)}\quad F(2m)+2F(2m-1^2)+2F(2m-2^2)+2F(2m-3^2)+\dots =2\Phi(m)+\phi(m),$$

$$\text{(III)}\quad F(2m)-2F(2m-1^2)+2F(2m-2^2)-2F(2m-3^2)+\dots=-\phi(m),$$

$$\text{(IV)}\quad 3G(m)+6G(m-1^2)+6G(m-2^2)+6G(m-3^2)+\dots =\Phi(m)+3\Psi(m)+3\phi(m)+2\psi(m),$$

$$\text{(V)}\quad 2F(m)+4F(m-1^2)+4F(m-2^2)+4F(m-3^2)+\dots =\Phi(m)+\Psi(m)+\phi(m),$$

$$\text{(VI)}\quad 2F(m)-4F(m-1^2)+4F(m-2^2)-4F(m-3^2)+\dots =(-1)^{\frac{m-1}{2}}[\Phi(m)-\Psi(m)]+\phi(m),$$

$$\text{(VII)}\quad 2F(r)-4F(r-4^2)+4F(r-8^2)-4F(r-12^2)+\dots =(-1)^{\frac{1}{8}(r-7)}[\Phi'(r)-\Psi'(r)],$$

$$\text{(VIII)}\quad 4\Sigma(-1)^{\frac{s-k^2}{16}}\left[2F\left(\frac{s-k^2}{16}\right)-3G\left(\frac{s-k^2}{16}\right)\right] =(-1)^{\frac{1}{8}(s-1)}[\Phi'(s)-\Psi'(s)]+\phi(s)+4\psi(s)-4\phi'(s)-8\psi'(s).$$

[54] Jour. für Math., 57, 1860, 248–255; Jour. de Math., (2), 5, 1860, 289–299.

[55] Demonstrated by the same method by H. J. S. Smith, Report Brit. Assoc., 1865, 349–359; Coll. Math. Papers, I, 325–37.[100]

[56] Later in the report of this paper will be noted the historical modification of Kronecker's F and G printed in Roman type.

In all recursion formulas (except those of G. Humbert[355]) of this chapter, the determinants are $\leqq 0$. In the above 8 formulas, $F(0)=0$, $G(0)=\frac{1}{2}$. The functions $\phi(n)$, $\psi(n)$, $\phi'(n)$, $\psi'(n)$ are removed hereafter from the formulas by replacing italic letters F and G throughout by Roman letters F and G, which agree respectively with the earlier symbols except that $\mathrm{F}(0)=0$, $\mathrm{G}(0)=-\frac{1}{12}$, and except that classes (1, 0, 1), (2, 1, 2) and classes derived from them are each counted as $\frac{1}{2}$ and $\frac{1}{3}$ of a class respectively. Later writers have commonly adopted these conventions but have not insisted on printing the symbols in Roman type.

The following also result[57] from the theory of elliptic functions:

$$\begin{aligned}&\mathrm{F}(4n)=F(4n), \text{ for all } n;\\&\mathrm{F}(4n)=2\mathrm{F}(n),\ \mathrm{G}(4n)=\mathrm{F}(4n)+\mathrm{G}(n), \text{ for all } n;\\&\mathrm{G}(n)=\mathrm{F}(n), \text{ if } n\equiv 1 \text{ or } 2 \pmod 4;\\&3\mathrm{G}(n)=[5-(-1)^{\frac{1}{4}(n-3)}]\mathrm{F}(n), \text{ if } n\equiv 3 \pmod 4.\end{aligned}$$

By means of these relations, Kronecker obtained from the original eight formulas the following[58]:

$$\text{(IX)}\quad \mathrm{F}(n)+\mathrm{F}(n-2)+\mathrm{F}(n-6)+\mathrm{F}(n-12)+\mathrm{F}(n-20)+\ldots=\tfrac{1}{3}\Psi(4n+1),$$

$$\text{(X)}\quad \mathrm{E}(n)+2\mathrm{E}(n-1)+2\mathrm{E}(n-4)+2\mathrm{E}(n-9)+\ldots=\tfrac{2}{3}[2+(-1)^n]X(n),$$

where $\mathrm{E}(n)=2\mathrm{F}(n)-\mathrm{G}(n)$. But

$$\Sigma(2\pm 1)X(n)q^n=\Sigma\frac{q^n}{(1\pm q^n)^2},\ n=1, 2, \ldots,$$

the plus or minus sign being taken on both sides according as n is even or odd. Hence formula (X) is equivalent to the important formula

$$\text{(XI)}\qquad 12\Sigma\mathrm{E}(n)q^n=\theta_3^3(q),\ \theta_3(q)=\sum_{n=-\infty}^{+\infty}q^{n^2},\ q=e^{\pi i\omega},$$

which implies that the number of representations[59] of n as the sum of three squares is $12\mathrm{E}(n)$. (Cf. this History, Vol. II, 265.)

By (VI) and (VII), Kronecker calculated $F(m)$ for m uneven from 1 to 10,000.

P. Joubert,[60] referring to a conjecture of Gauss,[61] proved that if n is a fixed prime and $\Delta>0$ grows through a range of values which are quadratic residues of n, then the number of classes in a genus of the forms of determinant $-\Delta$ has a lower limit for the range.

P. Joubert[62] considered the principal root ω of $P\omega^2+Q\omega+R=0$. If ω furnishes a root $\phi^2(\omega)$ of the modular equation for transformations of order 2^μ, μ arbitrary, he found that just two values of $\phi^2(\omega)$ are furnished as roots by all the forms (P, Q, R) of a given improperly primitive class which have third coefficients a

57 For the means of immediate arithmetical deduction, see Lipschitz[41] and H. J. S. Smith, Report Brit. Assoc., 1862, 514–519; Coll. Math. Papers, I, 246–51.

58 See H. J. S. Smith, Report Brit. Assoc., 1865, 348; Coll. Math. Papers, I, 323.

59 Cf. C. F. Gauss,[4] Disq. Arith., Arts 291–2. For a report, see this History, Vol. II, 262; while on pp. 263, 265, 269, are reports on papers by Dirichlet, Kronecker and Hermite giving applications of class-number to sums of three squares.

60 Comptes Rendus, Paris, 50, 1860, 832–837.

61 Disq. Arith.,[4] Art. 303.

62 Comptes Rendus, Paris, 50, 1860, 907–912.

multiple of 16. If (A, B, C) is a form of this kind and of negative determinant $-\Delta=(S^2-2^{\mu+2})/T^2$, in which S, T are odd, it is equivalent to $(2^\mu A, B, C/2^\mu)$, and these two forms give the same value of $\phi^2(\omega)$. Consequently, if in the ordinary modular equation we set $u^2=v^2=x$, the resulting equation $f(x)=0$ has a degree which is double the number of representative improperly primitive forms (A, B, C) of negative determinant $-\Delta$; and $f(x)$ can be decomposed into polynomial factors each of degree the double of the number of the improperly primitive classes of the corresponding determinant $-\Delta$.

For example, if $\mu=1$, the only possible determinant is -7. The modular equation for transformations of order 2 is $v^4=2u^2/(1+u^4)$, and becomes $x^2+x+2=0$. Therefore there is a single improperly primitive class of determinant -7. For somewhat larger values of determinant $-(8k-1)$, Hermite's[49] device is used for identifying common factors which belong to the same Δ and which occur in the left members of $f(x)=0$ for neighboring values of $n=2^\mu$.

In the modular equation $F(\lambda, \kappa)=0$ for transformations of odd prime order n, Joubert wrote $\lambda=2x/(1+x^2)$, $\kappa=x^2$, and obtained $f(x)=0$ in which $f(x)$ is a product of polynomials which have the same characteristic properties as in the former case. If ω is such that $\phi^2(\omega)=\sqrt{\kappa}$ is a root of $F(\lambda, \kappa)=0$, then ω is the principal root of an equation

$$A\omega^2+2B\omega+C=0,$$

where (A, B, C) is improperly primitive and the negative determinant $-\Delta$ has Δ equal to one of the numbers $8n-1^2$, $8n-3^2$, $8n-5^2$, ... Moreover, C is divisible by 16 and again there are therefore just two values of $\phi^2(\omega)$ for each improperly primitive class; and the roots $\phi^2(\omega)$ lead to forms (A, B, C) which just exhaust the classes of negative determinants $-(8n-\sigma^2)$. Hence the aggregate number of improperly primitive classes of the sequence of determinants is read off as in the following example. Let $n=3$; then $\Delta=23$, 15,

$$F(\lambda, \kappa)=\lambda^4-4\lambda^3(4\kappa^3-3\kappa)+6\lambda^2\kappa^2+4\lambda(3\kappa^3-4\kappa)+\kappa^4,$$
$$f(x)\equiv(x^4+4x^3+5x^2+2x+4)(x^6-x^5+9x^4+13x^3+18x^2+16x+8).$$

Since $15=2\cdot 8-1$, the first factor in $f(x)$ has already been associated with $\Delta=15$ by the use of $n=2^\mu=2$. The number of improperly primitive classes of determinant -23 may be read off as half the degree of the second factor and also as the index of its constant term regarded as a power of 2.

Joubert[63] illustrated his method by many examples.

Joubert,[64] in the modular equations for transformations of odd order $n=p^\alpha q^\beta r^\gamma$ (p, q, r different primes), and with the roots

$$v^8=\phi^8\left(\frac{g\omega+16m}{g_1}\right),$$

added to the usual conditions the restriction that g and g_1 be relatively prime. In the modular equation $f(x, y)=0$, he took $y=1/x$. Now $f(x, 1/y)$ is of degree

$$2N=2p^{\alpha-1}q^{\beta-1}r^{\gamma-1}(p+1)(q+1)(r+1)$$

[63] Comptes Rendus, Paris, 50, 1860, 940-4.
[64] *Ibid.*, 1040-1045.

and has $2\mathscr{N}+\phi\sqrt{n}$ or $2\mathscr{N}$ roots equal to unity, according as n is or is not a square, where

$$\mathscr{N}=\sum_{d}\sum_{\gamma}\gamma\cdot\phi(d),$$

d^2 ranging over the square divisors of n (n omitted if it is a perfect square), while $\gamma\gamma_1=n/d^2$, $\gamma<\gamma_1$, γ and γ_1 being relatively prime. Excluding the unity roots, he established a correspondence between the roots of $f(x, 1/x)=0$ and the roots of certain quadratic equations, and obtained the following formula when n is or is not a square respectively:

$$F(n)+2F(n-1^2)+2F(n-2^2)+2F(n-3^2)+\ldots=N-\mathscr{N} \text{ or } N-\mathscr{N}-\tfrac{1}{2}\phi(\sqrt{n}).$$

where $F(D)$ denotes the number of odd classes of determinant $-D$ which have all their divisors prime to n. If, however, a form is involved which is derived from (1, 0, 1), the right member in each case should be diminished by the number of proper decompositions of n into the sum of two squares. Numerous[65] other class-number relations in the modified F and a similarly modified G are obtained. Tables[66] verify the formulas in F. The interdependence of Joubert's and Kronecker's[54] class-number relations has been discussed by H. J. S. Smith.[67]

H. J. S. Smith[68] reproduced the principal parts of the researches of Gauss[4] and Dirichlet[19, 20, 28] on the class-number of binary quadratic forms. For $D>0$ and $\equiv 1 \pmod 4$, he wrote

$$h(D)=\left(\frac{2}{D}\right)\frac{2}{\log(T+U\sqrt{D})}\sum\left(\frac{D}{m}\right)\log\tan\frac{m\pi}{2D},$$

where m is positive, odd, prime to D, and $<D$. (Cf. Berger,[166] (3).)

C. Hermite[69] began with the factorization

$$\frac{H^2(z)\Theta_1(z)}{\Theta^2(z)}=\frac{H(z)\Theta_1(z)}{\Theta(z)}\cdot\frac{H(z)}{\Theta(z)}$$

and expanded each factor after C. G. J. Jacobi,[70] setting $z=2Kx/\pi$. In the product of the two expansions, the term independent of x is*

$$\mathscr{A}=\sum\frac{\sqrt{q^{2n+1}}}{1-q^{2n+1}}\cdot q^{\frac{1}{4}(2n+1)^2-a^2}\equiv\sum F(N)q^{N/4}, \tag{1}$$

where in the first sum, $a=0, \pm1, \pm2, \ldots \pm n$; while in the second sum, N ranges over all positive numbers $\equiv 3 \pmod 4$ which can be represented by (I), and hence by each of the three identically equal expressions

(I) $\quad (2n+1)(2n+4b+3)-4a^2,$

(II) $\quad (2n+1)(4n+4b+4-4a)-(2n+1-2a)^2,$

(III) $\quad (2n+1)(4n+4b+4+4a)-(2n+1+2a)^2.$

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* The expansion of the first fraction in (1) is $\sum q^k$, $k=\frac{1}{2}(2n+1)+(2n+1)b$, $b\geqq 0$.

[65] Comptes Rendus, Paris, 50, 1860, 1095–1100.

[66] *Ibid.*, 1147–1148.

[67] Report Brit. Assoc., 35, 1865, 364; Coll. Math. Papers, I, 343–4.

[68] Report Brit. Assoc., 1861, 324–340; Coll. Math. Papers, I, 1894, 163–228.

[69] Comptes Rendus, Paris, 53, 1861, 214–228; Jour. de Math., (2), 7, 1862, 25–44; Oeuvres, II, 1908, 109–124.

[70] Fundamenta Nova Funct. Ellipticarum, 1829, §§ 40–42; Werke, I, 1881, 159–170.

Thus $F(N)$ denotes the number of ways in which N can be represented by any one of the expressions (I), (II), (III). We represent N by (I), (II), or (III), according as

$$|a|<\tfrac{1}{4}(2n+1); \quad a \gtreqless \tfrac{1}{4}(2n+1); \quad a<0, \text{ but } |a| \gtreqless \tfrac{1}{4}(2n+1).$$

Now (I), (II), (III) are respectively the negatives of the determinants of the quadratic forms

$$(2n+1,\ 2a,\ 2n+4b+3),$$
$$(2n+1,\ 2n+1-2a,\ 4n+4b+4-4a),$$
$$(2n+1,\ 2n+1+2a,\ 4n+4b+4+4a).$$

Thus we have $F(N)$ forms which are reduced. Moreover, the $F(N)$ forms exhaust the reduced uneven forms of determinant $-N$. For, those of the first type constitute all uneven reduced forms of determinant $-N$ which have an even middle coefficient. Those of the second and third types constitute all forms (p, q, r) of determinant $-N$ in which p and q are uneven, $p>2q$, $r>2q>0$. Hence, since (p, q, r) is here never equivalent to $(p, -q, r)$, the number of forms of the three types together is $F(N)$, in the class-number sense.[54]

A second factorization yields[70]

$$(2) \qquad \frac{K}{2\pi}\sqrt{\frac{2kK}{\pi}}\cdot\frac{H^2(z)}{\Theta(z)}\cdot\frac{\Theta_1(z)}{\Theta(z)}$$
$$=\mathscr{A}\Theta_1(z)-\Sigma\cos 2nx\cdot q^{n^2}(\sqrt[4]{q^{-1}}+3\sqrt[4]{q^{-3}}+\ldots+(2n-1)\sqrt[4]{q^{-(2n-1)^2}}).$$

For $x=0$, the first member vanishes and the terms under the summation sign are of the type

$$q^{N/4}\cdot\tfrac{1}{2}(\Sigma d'-\Sigma d),$$

where $N\equiv 3 \pmod 4$, d' is any divisor $>\sqrt{N}$ of N and d is any divisor $<\sqrt{N}$. In Kronecker's[54] symbols, we get, by (1) and (2),

$$\Theta_1(0)\Sigma F(N)q^{N/4}=\tfrac{1}{2}\Sigma\Psi(N)q^{N/4}.$$

Or, since $\Theta_1(0)=1+2q+2q^4+2q^9+2q^{16}+\ldots$,

$$(3) \qquad F(N)+2F(N-2^2)+2F(N-4^2)+\ldots=\tfrac{1}{2}\Psi(N).$$

In Kronecker's[54] formulas this is (V) + (VI).

A third factorization combined with the first yields the following:

$$(4) \qquad F(4n-1)+F(4n-3^2)+F(4n-5^2)+\ldots=\Phi_1(n)-\Psi_1(n),$$

where $\Phi_1(n)$ denotes the sum of the divisors of n whose conjugates are odd, and $\Psi_1(n)$ denotes the sum of all the divisors $<\sqrt{n}$ and of different parity from their conjugates. Similarly,

$$(5) \quad F(N)-2F(N-2^2)+2F(N-4^2)-\ldots+2(-1)^kF(N-4k^2)\ldots$$
$$=(-1)^{\frac{1}{4}(N-3)}\Psi_2(N)=(-1)^{\frac{1}{4}(N-3)}\cdot\tfrac{1}{2}(\Phi(N)-\Psi(N)),\ N\equiv 3 \pmod 4,$$

where $\Psi_2(n)$ denotes the sum of the divisors of n which are $<\sqrt{n}$. Hermite's three class-number relations above are all derivable from Kronecker's[71] eight.

[71] See H. J. S. Smith, Rep. Brit. Assoc., 35, 1865, 364; Coll. Math. Papers, I, 1894, 343.

Since N in (1) is of the form $4n+3$, (1) implies

$$(6) \qquad \tfrac{1}{2}(\mathscr{A}-\epsilon\mathscr{A}_1)=\sum_0^\infty F(8n+3)q^{\frac{1}{4}(8n+3)},$$

where $\epsilon=(1+i)/\sqrt{2}$, $\epsilon^4=-1$, and $\mathscr{A}_1$ is the result of replacing q by $-q$ in $\mathscr{A}$. Another expression for $\frac{1}{2}(\mathscr{A}-\epsilon\mathscr{A}_1)$ is found by means of the integral of the product quoted at the beginning of this report; comparison of it with (6) gives

$$(7) \qquad \Sigma F(8n+3)q^{\frac{1}{4}(8n+3)}=(\sqrt[4]{q}+\sqrt[4]{q^9}+\sqrt[4]{q^{25}}+\dots)^3.$$

This result is implicitly included[72] in Kronecker's[54] (XI) and can be deduced from it by elementary algebra.[73] When the coefficients of equal powers of q are equated in the two members, this formula implies that the number of odd classes of determinant $-(8n+3)$ is the number of positive solutions of

$$8n+3=x^2+y^2+z^2.$$

L. Kronecker[74] referring to his[54] earlier memoir, multiplied formulas (I), (II), (V) respectively by q^{4n}, q^{2m}, $\frac{1}{2}q^m$, added the results, and summed for all values of n and m, and obtained

$$(1) \qquad \Sigma F(n)q^n=\tfrac{1}{2}\sqrt{\frac{\pi}{2K}}\,\Sigma\frac{n}{q^n-q^{-n}}(q^{n^2+n}-2+q^{n^2-n}).$$

Similarly from formulas (I), (III), (VI), he obtained

$$(2) \qquad \Sigma F(n)q^n=\tfrac{1}{2}\sqrt{\frac{\pi}{2k'K}}\,\Sigma n(-q)^{n^2}\cdot\frac{q^n-q^{-n}}{q^n+q^{-n}}.$$

Now (1) and (2) imply the following three formulas[75]:

$$\Sigma F(2m)q^{\frac{1}{2}m}=\frac{kK}{\pi}\cdot\sqrt{\frac{K}{2\pi}}, \qquad \Sigma F(4n+1)q^n=\frac{1}{q^{\frac{1}{4}}}\,\frac{K}{\pi}\cdot\sqrt{\frac{kK}{2\pi}},$$

$$\Sigma F(8n+3)q^{2n}=\frac{1}{q^{\frac{3}{4}}}\cdot\frac{kK}{2\pi}\sqrt{\frac{kK}{2\pi}};$$

and these imply Kronecker's[54] (IV).

By means of an expansion[76] of $\sin^2 \operatorname{am} 2Kx/\pi$ in terms of cosines of multiples of x, (1) takes the form

$$(3) \qquad \Sigma F(n)q^n=\frac{1}{q^{\frac{1}{4}}}\cdot\frac{k^2K}{2\pi^2}\sqrt{\frac{K}{2\pi}}\int_0^\pi \sin^2\operatorname{am}\frac{2Kx}{\pi}\cdot\theta_2(x)\cos x\,dx.$$

"From (3), all the formulas[54] (I)–(VIII) can be deduced." Other such relations are indicated by means of theta-functions, although the eight formulas "are algebraically-arithmetically independent."

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[72] Jour. für Math., 57, 1860, 253.
[73] Cf. L. J. Mordell, Messenger Math., 45, 1915, 79.
[74] Monatsber. Akad. Wiss. Berlin, 1862, 302–311. French transl., Annales Sc. Ecole Norm. Sup., 3, 1866, 287–294.
[75] Cf. C. Hermite,[69] Comptes Rendus, Paris, 53, 1861, 226.
[76] Cf. C. G. J. Jacobi, Fundamenta Nova, 1829, 110, (1), Werke, I, 1881, 166.

Kronecker stated that he had obtained arithmetical deductions of certain of his class-number relations by following the plan of Jacobi[77] who had first found by equating coefficients in two expansions, the number of expressions for n as the sum of four squares and had later translated the analytic method into an arithmetical one.[78] The following theorem, which Kronecker deduced from his formula (V), was offered as a suggestion for a means of deducing his class-number relations arithmetically: Let p be any odd prime and let

$$a_1z^2+2b_1z+c_1\equiv 0,\qquad a_2z^2+2b_2z+c_2\equiv 0\ldots \pmod p$$

be a succession of congruences corresponding to reduced forms of determinants, $-p$, $-(p-1^2)$, $-(p-2^2)$, ... respectively (with b taken negative in the reduced form if $a=c$); then the number of roots of the congruences is

$$F(p)+2F(p-1^2)+2F(p-2^2)+2F(p-3^2)+\ldots;$$

that is to say, by formula (V), the number is $p+1$ or p according as p is $\equiv 1$ or 3 (mod 4).

H. J. S. Smith[79] gave an account of Lipschitz's[41] method of obtaining the ratio of $h(D\cdot S^2)$ to $h(D)$.

C. Hermite[80] gave a list of expansions of quotients obtained from theta-functions and showed how the products and quotients of theta-functions lead to class-number relations (cf. Hermite[69]). This list of doubly periodic functions of the third kind has been extended by C. Biehler,[81] P. Appell,[81a] Petr,[252, 258] Humbert,[298] and E. T. Bell.[82] Finally, Hermite deduced Kronecker's[54] relation (XI).

Hermite[83] generalized a theorem of Legendre (this History, Vol. I, 115, (5)) into the *Lemma*: If $m=a^\alpha b^\beta c^\gamma\ldots k^\kappa$, where $a, b, c, \ldots, k$ are μ different primes, then the number of integers which are less than or equal to x and relatively prime to m is

$$\Phi(x)=E(x)-\Sigma E\left(\frac{x}{a}\right)+\Sigma E\left(\frac{x}{ab}\right)-\Sigma E\left(\frac{x}{abc}\right)+\ldots\pm E\left(\frac{x}{abc\ldots k}\right),$$

with the convention $\Phi(x)=E(x)$ if $m=1$. It follows that

$$\Phi(x)=\frac{x}{m}\phi(m)+2^{\mu-1}\epsilon,\qquad -1<\epsilon<+1. \tag{1}$$

Now $F(n)$ is defined by $F(n)=\Sigma_{i=1}^{i=n}f(i)$, where $f(i)=0$ if i is not a divisor of n or if i is a divisor of n but is not prime to m; also $F(n)=0$, if m and n are not relatively prime. Then, by definition,

$$\sum_{k=1}^{m}F(k)=\sum_{i=1}^{n}f(i)\Phi\left(\frac{n}{i}\right).$$

[77] Fundamenta Nova, 1829, Art. 66; Werke, I, 1881, 239.
[78] Jour für Math., 12, 1834, 167-172; Werke, VI, 1891, 245-251.
[79] Report Brit. Assoc., 1862, § 113; Coll. Math. Papers, I, 1904, 246-9.
[80] Comptes Rendus, Paris, 55, 1862, 11, 85; Jour. de Math., (2), 9, 1864, 145-159; Oeuvres, II, 1908, 241-254.
[81] Thesis, Paris, 1879.
[81a] Annales de l'École Normale, (3), 1, 1884, 135-164; 2, 1885, 9-36.
[82] Messenger Math., 49, 1919, 84.
[83] Comptes Rendus, Paris, 55, 1862, 684-692. Oeuvres, II, 1908, 255-263.

Now (cf. Dirichlet,[93] (1)), if $D=S^2D_0$, where D_0 is a fundamental determinant, and if n is any positive uneven integer relatively prime to D, then for $f(i)=(D_0/i)$ and D uneven, for example, the formula

$$(2)\qquad F(n)=k\sum_{i=1}^{n}\left(\frac{D_0}{i}\right)\Phi\left(\frac{n}{i}\right),\quad k=2 \text{ if } D<-3,\ k=1 \text{ if } D>0,\ m=2|D|,$$

gives the sum of the number of representations of integers from 1 to n which are uneven and relatively prime to D by the representative properly primitive forms of determinant D with the usual restriction[84] on x and y in case $D>0$.

Hermite omits the rather difficult proof that the term containing ϵ in (1) is negligible[85] for n very great and concludes from (1) and (2) that, for n very great,

$$F(n)=k\sum_{i=1}^{n}\left(\frac{D_0}{i}\right)\frac{n}{2i(-1)^{k+1}D}\,\phi((-1)^{k+1}\cdot 2D).$$

C. F. Gauss[86] and G. L. Dirichlet[87] had found geometrically the asymptotic mean number of such representations furnished by each form for n large. A comparison yields the class-number (Dirichlet,[19] (1)).

J. Liouville[88] stated that the number[89] of solutions of $yz+zx+xy=n$ in positive odd integers with $y+z\equiv 2 \pmod 4$, $n\equiv 3 \pmod 4$, is $F(n)$.

J. Liouville[90] obtained an arithmetical deduction of a Kronecker[54] recursion formula in the form

$$F(2m-1^2)+F(2m-3^2)+F(2m-5^2)+\ldots=\tfrac{1}{2}[\zeta_1(m)+\rho(m)],$$

where m is an arbitrary uneven integer, $\zeta_1(m)$ represents the sum of the divisors of m, and $\rho(m)$ is the excess of the number of divisors of m which are $\equiv 1 \pmod 4$ over the number of divisors $\equiv 3 \pmod 4$.

Lemma 1. Let any uneven integer m be subjected to the two types of partitions

$$(1)\qquad m=2m'^2+d''\delta'',\qquad 2m=m_1^2+d_2\delta_2+2^{\alpha_3+1}d_3\delta_3,$$

where m_1, d_2, d_3, δ_2, δ_3 are positive uneven integers; $\alpha_3>0$; while m' is any positive, negative, or zero integer. Then, if $f(x)$ is an even function,

$$(2)\qquad \begin{aligned}&\tfrac{1}{2}\Sigma[d''f(2m')-f(2m')-2f(2m'+2)-2f(2m'+4)-\ldots-2f(2m'+\delta''-1)]\\ &\qquad=\Sigma\left[f\left(\frac{d_2+\delta_2}{2}-d_3\right)-f\left(\frac{d_2+\delta_2}{2}+d_3\right)\right].\end{aligned}$$

Now take $f(x)$ so that $f(0)=1$, $f(x)=0$ if $x\neq 0$. Then the only partitions of the second type (1) which furnish terms in the right member of (2) are those in which $d_3=\frac{1}{2}(d_2+\delta_2)$. Hence the right member of (2) has for its value the number of solutions of

$$2m-m_1^2=d_2\delta_2+2^{\alpha_3}(d_2+\delta_2)\delta_3.$$

[84] G. L. Dirichlet,[19] Zahlentheorie, Art. 90, ed. 4, 1894, 225 and 226.
[85] Cf. T. Pepin, Annales Sc. de l'Ecole Norm. Sup., (2), 3, 1874, 165; M. Lerch, Acta Math., 29, 1905, 360.
[86] Werke, II, 1876, 281 (Gauss[4]).
[87] Jour. für Math.,[19] 19, 1839, 360 and 364.
[88] Jour. de Math., (2), 7, 1862, 44.
[89] Cf. Bell,[870] and Mordell.[872]
[90] Jour. de Math., (2), 7, 1862, 44–48.

We set $d_2+\delta_2=2u$, $d_2-\delta_2=4z$. Hence $u>2z$. Keeping m_1 fixed, Liouville followed the method of Hermite[69] and obtained the result that the number of solutions of $2m-m_1^2=u(u+4d)-4z^2$ is

$$\sum_{m_1} F(2m-m_1^2)-\tfrac{1}{2}\sum_{m_1}\zeta(2m-m_1^2)-\tfrac{1}{2}\sum_{s_1}\omega(2m-s_1^2),$$

in which $\zeta(n)$ denotes the number of divisors of n, $\omega(n)=1$ or 0 according as n is or not a perfect square. Hence $\Sigma\omega(2m-s_1^2)=\rho(m)$.

Now in the first member of (2), the summation of the first two terms in the bracket is equal to $\zeta_1(m)-\zeta(m)$. Furthermore the expression in (2):

$$f(2m'+2)+f(2m'+4)+\ldots+f(2m'+\delta''-1)$$

will have the value 1 for each pair of values $m'<0$, $2m'+\delta''>0$ and the value 0 for all other values m' and $2m'+\delta''$. Let A denote the number of pairs of values $m'<0$, $2m'+\delta''>0$ in the partition (1_1). We have now proved that

$$(3)\qquad \sum_{m_1} F(2m-m_1^2)-\tfrac{1}{2}\sum_{m_1}\zeta(2m-m_1^2)-\tfrac{1}{2}\rho(m)=\tfrac{1}{2}\{\zeta_1(m)-\zeta(m)\}-A,$$

Lemma 2. Let any uneven integer M be subjected to the two types of partitions

$$M=2M'^2+D''\Delta'',\qquad 2M=M_1^2+D_2\Delta_2,$$

where M_1, D_2, Δ_2, D'', Δ'' are positive odd integers, while M' is any integer. Then, if $f_1(x)$ is an uneven function,

$$(4)\qquad \Sigma f_1(D''+2M')=\Sigma f_1\left(\frac{D_2+\Delta_2}{2}\right).$$

To evaluate A, we identify m and M and specialize $f_1(x)$ so that $f_1(x)=1$ if $x>0$, $f_1(x)=0$ if $x=0$, $f_1(x)=-1$ if $x<0$. Since the number of solutions of $M=2M'^2+D''\Delta''$ with $M'>0$ is equal to the number with $M'<0$, the left member of (4) is composed of the following four parts:

$$\begin{aligned}
A&=\Sigma f_1(D''+2M'), & M'&<0, & D''+2M'&>0;\\
-B&=\Sigma f_1(D''+2M'), & M'&<0, & D''+2M'&<0;\\
A+B&=\Sigma f_1(D''+2M'), & M'&>0, & D''+2M'&>0;\\
\zeta(m)&=\Sigma f_1(D''+2M'), & M'&=0, & D''+2M'&>0.
\end{aligned}$$

Hence (4) implies that

$$2A+\zeta(m)=\Sigma f_1\left(\frac{D_2+\Delta_2}{2}\right)=\Sigma\zeta(2m-m_1^2).$$

Thus (3) becomes[91]

$$\sum_{m_1} F(2m-m_1^2)=\tfrac{1}{2}[\zeta_1(m)+\rho(m)].$$

This result has been established in detail by Bachmann[91] and Meissner.[292] From the same two lemmas, H. J. S. Smith[92] obtains a different form of the right member, for the case m odd.

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[91] Cf. P. Bachmann, Niedere Zahlentheorie, Leipzig, II, 1910, 423–433.
[92] Report Brit. Assoc., 35, 1865, 366; Coll. Math. Papers, I, 1894, 346–350.

Hermite's discovery[69] of the relation between the number of classes of determinant N and the number of certain decompositions of N, also enabled Liouville to announce that formulas exist analogous to those of Kronecker,[54] but in which the successive negative determinants are respectively $2s^2-n$, $3s^2-n$, $4s^2-n$, ..., where n is fixed and s has a sequence of values.[92a]

G. L. Dirichlet[93] reproduced in a text-book the theory of his memoirs[14, 19, 20, 28] of 1838, 1839, 1840. Continuing his former notation, he obtained (Arts. 105–110) new expressions for

$$N=-\frac{i^{\left(\frac{P-1}{2}\right)^2}}{8\sqrt{P}}\sum_r j^r(1+\delta i^{3r})[1+\epsilon(-1)^r]\int_0^1\sum_s\left(\frac{s}{P}\right)\frac{dx}{x-j^r\theta^s},$$

$j=e^{\pi i/4}$, $\theta=e^{2\pi i/P}$, while s ranges over a complete set of incongruent numbers (mod P) prime to P. The result[94] is, for $D>0$, $D\equiv 1 \pmod 4$, for example,

$$N\cdot 2\sqrt{P}=-\left\{1-\left(\frac{2}{P}\right)\tfrac{1}{2}\right\}\log\{F(1)^2\},\qquad F(x)=\Pi(x-\theta^s)^{(s/P)}.$$

Thence in the notation of the Pellian equation, for example,

$$(1)\quad D=P\equiv 1 \pmod 8,\quad (T+U\sqrt{P})^{h(D)}=(t+u\sqrt{D})^l,\qquad l=\left[2-\left(\frac{2}{P}\right)\right](2-\kappa),$$

where $\kappa=1$ or 0 according as P is prime or composite, and t, u are positive integers satisfying $t^2-Du^2=1$. From five such relations, Dirichlet points out divisibility properties of $h(D)$; e. g., if $D\equiv 1 \pmod 4$, $h(D)$ is odd or even according as P is prime or composite.

Incidentally (Art. 91), Dirichlet proved that the number of representations of a number σn by a system of primitive forms of determinant D is

$$(2)\qquad \tau\Sigma(D/\delta)$$

where $\sigma=1$ or 2 according as the forms are proper or improper, n is prime to $2D$, and δ ranges over the divisors of n.

This formula has been used by Hermite,[83] Pepin,[120] Poincaré[271] to evaluate the class-number.

V. Schemmel[95] denoted by p an arbitrary positive odd number which has no square divisors. By the use of Gauss sums he set up such identities as the following, when $p=4n+3$:

$$(1)\qquad \sum_1^{p-1}\left(\frac{m}{p}\right)\sin ma=\frac{1}{2\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\frac{\sin pa\sin 2m\pi/p}{\cos 2m\pi/p-\cos a},$$

where a is an arbitrary real number. He took $a=\pi/2$ in both members, then

$$A-B-C+D=-\frac{1}{2\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\tan\frac{2m\pi}{p},$$

[92a] Cf. Liouville,[107, 109] Gierster[145], Stieltjes,[154, 162] Hurwitz,[167, 184] Petr[258], Humbert[293], Chapelon[340].

[93] Vorlesungen über Zahlentheorie, Braunchweig, 1863, 1871, 1879, 1894, Ch. V.

[94] Cf. G. L. Dirichlet,[28] Jour. für Math., 21, 1840, 154; Werke, I, 1889, 495; Arndt.[53]

[95] De multitudine formarum secundi gradus disquisitiones, Diss., Breslau, 1863, 19 pp.

where A, B, C, D are the number of positive quadratic residues which are $<p$ and of the respective forms $4n+1$, $4n+2$, $4n+3$, $4n+4$. Whence,[96]

$$(2) \qquad h(-p)=-\frac{1}{2\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\tan\frac{2m\pi}{p}.$$

After differentiating both members of (1) with respect to a, he took $a=0$. The result is[28]

$$\sum_1^{p-1}\left(\frac{m}{p}\right)m=-\frac{p}{2\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\cot\frac{m\pi}{p},$$

whence follows Lebesgue's[36] class-number formula (1):

$$(3) \qquad p=4n+3, \qquad h(-p)=\frac{2-\left(\frac{2}{p}\right)}{2\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\cot\frac{m\pi}{p}.$$

Similarly to (3) are obtained

$$(4) \qquad p=4n+1, \qquad h(-p)=\frac{1}{2\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\sec\frac{2m\pi}{p};$$

$$p=4n+3, \qquad h(-2p)=-\frac{1}{\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\frac{\sin 2m\pi/p}{\cos 4m\pi/p};$$

$$p=4n+1, \qquad h(-2p)=\frac{1}{\sqrt{p}}\sum_1^{p-1}\left(\frac{m}{p}\right)\frac{\cos 2m\pi/p}{\cos 4m\pi/p}.$$

Schemmel, without discussing convergence, decomposed an infinite series by the identity

$$\sum_n\left(\frac{n}{p}\right)\cos na=\lim_{\kappa=\infty}\sum_1^{p-1}\left(\frac{m}{p}\right)\{\cos ma+\cos(p+m)a+\ldots+(\kappa p+m)a\},$$

where $p\equiv 3 \pmod 4$, and n is positive and relatively prime to p. After transforming the right member, he integrated both members between the limits 0 and $\frac{1}{2}\pi$, with Dirichlet's[28] formula (8_2) as the final result:

$$(5) \qquad h(p)=\frac{2}{\log(T+U\sqrt{p})}\log\Pi\frac{\cos(b\pi/p+\pi/4)}{\cos(a\pi/p+\pi/4)}.$$

Employing the usual cyclotomic notation,[58]

$$\psi(x)=\Pi(x-e^{2a\pi i/p}), \qquad \phi(x)=\Pi(x-e^{2b\pi i/p}),$$

Schemmel found that, for $p=4n+3>0$,

$$(6) \qquad \frac{\psi'(1)}{\psi(1)}-\frac{\phi'(1)}{\phi(1)}\equiv\Sigma\frac{1}{1-e^{2a\pi i/p}}-\Sigma\frac{1}{1-e^{2b\pi i/p}}=\frac{i}{2}\sum_1^{p-1}\left(\frac{m}{p}\right)\cot\frac{m\pi}{p},$$

which by (3) gives a new class-number formula for $-p$ (see H. Holden[280]). He noted that, for $p=4n+3>0$,

$$\frac{\phi(i)}{\psi(i)}=\Pi\frac{\cos\left(\frac{b\pi}{p}+\frac{\pi}{4}\right)e^{\Sigma b\pi/p}}{\cos\left(\frac{a\pi}{p}+\frac{\pi}{4}\right)e^{\Sigma a\pi/p}}; \quad e^{(\Sigma b-\Sigma a)\pi/p}=1 \text{ or } -1,$$

[96] G. L. Dirichlet,[28] Jour. für Math., 21, 1840, 152; Zahlentheorie, Art. 104, ed. 4, 1894, 264.

according as p is composite or prime. Hence by (5), if we set

$$F(x)=\log \pm \frac{\phi(x)}{\psi(x)},$$

we have, for $p=4n+3>0$,

$$h(p)=\frac{2}{\log(T+U\sqrt{p})}F(i).$$

Similarly for $p=4n+3>0$,

$$\frac{\psi'(i)}{\psi(i)}-\frac{\phi'(i)}{\phi(i)}=-\frac{1}{2}\sum_{1}^{p-1}\left(\frac{m}{p}\right)\tan\frac{2m\pi}{p}, \qquad h(-p)=\frac{1}{\sqrt{p}}F'(i).$$

Moreover, if $p=4n+3>0$,

$$h(2p)=\frac{2}{\log(T+U\sqrt{2p})}[F(\omega^3)-F(-\omega^3)];$$

$$h(-2p)=-\frac{i+1}{2\sqrt{p}}[F'(\omega^3)+F'(-\omega^3)],$$

and, if $p=4n+1$,

$$h(2p)=\frac{4}{\log(T+U\sqrt{2p})}[F(\omega)-F(-\omega)],\quad h(-2p)=\frac{i-1}{2\sqrt{p}}[F'(\omega)-F'(-\omega)],$$

where, in the last four formulas, $\omega=(1+i)/\sqrt{2}$.

L. Kronecker[97] obtained, more simply than had G. L. Dirichlet,[98] the fundamental equation (2) of Dirichlet,[20] and specialized it in the form

$$(1) \qquad \tau\Sigma\left(\frac{D}{n}\right)(nn')^{-1-\rho}=\sum_{a,b,c}\sum_{x,y}(ax^2+2bxy+cy^2)^{-1-\rho}.$$

For a particular (a, b, c), the sum

$$\sum_{x=s}^{\infty}(ax^2+2bxy+cy^2)^{-1-\rho}\equiv\sum_{x=s}^{\infty}\phi(x,y)$$

lies between the two values

$$\pm\phi(hy,y)+\int_{hy}^{\infty}\phi(x,y)\,dx,$$

if $hy<s<hy+1$. Hence

$$\lim_{\rho=0}\rho\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\phi(x,y)=\lim_{\rho=0}\rho\sum_{y=1}^{\infty}\int_{hy}^{\infty}\phi(x,y)\,dx,$$

where h is taken so that $ah^2+2bh+c\neq 0$. When we set $ax+by=zy$, this limit is given by

$$\lim_{\rho=0}\tfrac{1}{2}\int_{ah+b}^{\infty}\frac{dz}{z^2-D}=\frac{1}{4\sqrt{D}}\log\frac{t+u\sqrt{D}}{t-u\sqrt{D}},\quad ah+b=\frac{t}{u},\ D>0$$

$$=\frac{\pi}{\sqrt{-D}} \qquad\qquad , D<0$$

Hence, when we exclude[99] from the final sum (1) those terms for which the form takes values not prime to P, (1) implies, for $\rho=0$,

$$\rho\Sigma\left(\frac{D}{n}\right)(nn')^{-1-\rho}=\tfrac{1}{2}\cdot\frac{1}{\sqrt{D}}\log(t_1+u_1\sqrt{D})\cdot h(D)\Pi\left(1-\frac{1}{p}\right)\left[1-\left(\frac{D}{p}\right)\frac{1}{p}\right],$$

[97] Monatsber. Akad. Wiss. Berlin, 1864, 285–295.
[98] Jour. für Math., 21, 1840, 7; Werke, I, 1889, 467.
[99] Cf. R. Dedekind, Remarks on Gauss' Untersuchungen über höhere Arithmetik, Berlin, 1889, 685–686; Gauss' Werke, II, 293–4.

where p ranges over the distinct prime divisions of P, and t_1, u_1 are fundamental solutions of $t^2-Du^2=1$. For $D>0$, this proves that $h(D)$ is finite, since the left member is a definite number.

H. J. S. Smith[100] discussed the researches of Kronecker,[54, 74] Hermite,[49, 69, 80] Joubert,[62, 64] and Liouville[90] in class-number relations. He found proofs of Kronecker's class-number relations[64] by means of the complex multiplication of elliptic functions. The details are based on the methods used by Joubert and Hermite. L. Kronecker[101] has commended the report for its mastery and insight.

For instance, formula (V) of Kronecker is proved by putting $x=\kappa^2$ and $1-x=\lambda^2$ in the ordinary modular equation $f_8(\kappa^2, \lambda^2)=0$ for transformations of uneven order m. The right member of the desired formula is found as the order of the infinity of $f_8(x, 1-x)$ as x increases without limit. The left member is the aggregate multiplicity of the roots of $f_8(x, 1-x)=0$.

R. Lipschitz[102] developed a general theory of asymptotic expansions for number-theoretic functions and found that, in the special case of the number of properly primitive classes, the asymptotic expression is

$$h(-m)=\frac{2\pi}{7\Sigma s^{-3}}m^{\frac{3}{2}},\quad s=1, 2, 3, \ldots;\ m>0.$$

This agrees with C. F. Gauss[103] since

$$\frac{7}{2}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)=4\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\ldots\right).$$

And asymptotically,

$$h(m)=\frac{24\log 2}{7\Sigma s^{-3}}m^{\frac{3}{2}},\quad s=1, 2, 3, \ldots;\ m>0.$$

The method of Lipschitz is illustrated by C. Hermite.[104]

J. Liouville[105] stated without proof that if a and a' denote respectively the [odd] minimum and second [odd] minimum of the forms of a properly primitive class of determinant $-k=-(8n+3)<0$, then

$$\sum_{Cl} a(a'-a)=\tfrac{2}{3}k\cdot h(-k).$$

He discussed as examples the cases $k=3, 11, 19, 27$. The theorem has been proved arithmetically by Humbert.[298]

Liouville[106] let m be an arbitrary number of the form $8n+3$, whence the only reduced ambiguous forms of negative determinant $-(m-4\sigma^2)$ are $(d, 0, \delta)$, where $d\delta=m+4\sigma^2$ and $d\leqq\sqrt{m-4\sigma^2}$. Hence the d's are the values of the minima of the uneven ambiguous classes of determinant $-(m-4\sigma^2)$. And hence, if n_1 [and n_2] denotes the number of ambiguous classes of determinant $-m$ whose minima are $\equiv 1$

100 Report Brit. Assoc., 35, 1865, 322–375; Collected Papers, I, 1894, 289–358.
101 Sitzungsber. Akad. Wiss. Berlin, 1875, 234.
102 Sitzungsber. Akad. Berlin, 1865, 174–185. Reproduced by P. Bachmann, Zahlentheorie, Leipzig, II, 1894, 438–459.
103 Werke, II, 1876, 284; Untersuchungen,[9] Berlin, 1889, 670.
104 Bull. des Sc. Math.,[204] (2), 10, I, 1886, 29; Oeuvres, IV, 220–222.
105 Jour. de Math. (2), 11, 1866, 191–192.
106 Jour. de Math. (2), 11, 1866, 221–224.

(mod 4) [and $\equiv 3$ (mod 4)], and if p_1 [and p_2] denotes the number of uneven ambiguous classes of determinants $-(m-4\sigma^2)$ excluding $\sigma=0$, whose minima are $\equiv 1$ (mod 4) [and $\equiv 3$ (mod 4)], then in the notation of this History, Vol. II, p. 265 (Liouville[33a]),

$$n_1-n_2+2(p_1-p_2)=\rho'(m)+2\rho'(m-4\cdot 1^2)+2\rho'(m-4\cdot 2^2)+\dots.$$

By the theorem there stated, it follows from Hermite[69] that

$$F(m)=n_1-n_2+2(p_1-p_2).$$

Liouville[107] stated that he had obtained the following results arithmetically. He generalized Hermite's[69] formula (4) both to

$$\sum_i F(2^{\alpha+2}m-i^2)=2^\alpha\Sigma d-\Sigma D,\quad i>0, \tag{1}$$

in which i and m are odd; and to

$$\sum_i i^2F(2^{\alpha+2}m-i^2)=2^\alpha m(2^\alpha\Sigma d-\Sigma D)-\Sigma D^3,\quad i>0 \tag{2}$$

where α is an integer $\gtreqless 0$, d denotes a divisor of m; and D is a divisor of $2^\alpha m$ which is of opposite party to its conjugate divisor. By the nature of their second members, these formulas represent what Humbert[293] has called the second type of Liouville's formulas.

For $m=d\delta=12g+7$ or $12g+11$, he gave

$$\sum_i F(2m-3i^2)=\frac{1}{8}\left[3+\left(\frac{m}{3}\right)\right]\sum_d\left(\frac{3}{\delta}\right)d, \tag{3}$$

where $i=1, 3, 5\dots.$ He stated that if m denotes an odd positive number prime to 5; and α, β are given positive numbers or zero, and $m=d\delta$, then

$$\sum_i F(8\cdot 2^\alpha 5^\beta m-5i^2)=2^{\alpha-2}\left[5^{\beta+1}-(-1)^\alpha\left(\frac{m}{5}\right)\right]\sum_d\left(\frac{5}{\delta}\right)d, \tag{4}$$

where $i=1, 3, 5, \dots, m=d\delta$. A special case of this relation is proved by Chapelon[340] as his formula (3) below.

If m is a positive integer of the form $24g+11$, then

$$F(m)+2\Sigma F(m-48\cdot s^2)=\tfrac{1}{4}\sum_d\left(\frac{3}{\delta}\right)d,\quad s>0.$$

Finally, if $m=4g+3$ and $t=g-s$, then

$$\sum_{s=0}^{g}(8s+3)F(8t+3)=\tfrac{1}{8}\Sigma(-1)^{(\delta-1)/2}d^2.$$

The right members here characterize what Humbert has called the first type of Liouville's formulas. G. Humbert[108] has deduced formulas of this type, by C. Hermite's method, from elliptic function theory.

[107] Comptes Rendus, Paris, 62, 1866, 1350; Jour. de Math., (2), 12, 1867, 98–103.
[108] Jour. de Math.,[293] (6), 3, 1907, 366–368, 446–447.

Liouville[109] by replacing n by $3m$ in Hermite's[69] formula (4), decomposed it into two class-number relations

$$\Sigma' F(12m-i^2)=\zeta_1(3m)-\zeta_1(m), \qquad i\not\equiv 0 \pmod 3,\ i\equiv 0 \pmod 2$$
$$\Sigma F(12m-9i^2)=\zeta_1(m), \qquad i\equiv 0 \pmod 3,$$

where $\zeta_1(n)$ is the sum of the divisors of n; and m is odd.

Liouville[110] announced without proof the relation[111]

$$\Sigma\left(\frac{-1}{i}\right)iF(4m-i^2)=\Sigma(a^2-4b^2),$$

where $i=1, 3, 5, 7, \ldots$; a is positive and uneven; and a, b range over the integral solutions of $m=a^2+4b^2$; m odd.

Stieltjes[160] and G. Humbert[112] have each given a proof by Hermite's method of equating coefficients in expansions of doubly periodic functions of the third kind.

Liouville[113] stated for $m\equiv 5 \pmod{12}$ that

$$\Sigma F\left(\frac{2m-i^2}{3}\right)=\tfrac{1}{4}\Sigma\left(\frac{3}{\delta}\right)d, \tag{1}$$

where $i=1, 5, 7, 11, 13, 17, \ldots$ is relatively prime to 6; $m=\delta d$. For[114] m odd and relatively prime to 5,

$$F(10m)+2\Sigma F(10m-25t^2)=2\Phi(m),$$

where $t=1, 2, 3, \ldots$; $\Phi(m)$ denotes the sum of the divisors of m.

R. Dedekind,[115] by the composition of classes, solved completely the Gauss[4] problem, obtaining the results of Dirichlet.[20]

R. Götting,[116] to evaluate Dirichlet's[14] formula (4) for $h(-p)$, p a prime of the form $4n+3$, proved that

$$\sum_{a=0}^{\frac{1}{2}p}\left(\frac{a}{p}\right)=-\frac{p-1}{2}+2\sum_{j=0}^{\frac{1}{4}(p-3)}\sigma_j-2\sum_{j=1}^{\frac{1}{4}(p-3)}\rho_j,$$
$$\sum_{j=0}^{\frac{1}{4}(p-3)}\sigma_j=\frac{1}{p}\sum_{a=0}^{\frac{1}{2}p}\left(\frac{a}{p}\right)a+\frac{p^2+1}{12},$$

where

$$\sigma_j=\left[\sqrt{pj+\frac{p}{2}}\right], \qquad \rho_j=[\sqrt{pj}].$$

Hence if $\quad p=8n+7, \qquad \Sigma\sigma_j=\frac{1}{12}(p^2-1)$;

if $\quad p=8n+3, \qquad \Sigma\sigma_j+2\Sigma\rho_j=\frac{1}{4}(p^2-1)$.

He obtained numerous formulas for computing $\Sigma(a'/p)$.

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109 Jour. de Math., (2), 13, 1868, 1–4.
110 Jour. de Math. (2), 14, 1869, 1–6.
111 Cf. *T. Pepin, Memoire della Pontifica Accad. Nuovi Lincei, 5, 1889, 131–151.
112 Jour. de Math.,[293] (6), 3, 1907, 367, Art. 30.
113 Jour. de Math., (2), 14, 1869, 7.
114 *Ibid.*, 260–262. Proved on p. 171 of Chapelon's[340] Thesis.
115 Supplement X to G. L. Dirichlet's Zahlentheorie, ed. 3, 1871; ed. 4, 1894, §§ 150–151.
116 Ueber Klassenzahl quadratischen Formen. Sub-title: Ueber den Werth des Ausdrucks $\Sigma(a'/p)$ wenn p eine Primzahl von der Form $4n+3$ und a' jede ganze Zahl zwischen 0 und $\frac{1}{2}p$ bedeutet. Prog., Torgau, 1871, 20 pp.

F. Mertens[117] denoted by $\psi(s, x)$ the number of positive classes of negative determinants 1, 2, 3, ..., x which have reduced forms with middle coefficient $\pm s$; by $\chi(s, x)$ the number of these classes which are even. By a study of the coefficients of reduced forms, it is found that the number of uneven classes of negative determinants 1, 2, 3, ..., x is[118]

$$F(x)=\sum_0^{\sqrt{x/3}}[\psi(s,x)-\chi(s,x)],$$

where, except for terms of the order of x,

$$\sum_0^{\sqrt{x/3}}\psi(s,x)=\frac{2\pi}{9}x^{\frac{3}{2}},\qquad \sum_0^{\sqrt{x/3}}\chi(s,x)=\frac{\pi}{18}x^{\frac{3}{2}}.$$

If we set $f(N)=\Sigma_1^N h(-n)$, we have

$$\begin{array}{rl} F(x) = & f(x)+f(x/3^2)+f(x/5^2)+f(x/7^2)+f(x/9^2)+\dots \\ F(x/3^2) = & \qquad f(x/3^2) \qquad\qquad\qquad\qquad +f(x/9^2)+\dots \\ F(x/5^2) = & \qquad\qquad\qquad f(x/5^2) \qquad\qquad\qquad\quad +\dots \\ \dots & \dots\dots\dots\dots\dots\dots\dots\dots \end{array}$$

and we solve for $f(x)$ by multiplying the respective equations by $\mu(1)$, $\mu(3)$, $\mu(5)$, ..., where $\mu(n)$ is the Moebius function (this History, Vol I, Ch. XIX). Thus

$$f(x)=\sum_{n=1}^{x}\mu(n)F(x/n^2).$$

But

$$F\left(\frac{x}{n^2}\right)=\frac{\pi x^{\frac{3}{2}}}{6n^3}+O\left(\frac{x}{n^2}\right),$$

where $Of(x)$ denotes a function of the order of $f(x)$, or more exactly a function whose quotient by $f(x)$ remains numerically less than a fixed finite value for all sufficiently large values of x.

Hence, when terms of the order of x are neglected,

$$f(x)=\frac{\pi}{6}x^{\frac{3}{2}}\sum_1^\infty\frac{\mu(n)}{n^3}=\frac{\pi}{6}x^{\frac{3}{2}}\left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right)\cdots.$$

Then, asymptotically,

$$\sum_{n=1}^{N}h(-n)=\frac{4\pi}{21S_3}N^{\frac{3}{2}},\qquad S_3=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\dots$$

And therefore the asymptotic median class number is[119] $2\pi\sqrt{N}/(7S_3)$.

T. Pepin[120] let Σm be the total number of representations of numbers n relatively prime to a given number Δ, $0\leqq n\leqq M$, M being an arbitrary positive integer, by a system of properly primitive forms of negative determinant D. He also let Σm be the total number of representations of numbers $2n$, n relatively prime to Δ,

[117] Jour. für Math., 77, 1874, 312-319. Reproduced by P. Bachmann, Zahlentheorie, Leipsig, II, 1894, 459.
[118] Cf. C. F. Gauss, Disq. Arith., Art. 171.
[119] Cf. C. F. Gauss, Disq. Arith., Art. 302; Werke, II, 1876, 284. Cf. R. Lipschitz,[102] Sitzungsber. Akad., Berlin, 1865, 174-185.
[120] Annales Sc. de l'École Norm. Sup., (2), 3, 1874, 165-208.

$0 \leqq n \leqq M$, by a system of improperly primitive forms of determinant D. In every representation, let

$$x=ax_1+\gamma, \quad y=\beta y_1+\delta, \quad \gamma<a, \quad \delta<\beta, \quad a, \beta, \gamma, \delta \text{ each} \gtreqless 0;$$

and in each of the two cases above, let K, K' be respectively the number of pairs of values γ, δ possible for given a, β. Then[121]

$$\frac{Kh(D)\pi M}{a\beta}+M\epsilon=\frac{K'h'(D)\pi M}{a\beta}+M\eta=\Sigma m,$$

where the limits of $M\epsilon$ and $M\eta$ for $M=\infty$ are finite.

A comparison of K and K' for $a=\beta=\Delta=2$ gives Dirichlet's[20] ratio h/h'. The corresponding result is obtained for the other orders and for the positive determinant.

Pepin avoids the convergence difficulty of Hermite[88] and obtains Dirichlet's[28] classic closed expression (5) for $h(D)$, $D<0$, by extending a theorem of Dirichlet[93] (2), to give

$$\Sigma m=\kappa\sum_n\sum_i(D_0/i),$$

in which κ is the automorph factor 2, 4 or 6; D_0 is a fundamental determinant, $D=D_0S^2$; i ranges over all divisors of n, while n ranges over all odd numbers $\leqq M$; and (D_0/i) is the Jacobi-Legendre symbol.

Pepin translating certain results of A. Cauchy[122] on the location of quadratic residues, found in Dirichlet's[28] notation

$$(1)\qquad h(-n)=\left[2-\left(\frac{2}{n}\right)\right]\frac{\Sigma b-\Sigma a}{n}=\left[2-\left(\frac{2}{n}\right)\right]\frac{\Sigma b^2-\Sigma a^2}{n^2},$$

where $-n=-(4\mu+3)$ is a fundamental negative determinant. This latter class-number formula, called Cauchy's, has been simply deduced by M. Lerch, Acta Math., 29, 1905, 381. Other results of Cauchy[123] give, in terms of Bernoullian numbers,

$$h(-n)\equiv 2B_{(n+1)/4} \text{ if } n=8l+7; \quad \equiv -6B_{(n+1)/4} \text{ if } n=8l+3,$$

modulo n a prime. And without proof Pepin states, for $n>0$, that

$$h(-n)=\left[2-\left(\frac{2}{n}\right)\right]\left\{\tfrac{1}{3}[2l+1][4l+1]-2\sum_{i=1} i[\sqrt{in}]\right\}, \quad l=\frac{n-3}{4}$$

L. Kronecker[124] obtained from his[54] eight classic relations new ones, as, for example, by combining (IV), (V), (VI), the following:

$$\sum_h(-1)^hF(n-4h^2)=\tfrac{1}{2}(-1)^{\frac{1}{8}(n-8)}\{\Phi(n)+\Psi(n)\}, \quad n\equiv 3 \pmod 4, \quad h\gtrless 0$$

By means of[125]

$$(1)\qquad 4\sum_0^\infty F(4n+2)q^{n+\frac{1}{2}}=\theta_2^3(q)\theta_3(q),$$

[121] C. F. Gauss, Werke, II, 1876, 280; Untersuchungen über höhere Arithmetik, Berlin, 1889, 666.
[122] Mém. Institut de France,[29] 17, 1840, 697; Oeuvres, (1), III, 388.
[123] Mém. Institut de France, 17, 1840, 445 (Cauchy [28]); Oeuvres, (1), III, 172.
[124] Monatsber. Akad. Wiss. Berlin, 1875, 223-236.
[125] Cf. Monatsber. Akad. Wiss. Berlin,[74] 1862, 309.

he obtained formulas for

$$\Sigma G\left(\frac{s-h^2}{16}\right), \quad \sum_h F\left(\frac{s-h^2}{16}\right), \quad s\equiv 1 \pmod 8.$$

He obtained two analogues[156] of (1), and stated that, in his[54] classic relations, $\frac{1}{4}(\mathrm{IV})-\frac{15}{32}(\mathrm{V})+\frac{3}{32}(\mathrm{VI})-\frac{1}{8}(\mathrm{VIII})$ is, when m is the square of a prime, equivalent to Hermite's[49] first class-number relation.

R. Dedekind[126] supplied the details of Gauss'[9] fragmentary deduction of formulas for $h(D)$ and $h(-D)$. He also[127] deduced and complemented Gauss'[9] set of theorems which state, in terms of the class-number of the determinant $-p$, the distribution of quadratic residues and non-residues of p in octants and 12th intervals of p, where p is an odd prime.

Dedekind,[127a] in a study of ideals, obtained results which he translated[378] immediately into the solution of the Gauss Problem.[4]

Dedekind[128] extended the notion of equivalence in modular function theory by removing the condition[129] that β and γ be even in the unitary substitution $\binom{\gamma\ \delta}{\alpha\ \beta}$. Each point ω in the upper half of the complex plane is equivalent to just one point ω_0, called a reduced point, in a fundamental triangle defined as lying above the circle $x^2+y^2=1$ and between the lines $x=\pm\frac{1}{2}$ and including only the right half of the boundary (cf. Smith[95] of Ch. I). The function, called the *valence* of ω,

$$v=val(\omega)=\tfrac{4}{27}\,\frac{(k+\rho)^3(k+\rho^2)^3}{k^2(1-k)^2}, \quad k=\kappa^2, \tag{1}$$

where ρ is an imaginary cube root of unity, is invariant[130] under the general unitary substitution. Dedekind's v is $-4/27$ times C. Hermite's[131] a. Let

$$v_n=val\left(\frac{C+D\omega}{A+B\omega}\right), \qquad \begin{vmatrix} C & D \\ A & B \end{vmatrix}=-n,$$

where A, B, C, D are integers without common divisor. Then v_n ranges exactly over the values

$$val\left(\frac{c+d\omega}{a}\right),$$

where a, c, d are integers $\gtreqless 0$ and $ad=n$; moreover, if e is the g.c.d. of a and d, then c ranges over those of the numbers $0, 1, 2, \dots, a$ which are relatively prime to e. Hence the number of distinct values of v_n is

$$\nu=\sum_a \frac{a}{e}\,\phi(e)=n\Pi\left(1+\frac{1}{p}\right), \tag{2}$$

where p ranges over the distinct prime divisors of n.

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126 Remark on Disq. Arith., in Gauss's Werke, II, 1876, 293–296; Untersuchungen über Hohere Arithmetik, 1889, 686–688.

127 Gauss's Werke, II, 1876, 301–303; Untersuchungen, 1889, 693–695.

127a Über die Anzahl der Ideal-classen in der verschiedenen Ordnungen eines endlichen Körpers. Festschrift zur Saecularfeier des Geburstages von Carl Frederich Gauss, Braunschweig, 1877, 55 pp.

128 Jour. für Math., 83, 1877, 265–292.

129 Cf. H. J. S. Smith,[100] Rep. Brit. Assoc., 35, 1865, 330; Coll. Math. Papers, I, 299.

130 Cf. C. F. Gauss, Werke, III, 1876, 386.

131 Oeuvres, II, 1908, 58 (Hermite [49]).

Dedekind discussed the equations whose roots are the ν values of v_n.

H. J. S. Smith[182] called the totality of those indefinite forms which are equivalent with respect to his normal substitution (Smith[95] of Ch. I) a *subaltern class*. He found that if σ denotes 2 or 1, according as U is even or uneven in $T^2-NU^2=1$, the circles of each properly primitive class of determinant N are divided into 3σ subaltern classes which in sets of σ satisfy the respective conditions

$$\text{(A)}\ a\equiv c\equiv 1 \pmod 2;\qquad \text{(B)}\ a\equiv 0,\ c\equiv 1 \pmod 2;\qquad \text{(C)}\ a\equiv 1,\ c\equiv 0 \pmod 2.$$

Since the circle $[a, b, c]$ corresponds to both (a, b, c) and $(-a, -b, -c)$, the number of subaltern classes of properly primitive circles of determinant N is $H=\frac{3}{2}\sigma h(N)$. There is a similar relation for the improperly primitive circles.

Now $\omega=x+iy$, representing a point in the fundamental region Σ, is inserted in

$$\phi^8(\omega)=\tfrac{1}{2}+X+iY,\qquad \psi^8(\omega)=\tfrac{1}{2}-X-iY,$$

where $\phi^8(\omega)$, $\psi^8(\omega)$ are Hermite's[49] symbols in elliptic function theory. Then if the circle $[a, b, c]$ satisfy (A), for example, the arcs within Σ of all and only circles (completely) equivalent to $[a, b, c]$ are transformed by the modular equation $F(k^2, \lambda^2)=0$ of order N into a certain algebraic curve, an interlaced lemmiscatic spiral. Hence all the circles of determinant N that satisfy (A) go over into a *modular curve* consisting of $\frac{1}{3}H$ distinct algebraic branches. This is called by F. Klein the *Smith-curve*.[183]

The number of improperly primitive subaltern classes of determinant N (not a square) is just the number of branches of a modular curve which is derived as the preceding from circles of determinant N, in which $a\equiv c\equiv 0 \pmod 2$.

F. Klein[184] called Dedekind's[128] v the absolute invariant J and, instead of v_n, he wrote J'. The equation, $\Pi(J-J')=0$ is called the transformation equation of order n. He gave an account of its Galois group, fundamental polygon, and Riemann surface. Simplest forms of Galois resolvents are found for $n=2, 3, 4, 5$. For example, the simplest resolvent for $n=5$ is the icosahedron equation.

Define $\eta(\omega)$ as a modular function if it is invariant under a subgroup of the group of unitary substitutions $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Then ω_1 and ω_2 are *relatively equivalent* if $\eta(\omega_1)=\eta(\omega_2)$. A subgroup $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is said to be of *grade* (*stufe*) q if

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}a & b\\ c & d\end{pmatrix}\pmod{q},$$

where a, b, c, d are constants. Klein ascribed the grade q to any modular function which is invariant under only (that is, *belongs* to) such a subgroup. The subgroup

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\pmod{q}$$

is called the *principal* subgroup; and it is found that the icosahedron irrationality belongs to this subgroup if $q=5$. This result for the case of $n=5$ is extended to all

182 Atti della R. Accad. Lincei, fis. math. nat. (3), 1, 1877, 134–149; Coll. Math. Papers, II, 1894, 224–239; Abstract, Transunti, (3), 1, 68–69.
183 Elliptische Modulfunctionen,[217] II, 1892, 167 and 205.
184 Math. Annalen, 14, 1879, 111–162.

odd primes n. A modular function which belongs to the principal subgroup is called a principal modular function.

If n is an odd prime, the simplest Galois resolvent is of order $\frac{1}{2}n(n^2-1)$ and its Riemann surface is equivalent to $\frac{1}{2}n(n^2-1)$ triangles in the modular division of the plane. These triangles are chosen so as to form a polygon; and the surface of the resolvent is formed from the polygon by joining the points in the boundary which are relatively equivalent. The genus of the surface is

$$p=\tfrac{1}{24}(n-3)(n-5)(n+2).$$

Klein hereafter ascribes the p of the surface to η itself. Hence if a principal modular function η has $q=3$ or 5 then $p=0$; but if $q=7$, then $p=3$. It follows that if q is an odd prime, J is a rational function of η if and only if $q=3$ or 5. It is found similarly that if $q=2$ or 4, J is a rational function of η.

The modular equation of prime order $n\neq 5$ and of grade 5 is written as

$$\Pi[\eta(\omega)-\eta(\omega')]=0, \tag{1}$$

where $\eta(\omega)$ is the icosahedron function, and the $n+1$ relatively non-equivalent representatives ω' are displayed in detail.

J. Gierster[185] wrote a set of eight class-number relations which he stated he had found from the icosahedron equation (Klein,[184] (1)) by the method of L. Kronecker[186] and Smith.[100] For example,

$$3\Sigma H(4n-k_1^2)=\Phi(n), \qquad n\equiv \pm 1 \pmod 5,$$

where, as always hereafter, $H(m)$ denotes the number of even classes of determinant $-m$ with the usual conventions[54]; k_1 ranges over positive quadratic residues of 5 which are $\leqq \sqrt{4n}$.

A combination of these eight relations gives

$$\Sigma H(4n-k^2)=\Phi(n)+\Psi(n), \tag{A}$$

which may be expressed in terms of Kronecker's[54] original eight:

$$\mathrm{I}(n)-\mathrm{II}(m) \text{ or } \mathrm{I}(n)-\tfrac{2}{3}\mathrm{I}(m)+\tfrac{1}{3}\mathrm{IV}(m)-\tfrac{1}{3}\mathrm{V}(m),$$

according as μ is odd or even in $n=2^\mu m$, where m is odd.

T. Pepin[187] completed the solution of Gauss'[4] problem. He accomplished this by finding the number of properly primitive classes of determinant $S^2\cdot D$ which when compounded with $(S, 0, -D\cdot S)$ reproduce that class. Similarly he found the ratio between the number of properly and improperly primitive classes of the same determinant.

F. Klein[188] emphasized the importance of the study of the modular functions (cf. Klein[184]) which are invariants of subgroups of finite index (i. e., subgroups whose substitutions are in $(1, k)$ correspondence with those of the modular group) and in particular those in which the subgroups are at once (a) congruence sub-

[185] Göttingen Nach., 1879, 277–81; Math. Annalen, 17, 1880, 71–3.
[186] Monatsber. Akad. Wiss. Berlin, 1875, 235.
[187] Atti Accad. Pont. Nuovi Lincei, 33, 1879–80, 356–370.
[188] Math. Annalen, 17, 1880, 62–70.

groups, (b) invariant subgroups, and (c) of genus zero. In the last case, a (1, 1) correspondence can be set up between the points of the fundamental polygon of the sub-group in the ω plane and the points of the complex plane by means of the equation $J=f(\eta)$ of genus zero where $\eta(\omega)$ is called a *haupt modul.* But if the genus p is >0, $\eta(\omega)$ must be replaced by a *system* of modular functions $M_1(\omega)$, $M_2(\omega)$, ... Klein and after him A. Hurwitz and J. Giester always chose $M_i(\omega)$ so that

$$M_i\left(\frac{a\omega+\beta}{\gamma\omega+\delta}\right),\ a\delta-\beta\gamma=1,$$

for all values of i, is a linear combination of $M_1(\omega)$, $M_2(\omega)$,.... The representatives ω' are $(A\omega+B)/D$, with $AD=n$, $0\leqq B<D$, B having no factor common to A and D. The analogue of the vanishing of $\Pi[\eta(\omega)-\eta(\omega')]$ in the modular equation[134] for the case $p=0$, is for the case $p>0$ the coincidence of the values of $M_1(\omega)$, $M_2(\omega)$, ... with those of $M_1(\omega')$, $M_2(\omega')$, ... respectively. This analogue of the modular equation is called the *modular correspondence* and it is said to be *grade* q if the M's are of grade q.

J. Gierster[139] stated that all of F. Kronecker's[54] eight class-number relations are obtainable as formulas of grades 2, 4, 8, 16. From F. Klein's[140] correspondence of order n and grade $q>2$, Gierster obtained $r=\frac{1}{2}q(q^2-1)$ correspondences by means of the unitary substitutions. He also considered the case where A, B, D have a common factor, i. e., the reducible correspondence. The number of coincidences of a reducible correspondence at points ω in the fundamental polygon[134] for q can be determined arithmetically in terms of class-number and algebraically in terms of the divisors of n. Excluding the coincidences which occur at the vertices, in the real axis, of the fundamental polygon, he gave briefly the chief material for the arithmetical determination. This he[145] made complete later.

If a given congruence subgroup G is not invariant, Gierster indicated a method of finding the number of coincidences of a correspondence for G in terms of the number of coincidences of the r reducible correspondences for the largest invariant subgroup under G and hence in terms of a class-number aggregate (cf. Gierster[148] for details).

He here stated (but later[141] proved) a full set of class-number relations of grade 7 (failing to evaluate just one arithmetical function $\xi(n)$ which occurs in several of the relations). These relations for the case when n is relatively prime to 7 were derived in detail later by Gierster[148] and A. Hurwitz[142] by different methods, Gierster employing modular functions which belong to other than invariant congruence subgroups.

A. Hurwitz[143] denoted by D any positive or negative integer which has no square factor other than 1, and wrote

$$F(s,D)=\left[1-(-1)^{\frac{1}{8}(D^2-1)}\frac{1}{2^s}\right]^{-1}\Sigma\left(\frac{D}{n}\right)\frac{1}{n^s},\text{ if } D\equiv 1 \pmod 4,$$

$$F(s,D)=\Sigma\left(\frac{D}{n}\right)\frac{1}{n^s}\quad\text{in all other cases,}$$

[139] Sitzungsber. Münchener Akad., 1880, 147–63; Math. Annalen, 17, 1880, 74–82.
[140] Math. Annalen, 17, 1880, 68 (Klein[138]).
[141] *Ibid.*, 22, 1883, 190–210 (Giester[148]).
[142] *Ibid.*, 25, 1885, 183–196 (Hurwitz[184]).
[143] Zeitschrift Math. Phys., 27, 1882, 86–101.

where the summation extends over all integers $n>0$ prime to $2D$. (Cf. Dirichlet,[19] (1).) He proved the following four theorems:

(I) The functions $F(s, D)$ are everywhere one-valued functions of the complex variable s.

(II) Every function $F(s, D)$, except $F(s, 1)$, has a finite value for every finite value of s.

(III) For every finite value of s, the function $F(s, 1)$ has a finite value except when $s=1$. Then $F(s, 1)$ becomes infinite in such a way that

$$\lim_{s\to 1}[(s-1)F(s, 1)]=1.$$

(IV) If $D>0$,

$$F(1-s,\ D)=\left(\frac{2\pi}{\kappa D}\right)^{1-s}\frac{\Gamma(s)}{\pi}\sqrt{\kappa D}\cos\tfrac{1}{2}s\pi\cdot F(s, D)$$

$$=\left(\frac{\kappa D}{\pi}\right)^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}F(s, D);$$

if $D<0$,

$$F(1-s,\ D)=\left(\frac{2\pi}{-\kappa D}\right)^{1-s}\frac{\Gamma(s)}{\pi}\sqrt{-\kappa D}\sin\tfrac{1}{2}s\pi\cdot F(s, D)$$

$$=\left(\frac{-\kappa D}{2\pi}\right)^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1-s}{2}+\frac{1}{2}\right)}\cdot F(s, D),$$

where $\kappa=1$ if $D\equiv 1 \pmod 4$, $\kappa=4$ in all other cases. These four results are extended to $D=D'\cdot S^2$ by the use of Dirichlet's identity

$$\Sigma\left(\frac{D}{n}\right)\frac{1}{n^s}=\Sigma\left(\frac{D'}{n'}\right)\frac{1}{n'^s}\Pi\left[1-\left(\frac{D'}{r}\right)\frac{1}{r^s}\right],$$

where n' ranges over all positive integers prime to $2D'$, and r ranges over all prime numbers which are divisors of D' but not of D (cf. Dirichlet, Zahlentheorie, § 100).

The memoir ends with an ingenious proof of the three following theorems:

$$\text{If } D>0 \text{ and } D\neq 1,\ F(s, D)=0,$$

for $s=0$ and for all negative even integral values of s. If $D<0$, $F(s, D)=0$ for all negative odd integral values of s.

$$F(s, D)\cdot\Gamma\left(\frac{s}{2}\right)\cdot\left(\frac{\kappa D}{\pi}\right)^{s/2}(D>0),\quad F(s, D)\Gamma\left(\frac{s+1}{2}\right)\cdot\left(\frac{-\kappa D}{\pi}\right)^{s/2}(D<0)$$

are not altered in value when s is replaced by $1-s$.

L. Kronecker[144] proved six of his[54] eight classic relations by means of a formula for the class-number of bilinear forms and a correspondence between classes of bilinear forms and classes of quadratic forms (Kronecker[14] of Ch. XVII).

Two quadratic forms are completely equivalent if and only if one is transformed into the other by a unitary substitution congruent to the identity (mod 2). (For

[144] Abhand. Akad. Wiss. Berlin, 1883, II, No. 2; Werke, II, 1897, 425–490.

more details, see Kronecker[113] of Ch. I.) Whence $12G(n)$ and $12F(n)$ are the number of classes and of odd classes respectively of determinant $-n$ under this new definition of equivalence. Two bilinear forms are likewise completely equivalent if they are transformed into each other by cogredient substitutions of the above kind. Then the number of representative bilinear forms $Ax_1y_1+Bx_1y_2-Cx_2y_1+Dx_2y_2$ having a determinant $\Delta=AD+BC$ is $12(G(n)-F(n))$ or $12G(n)$, according as $B+C$ is odd or even where $n=-\Delta+\frac{1}{4}(B+C)^2$ is the determinant of the quadratic form $(A, \frac{1}{2}(B-C), D)$. But since $G(4n)-F(4n)=G(n)$, the number of classes of bilinear forms of determinant Δ is

$$12\sum_h[G(4\Delta-h^2)-F(4\Delta-h^2)], \quad -2\sqrt{\Delta}<h<2\sqrt{\Delta}.$$

And there are $12\Sigma F(\Delta-h^2)$ classes of those bilinear forms of determinant Δ, for which at least one of the outer coefficients A and D is odd and the sum of the middle coefficients B and C is even.

The class-number of bilinear forms is now obtained in terms of $\Phi(\Delta)$, $\Psi(\Delta)$ and $X(\Delta)$. This gives immediately such class-number relations as

$$\sum_h[G(4\Delta-h^2)-F(4\Delta-h^2)]=\Phi(\Delta)+\Psi(\Delta);$$

and so (I)–(VI) of Kronecker.[54]

J. Gierster[145] gave a serviceable introductory account of the modular equation $f(J', J)\equiv\Pi(J-J')=0$ and of the congruencial modular equation, and also of the congruencial modular correspondence. He determined (p. 11) the location and order of the branch-points of the Riemann surface of the transformed congruencial modular function $\mu(\omega')$ as a function of $\mu(\omega)$, for the case q a prime, n prime to q, and $\mu(\omega)$ belonging to the unitary sub-group,

$$(1) \qquad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \pmod q.$$

From the condition that ω furnish a root of the reducible modular equation[134] $f(J', J)=0$, namely, that integers a, b, c, d exist such that

$$(2) \qquad \omega=\frac{a\omega+b}{c\omega+d},\ ad-bc=n,$$

he established (p. 17) a correspondence between the roots of $f(J', J)=0$ and the roots of certain quadratic equations $P\omega^2+Q\omega+R=0$ of all discriminants $-\Delta=(d+a)^2-4n<0$. Whence the number of zeros of $f(J', J)$ in the fundamental triangle is

$$\Sigma H(4n-\kappa^2), \qquad \kappa=0, \pm1, \pm2, \ldots, \qquad \kappa^2<4n.$$

To study the infinities of $f(J, J')$ in the fundamental triangle, Gierster (after Dedekind[128]) took $\omega'=(A\omega+B)/D$, noted the initial terms in the expansion of J and J' in powers of $q=e^{\pi i\omega}$, and found that

$$(J-J')^{D/T}=\text{const. } J^{g/T}$$

in the neighborhood of $\omega=i\infty$; in which g is the greater of A and D, and T is the

[145] Math. Annalen, 21, 1883, 1–50. Cf. Gierster[189]; Klein-Fricke, Elliptische Modulfunctionen, II, 160–235.

g.c.d. of A and D. Whence, taking into account the number of values of B, he arrived at the class-number relation

$$\Sigma H(4n-\kappa^2)=\Phi(n)+\Psi(n), \qquad \kappa=0,\ \pm1,\ \pm2,\ \ldots.$$

The result also follows from the Chasles correspondence principle.[146]

The irreducible correspondence[189] is now studied (p. 29) between $\mu_i(\omega)$ and $\mu_i(\omega')$, where the $\mu_i(\omega)$ are a system of functions invariant only of the subgroup of unitary substitutions (1), and ω' ranges over a complete set of relatively non-equivalent representatives

$$\frac{a\omega+b}{c\omega+d}, \quad ad-bc=n,$$

where n is prime to q, and a, b, c, d have fixed residues (mod q). Now ω in the fundamental polygon[184] furnishes a finite coincidence if and only if there exist integers a, b, c, d satisfying (2). Hence the condition is that ω be the vanishing point for some form $P\omega^2+Q\omega+R$, for which

(3) $$\pm P\equiv\pm c, \qquad \pm Q\equiv\pm(d-a), \qquad \pm R\equiv\mp b \pmod q.$$

For an arbitrary reduced form $P_0\omega^2+Q_0\omega+R_0$, let g be the number of equivalent forms $P_\nu\omega^2+Q_\nu\omega+R_\nu$ which have roots in the fundamental polygon and which satisfy both (3) and

(4) $$(P_\nu,\ Q_\nu,\ R_\nu)\binom{\alpha\ \ \beta}{\gamma\ \ \delta}=(P_0,\ Q_0,\ R_0).$$

In the particular case, $b\equiv c\equiv 0$, $d\equiv a\equiv\sqrt{n}$, we have $a+d\equiv 2\sqrt{n}$, $0\equiv P_0\equiv Q_0\equiv R_0\equiv\Delta=4n-(a+d)^2 \pmod q$; and (3) and (4) impose no condition on α, β, γ, δ. Hence (Klein[184]), $g=\frac{1}{2}q(q^2-1)$ and the number of finite coincidences is

$$\tfrac{1}{2}q(q^2-1)\Sigma H'\left(\frac{4n-l^2}{q^2}\right),$$

where l ranges over the positive and negative integers for which $4n-l^2$ is positive and divisible by q^2, while $H'(m)$ is the number of classes of forms of discriminant $-m$ which have no divisor which is a divisor of n. The number of finite coincidences of the reducible correspondence of order n is therefore

$$z_\infty=q(q^2-1)\Sigma H\left(\frac{4n-\kappa^2}{q^2}\right),$$

where $\kappa\equiv\kappa_{2\sqrt{n}}$ ranges over the positive integers $\leqq 2\sqrt{n}$ which are $\equiv\pm2\sqrt{n} \pmod q$.

Gierster now finds for the reducible correspondence the number of infinite coincidences in the fundamental polygon. For the above particular case, this is

$$\sigma_\infty=z_\infty+(q^2-1)U_{\sqrt{n}},$$

where U_i denotes the sum of the divisors of n which are $<\sqrt{n}$ and $\equiv\pm i \pmod q$, provided that, if $\sqrt{n}$ is an integer $\equiv\pm i \pmod q$ then $\frac{1}{2}\sqrt{n}$ is to be added to the sum. He evaluated σ in many further cases.

[146] M. Chasles, Comptes Rendus Paris, 58, 1864, 1775. A. Cayley, On the Correspondence of Two Points on a Curve, Proc. London Math. Soc., 1, 1865–6, Pt. VII; Coll. Math. Papers, VI, 9–13.

For q's such that[134] $p=0$, the σ's are evaluated also by the principle of Chasles.[146] And so for $q=3$ and 5, twelve exhaustive class-number relations are written such as (for our particular case above):

$$q=5,\ \left(\frac{n}{5}\right)=1,\quad 60\Sigma H\left(\frac{4n-\kappa^2_{2\sqrt{n}}}{25}\right)=\Phi(n)-12U_{\sqrt{n}}.$$

J. Gierster[147] tabulated congruence sub-groups of prime grade q of the modular group and calculated their genus (Klein[134]) for $q\leqq 13$.

Gierster[148] continued his[145] investigation but now replaced his former invariant subgroup of grade q by any one not invariant. There the total number of coincidences in the correspondences was expressed as a sum σ of class-numbers. Here the analogues of the σ's are found to be mere linear combinations of the former σ's. Employing congruence groups of grade 7, 11, 13 and genus[134, 138] zero, he deduced class-number relations[149] including for example

$$4\Sigma H(4n-\kappa_s^2)=\Phi(n),\quad q=7,\quad s=4\sqrt{-n},\quad (n/7)=-1.$$

A. Berger[150] employed an odd prime p, integers m, n and put

$$U_n=(-1)^{[\sqrt{n}]}=4[\tfrac{1}{2}\sqrt{n}]-2[\sqrt{n}]+1,\qquad 0\leqq n<p^2,$$

$$S_m=\sum_{k=}^{p} U_{m+4(k-1)p},\qquad 0\leqq m<4p,$$

where $[x]$ denotes the largest integer $\leqq x$. Various expressions for S_m are found. For example, if $p\equiv 1 \pmod 4$,

$$S_m=\epsilon+2\sum_{k>(m-p)/4}^{k\leqq m/4}\left(\frac{k}{p}\right), \tag{1}$$

where $\epsilon=+1$ if $m\equiv 0 \pmod 4$, $\epsilon=-1$ if $m\equiv 1, 2,$ or $3 \pmod 4$. Write

$$L_r=\sum_{k>(r-1)p/8}^{k<rp/8}\left(\frac{k}{p}\right).$$

Let K_1 be the number of properly primitive classes of determinant $-p$, and K_2 that of determinant $-2p$. A study of L_r and Dirichlet's[28] formula (5) give

$$K_1=2(L_1+L_2),\qquad K_2=2(L_1-L_4),\qquad \text{if } p\equiv 1 \pmod 4;$$
$$L_1=L_8=\tfrac{1}{2}(K_1+K_2),\quad L_2=L_4=L_5=L_7=\tfrac{1}{2}(K_1-K_2),\quad \text{if } p\equiv 1 \pmod 8.$$

Whence, for $p\equiv 1 \pmod 8$, he found by (1) such relations as

$$S_0=1+K_1,\qquad S_p=-1+K_1,\qquad S_{2p}=-1-K_1,$$
$$S_{3p}=-1-K_1,\qquad S_{(p-1)/2}=1+K_1+K_2,\qquad S_{(3p-1)/2}=-1-K_1.$$

Similar relations are obtained for $p\equiv 3, 5, 7 \pmod 8$.

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[147] Math. Annalen, 22, 1883, 177–189.
[148] *Ibid.*, 190–210.
[149] Notations of Gierster[145] (2), or more fully in Math. Ann., 22, 1883, 43–50.
[150] Nova Acta Reg. Soc. Sc. Upsaliensis, (3), 11, 1883, No. 7, 22 pp. For some details of the proof of (1), see Fortschritte Math., 14, 1882, 143, where the denotation of (1) is incorrectly given.

Berger wrote $Q(x)$ for the largest square $\leqq x$ and deduced eight theorems like the following: Among the p squares

$$(2) \qquad Q(0),\ Q(4p),\ Q(8p),\ \ldots,\ Q\{4(p-1)p\},$$

there are $\frac{1}{2}(p+1+K_1)$, $\frac{1}{2}(p+1)$, $\frac{1}{2}(p+1+K_1)$, or $\frac{1}{2}(p+1-2K_1)$ even numbers, according as $p\equiv 1, 3, 5$, or $7 \pmod 8$. Since K_1 and K_2 are positive, the squares (2) include at least $\frac{1}{2}(p+5)$, $\frac{1}{2}(p+1)$, $\frac{1}{2}(p+3)$, or $\frac{1}{2}(p-1)$ even numbers in the respective cases.

C. Hermite[151] communicated to Stieltjes and Kronecker the fact that if $F(D)$ denotes the number of uneven classes of determinant $-D$, then (cf. Hermite,[164] (2))

$$F(3)+F(7)+\ldots+F(4n-1)=\Sigma E\left(\frac{n-\nu^2}{2\nu+1}\right)+2\Sigma E\left(\frac{n-\nu^2-2\nu}{2\nu+3}\right)$$
$$+2\Sigma E\left(\frac{n-\nu^2-4\nu}{2\nu+5}\right)+\ldots+2\Sigma E\left(\frac{n-\nu^2-2k\nu}{2\nu+2k+1}\right);$$

in which $n-\nu^2-2k\nu \gtreqqless 2\nu+2k+1$.

Hermite[152] stated Oct. 24, 1883, that if $F(N)$ denotes the number of properly primitive [he meant uneven] classes of determinant $-N$ and $\psi(n)=\Sigma(-1)^{(d-1)/2}$, where d ranges over all divisors $<\sqrt{n}$ of n, then

$$F(3)+F(11)+F(19)+\ldots+F(n)=\psi(3)+\psi(11)+\ldots+\psi(n)$$
$$+2\Sigma\psi(k)E(\tfrac{1}{2}\sqrt{n-k})+2\Sigma\psi(l)E(\tfrac{1}{2}\sqrt{n-l}+\tfrac{1}{2});$$

$k=3, 11, 19, \ldots, n$; $l=7, 15, 23, \ldots, n-4$.

T. J. Stieltjes[153] observed that this result is equivalent to

$$F(n)=\psi(n)+2\psi(n-4\cdot 1^2)+2\psi(n-4\cdot 2^2)+2\psi(n-4\cdot 3^2)+\ldots;$$

and this is equivalent to an earlier result of J. Liouville, Jour. de Math., (2), 7, 1862, 43–44. [For, by definition, Liouville's $\rho'(n)$ is Hermite's $\psi(n)$; see this History, Vol. II, Ch. VII, 265, note 33a.]

Stieltjes[154] let $F(n)$ denote generally the number of classes of determinant $-n$ with positive outer coefficients, but in case $n=8k+3$ with even forms excluded. Then he found, when $n\equiv 5 \pmod 8$, that $\frac{1}{2}F(n)$ is the number of solutions of $n=x^2+2y^2+2z^2$, x, y, z each >0 and uneven. Consequently setting $\phi(n)=\Sigma(2/d_1)d$, $dd_1=n$, he found that

$$F(n)+2F(n-8\cdot 1^2)+2F(n-8\cdot 2^2)+\ldots=\tfrac{1}{2}\phi(n),\ n\equiv 3 \text{ or } 4 \pmod 8;$$
$$F(n-2\cdot 1^2)+F(n-2\cdot 3^2)+F(n-2\cdot 5^2)+\ldots=\tfrac{1}{4}\phi(n),\ n\equiv 5 \text{ or } 7 \pmod 8.$$

On Nov. 15, 1883, Stieltjes[155] observed that the former of the last two theorems is a corollary to Gauss, Disq. Arith., Art. 292. For $i=1, 2, 3, 5$ or 6, he found that

$$\sum_{n=1}^{N} F(8n+i)=\frac{\pi}{48}N^{\frac{3}{2}},$$

151 Aug., 1883, Correspondance d'Hermite et Stieltjes, Paris, I, 1905, 26.
152 Correspondance d'Hermite et Stieltjes, Paris, I, 1905, 43.
153 *Ibid.*, 45; Oct. 28, 1883.
154 Correspondance d'Hermite et Stieltjes, Paris, 1905, I, 50–52, Nov. 12, 1883.
155 *Ibid.*, 52–54.

asymptotically (cf. Gauss,[4] Disq. Arith., Art. 302, Mertens,[117] Gegenbauer,[199] Lipschitz[102]).

Stieltjes,[156] by the use of the two Kronecker[124] formulas,

$$4\sum_0^\infty F(4n+1)q^{n+\frac{1}{4}}=\theta_2(q)\theta_3^2(q),\qquad 8\sum_0^\infty F(8n+3)q^{2n+\frac{3}{4}}=\theta_2^3(q),$$

obtained the following three results: Let

$$\Phi(n)=\Sigma(2/d')d,\ dd'=n;\qquad \Psi(n)=\Sigma(-2/d),$$

whence $2\Psi(n)$ is the total number of representations of n by x^2+2y^2; then

$$\left.\begin{array}{ll} n\equiv 1 \pmod 8, & \Sigma F(n-8r^2)=\frac{1}{2}\Phi(n)+\frac{1}{2}\Psi(n) \\ n\equiv 3,\ 5 \pmod 8, & \Sigma F(n-8r^2)=\frac{1}{2}\Phi(n) \end{array}\right\} r=0,\ \pm 1,\ \pm 2,\ \ldots$$

$$n\equiv 3,\ 5,\ 7 \pmod 8,\quad \Sigma F(n-2s^2)=\tfrac{1}{4}\Phi(n)+\tfrac{1}{4}\Psi(n),\quad (s=1,\ 3,\ 5,\ \ldots).$$

Stieltjes[157] stated that he had deduced Liouville's[110] class-number relation of 1869 and other similar formulas both by arithmetical methods and by the theory of elliptic functions. For example, for $N>0$,

$$2\Sigma(-1)^{\frac{1}{2}(s-1)}sF(4N-2s^2)=(-1)^{\frac{1}{2}N(N-1)}\Sigma(x^2-2y^2),\quad s=1,\ 3,\ 5,\ \ldots,$$

summed for all integral solutions of $x^2+2y^2=N$. This he[158] later proved in detail. For $N>0$,

$$\Sigma(-1)^{\frac{1}{2}(s-1)}sF(16N-3s^2)=\Sigma(x^2-3y^2),\quad s=1,\ 3,\ 5,\ \ldots,$$

summed for all integral solutions of different parity of $x^2+3y^2=N$. The method of verifying this formula was indicated[159] later.

Stieltjes[160] obtained from classic expansions the expansion

$$(1)\quad \theta(q)\theta_2^4(q)\theta_3(q)=16\Sigma(x^2-y^2)q^{x^2+y^2},\quad x=1,\ 3,\ 5,\ 7,\ \ldots,\quad y=0,\ \pm 2,\ \pm 4,\ \ldots.$$

But

$$(2)\qquad \theta(q)\theta_2(q)\theta_3(q)=2(q^{1/4}+3q^{9/4}+5q^{25/4}+\ldots);$$

and (cf. Hermite,[69] (7))

$$(3)\qquad \theta_2^3(q)=8\sum_0^\infty F(8n+3)q^{\frac{1}{4}(8n+3)}.$$

A comparison of (1), (2), (3) gives at once a Liouville[110] class-number relation. Stieltjes added three new relations of the same type; e. g., for $N=8k+1$,

$$2\Sigma(-1)^{\frac{1}{2}(s-1)+\frac{1}{8}(s^2-1)}sF(2N-s^2)=\Sigma(-1)^y(x^2-8y^2),$$

summed for all integral solutions of $x^2+8y^2=N$ in which $x>0$ and uneven.

Stieltjes[161] stated, for the Kronecker[54] symbol $F(n)$, that

$$(1)\qquad F(np^{2k})=\left[p^k+p^{k-1}+\ldots-\left(\frac{-n}{p}\right)(p^{k-1}+p^{k-2}+\ldots)\right]F(n).$$

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156 Correspondance d'Hermite et Stieltjes, Paris, I, 1905, 54, Nov. 24, 1883.
157 Comptes Rendus, Paris, 97, 1883, 1358–1359; Oeuvres, I, 1914, 324–5.
158 Correspondance d'Hermite et Stieltjes, Paris, I, 1905, 63, Nov. 27, 1883.
159 *Ibid.*, 69–70, Dec. 8, 1883.
160 Comptes Rendus, Paris, 97, 1883, 1415–1418; Oeuvres, I, 1914, 326–8.
161 Correspondance d'Hermite et Stieltjes, Paris, I, 1905, 81, 85–87, letter to Hermite, Jan. 6, 1884.

He gave[162] a proof depending on the fact that $F(n)=\rho\Sigma h(n/d)$, where d ranges over the odd square divisors of n; $\rho=\frac{1}{2}$ or 1, according as n is or is not an uneven square; $h(m)$ denotes the number of properly primitive classes of determinant $-m$.

Stieltjes[162] put

$$\psi(n)=\Sigma(-1)^{(d_1^2-1)/8}d \quad (dd_1=n); \quad \chi(n)=\Sigma x,$$

where x ranges over the solutions of $n=x^2-2y^2>0$, $x>0$, $|y|<\frac{1}{2}x$; and stated that, when n is odd,

$$2\Sigma(-1)^r F(n-2r^2)=(-1)^{(n-1)/2}\chi(n), \qquad r=0, \pm 1, \pm 2, \ldots;$$
$$2\Sigma F(n-2r^2)=2\psi(n)-\chi(n), \qquad r=0, \pm 1, \pm 2, \ldots.$$

These and two similar formulas he was unable to deduce by equating coefficients of powers of q in expansions. This was later done for formulas which include these as special cases by Petr,[258] Humbert,[293] and Mordell.[352]

C. Hermite[163] imparted to Stieltjes in advance the outline of the deduction of Hermite's[164] formula (1).

Hermite,[164] by the same study of the conditions on the coefficients of reduced forms as he employed[69] in 1861, found that

$$\theta_2^3(q)=24\Sigma[(N)+2f(N)]q^{N/4}-16\epsilon,$$

where (N) denotes the number of ambiguous, and $f(N)$ the number of unambiguous, even classes of determinant $-N$; while $\epsilon=1$ or 0, according as N is or is not the treble of a square. For the case $N\equiv 3 \pmod 8$ a comparison of this with his earlier result[69] $\theta_2^3(q)=8\Sigma F(N)q^{N/4}$, where $F(N)$ is the number of uneven classes of $-N$, gives at once the ratio between the number of classes of the two primitive orders (cf. Gauss,[4] Disq. Arith., Art. 256, VI).

Kronecker's[124] formula (1) implies that

$$(1) \quad \frac{\theta_2^2(q)\theta_3(q)}{1-q}=4\Big(\sum_0^\infty q^n\Big)\sum_0^\infty F(4n+2)q^{n+\frac{1}{2}}=4\sum_n[F(2)+F(6)+\ldots+F(4n+2)]q^{n+\frac{1}{2}}.$$

But obviously

$$\theta_2^2(q)\equiv\Big[2\sum_0^\infty q^{\frac{1}{4}(2n+1)^2}\Big]^2=\sum_0^\infty f(8c+2)q^{2c+\frac{1}{2}},$$

where $f(n)$ denotes the number of solutions of $x^2+y^2=n$. Moreover,

$$\theta_3(q)=1+2\sum_1^\infty q^{n^2}.$$

Therefore, in the identity

$$\frac{\theta_2^2(q)\theta_3(q)}{1-q}=\frac{\theta_2^2(q)}{1-q}+2\frac{(\Sigma q^{n^2})\theta_2^2(q)}{1-q},$$

the first term of the right member is $\Sigma f(8c+2)q^{n+\frac{1}{2}}$, summed for $n=0, 1, 2, \ldots$;

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[162] Correspondance d'Hermite et Stieltjes, Paris, I, 1905, 82–85, Jan. 15, 1884.
[163] *Ibid.*, I, 88–89, Feb. 28, 1884.
[164] Bull. de l'Acad. des Sc. St. Petersburg, 29, 1884, 325–352; Acta Math., 5, 1884–5, 297–330; Oeuvres, IV, 1917, 138–168.

$c=0, 1, 2, \ldots, [\frac{1}{2}n]$; the second term, by a lemma on the Legendre greatest-integer symbol, is

$$2\Sigma f(8c+2)\cdot[\sqrt{n-2c}]\cdot q^{n+\frac{1}{4}},$$

summed for $n=0, 1, 2, \ldots, c=0, 1, 2, \ldots, [\frac{1}{2}(n-1)]$. Hence a comparison with (1) gives

$$4[F(2)+F(6)+\ldots+F(4n+2)]=\sum_{c_1=0}^{n} f(8c_1+2)+2\Sigma f(8c+2)\cdot[\sqrt{n-2c}].$$

By the use of Jacobi's expansion formula:

$$\theta_2^2(q)=4\sqrt{q}\frac{1+q^2}{1-q^2}-4\sqrt{q^9}\frac{1+q^6}{1-q^6}+4\sqrt{q^{25}}\frac{1+q^{10}}{1-q^{10}}-\ldots,$$

Hermite found similarly other expressions for $F(2)+F(6)+\ldots+F(4n+2)$, such as

(1) $$\Sigma(-1)^{\frac{1}{2}(a-1)}+2\Sigma(-1)^{\frac{1}{2}(a-1)}\left[\frac{4n+2-a^2-b^2}{4a}\right],$$

where a and b range over all odd positive integers satisfying

$$4n+2-a^2-b^2 \gtreqqless 0.$$

By means of two other formulas of Kronecker, Hermite evaluated similarly

$$F(1)+F(5)+\ldots+F(4n+1), \qquad F(3)+F(11)+\ldots+F(8n+3).$$

He announced without proof that

(2) $$F(3)+F(7)+\ldots+F(4n+3)=2\Sigma\left[\frac{n+1-c^2-2cc'}{2c+2c'+1}\right],$$

$c>0$, $c'>0$ and satisfying $(c+1)(2c+2c'+1)\leqq n+1$, counting half of each term in which $c'=0$.

T. J. Stieltjes[165] stated that by the theory of elliptic functions he obtained the theorem: If d range over the odd divisors of n and

$$\psi(n)=\Sigma(-1)^{\frac{1}{2}(d-1)+\frac{1}{8}(d^2-1)}=\Sigma\left(\frac{-2}{d}\right), \quad \psi(0)=\tfrac{1}{2},$$

then, for $n\equiv 2 \pmod 4$, in Kronecker's[54] notation,

$$F(n)=\tfrac{1}{2}\Sigma\psi(n-2r^2)=\Sigma\psi(n-8r^2), \quad r=0, \pm 1, \pm 2, \ldots.$$

Thence he verified his[161] earlier theorem (1) for the cases $n=k^2$ and $n=2k^2$ by the method used by Hurwitz in finding the number of decompositions of a square into the sum of five squares (see this History, Vol. II, 311).

A. Berger,[166] to evaluate Dirichlet's[14] series (2), namely,

$$V=\sum_{k=1}^{\infty}\left(\frac{\Delta}{k}\right)\frac{1}{k},$$

[165] Comptes Rendus, Paris, 98, 1884, 663–664; Oeuvres, I, 1914, 360–1.
[166] Nova Acta Regiae Soc. Sc. Upsaliensis, (3), 12, 1884–5, No. 7, 31 pp.

Δ being a fundamental discriminant, started from Kronecker's[171] identity (4a) in the form

$$\sum_{h=1}^{\epsilon\Delta-1}\left(\frac{\Delta}{h}\right)e^{2hk\pi i/(\epsilon\Delta)}=(\sqrt{\Delta})\left(\frac{\Delta}{k}\right),$$

where ϵ is the sign of Δ, and $k>0$. By separating the real from the imaginary and by a study of quadratic residues and non-residues, he obtained

$$(1)\qquad \Delta<0,\ \sum_{h=1}^{-\Delta/2}\left(\frac{\Delta}{h}\right)\sin\frac{2hk\pi}{-\Delta}=-\tfrac{1}{2}\sqrt{-\Delta}\cdot\left(\frac{\Delta}{k}\right),\quad k>0.$$

Since (cf. Dirichlet[14])

$$(2)\qquad \sum_{n=1}^{\infty}\frac{\sin nu}{n}=\frac{\pi-u}{2},\quad 0<u<2\pi,$$

we get, by dividing (1) by k and summing, Dirichlet's[28] formula (6) for $\Delta<0$.

Similarly by the use of the identity

$$-\log\left(2\sin\frac{u}{2}\right)=\sum_{n=1}^{\infty}\frac{\cos nu}{n},$$

Berger obtained Dirichlet's[28] closed formula (8), for $\Delta>0$.

To obtain Dirichlet's[28] second closed form, Berger took, for $\Delta<0$ (cf. Dirichlet, Zahlentheorie, § 89, ed. 4, p. 224)

$$V=\frac{1}{r}\Pi\left[1-\left(\frac{\Delta}{p}\right)\frac{1}{p}\right]^{-1}=\frac{1}{r}\sum_{k=1}^{\infty}\left(\frac{\Delta}{2k-1}\right)\frac{1}{2k-1},$$

where[171] $r=1-\frac{1}{2}(\Delta/2)$, and p ranges over all odd positive primes. By means of (1), this becomes

$$r\sum_{k=1}^{\infty}\left(\frac{\Delta}{k}\right)\frac{1}{k}=\frac{-2}{\sqrt{-\Delta}}\sum_{h=1}^{h<-\Delta/2}\left(\frac{\Delta}{h}\right)\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin 2h(2k-1)\pi/\Delta.$$

But (2) implies that the final factor is $\pi/4$. Hence we get Dirichlet's[28] classic formula (5). By parallel procedure, Berger obtained, for $\Delta>0$,

$$(3)\qquad V=\frac{2}{\left\{2-\left(\frac{\Delta}{2}\right)\right\}\sqrt{\Delta}}\sum_{h=1}^{h<\Delta/2}\left(\frac{\Delta}{h}\right)\log\cot\frac{h\pi}{\Delta}.$$

Cf. Dirichlet,[28] (8).

A. Hurwitz[167] gave without proof[168] thirteen class-number relations of the 11th grade which he had deduced by the method which he had used to obtain relations of the 7th grade.[169]

For example,

$$6\sum_{k}H(4n-\kappa^2)=\Phi(n)+\psi_1(n)+\psi_2(n)-\psi_3(n),\qquad\left(\frac{n}{11}\right)=1,$$

where κ ranges over all positive integers whose square is $\equiv n$ (mod 11); while

[167] Berichte Sächs. Gesells., Math-Phys. Classe, 36, 1884, 193–197.
[168] For proof, see F. Klein and R. Fricke, Vorlesungen über Elliptischen Functionen,[217] Leipzig, II, 1892, 663–664.
[169] Math. Annalen,[184] 25, 1885, 157–196.

$\psi_1(n)=\frac{1}{2}\Sigma x$, where x ranges over those solutions of $4n=x^2+11y^2$ in which x and y are not 0, and $(x/11)=-1$;

$$\psi_2(n)=\tfrac{1}{8}[5Z(n)-12\Phi(n)], \qquad \psi_3(n)=\tfrac{1}{8}[3Z_0(n)-Z(n)],$$

in which $Z(n)$ denotes the number of solutions of $4n=x^2+11y^2+z^2+11u^2$ for which $x+y$ is even; $Z_0(n)$, the number for which one of $x, z, x-z, x+z$ is divisible by 11.

By eliminating ψ_2 and ψ_3 from his set, Hurwitz obtained a new set which he showed to include J. Gierster's[170] class-number relations of grade 11.

L. Kronecker,[171] unlike Gauss, studied quadratic forms $ax^2+bxy+cy^2$ in which b may be even or uneven. He defined primitive forms as those in which a, b, c have no common factor. He denoted by $K(D)$ the number of primitive classes of discriminant $D=b^2-4ac$. He put

$$H(D)=\sum_{h=1}^{\infty}\left(\frac{D}{h}\right)\frac{1}{h}, \qquad \left(\frac{D}{h}\right)=\left(\frac{2^{\rho}}{D}\right)\left(\frac{D}{h'}\right),$$

if $h=2^{\rho}h'$, h' uneven, in which the symbols of the last right member are the Jacobi-Legendre signs.

Dirichlet's[20] fundamental formula (2) is specialized as follows:

$$\tau\sum_{h,k}\left(\frac{Q^2}{h}\right)\left(\frac{D}{k}\right)F(hk)=\sum_{a,b,c}\sum_{m,n}\left(\frac{Q^2}{m}\right)F(am^2+bmn+cn^2), \tag{1}$$

where h, k range over all positive integers; m, n over all integers not both zero; a, b, c over the coefficients of a system of representative forms (a, b, c) of the primitive classes of the discriminant $D=D_0\cdot Q^2$ (D_0 being fundamental); $a>0$ is relatively prime to Q; and b and c are divisible by all the prime divisors of Q; $F(x)$ is any function for which the series in each member is convergent.

By Dirichlet's methods (Zahlentheorie, Arts. 93–98) are obtained the following results:

$$\tau H(D)=\frac{2\pi}{\sqrt{-D}}K(D),\quad D<0; \qquad H(D)=\frac{K(D)}{2\sqrt{D}}\log\frac{T+U\sqrt{D}}{T-U\sqrt{D}},\quad D>0. \tag{2}$$

These are combined into one formula

$$H(D)=K(D)\int_{T/U}^{\infty}\frac{dz}{z^2-D},$$

where T, U denote that fundamental solution of $T^2-DU^2=1$ or 4 for which T/U is the greater. This is equivalent to

$$H(D)=\frac{K(D)}{\sqrt{D}}\log E(D), \qquad E(D)=\frac{1}{r}\,(T+U\sqrt{D}),\quad r=1 \text{ or } 2. \tag{3}$$

But (cf. Dirichlet, Zahlentheorie, Art. 100),

$$H(D)=H(D_0)\Pi\left(1-\left(\frac{D_0}{q}\right)\frac{1}{q}\right),$$

[170] Math. Annalen,[148] 22, 1883, 203–206.
[171] Sitzungsber. Akad. Wiss. Berlin, 1885, II, 768–780.

q ranging over the prime divisors of Q. Hence,

$$(4) \qquad \frac{K(D)}{K(D_0)} = Q\Pi\left\{1-\left(\frac{D_0}{q}\right)\frac{1}{q}\right\}\frac{\log E(D_0)}{\log E(D)}.$$

In the light of the identity (p. 780)

$$(4a) \qquad \left(\frac{D_0}{r}\right) = \frac{1}{\sqrt{D_0}}\sum_k \left(\frac{D_0}{k}\right) e^{2rk\pi i/|D_0|},\ k=1, 3, 5, \ldots, 2|D_0|-1;\ r>0,$$

(2) implies

$$(5) \qquad \begin{cases} K(D_0) = \dfrac{\tau}{2D_0}\displaystyle\sum_{k=1}^{-D_0+1}\left(\frac{D_0}{k}\right)k, \quad D_0<0, \\ K(D_0)\log E(D_0) = -\displaystyle\sum_{k=1}^{D_0-1}\left(\frac{D_0}{k}\right)\log(1-e^{2k\pi i/D_0}), \quad D_0>0. \end{cases}$$

H. Weber[172] and J. de Séguier[173] have modified the above identity (4a) so as to be true also for $D_0 \equiv 0 \pmod 4$, which is not the case in Kronecker's form of it. De Séguier has given the deduction in full of (5) and has shown that (5_2) holds also for $D_0<0$. Dirichlet[174] at this point needed to treat eight cases instead of Kronecker's two and de Séguier's one.

Kronecker[175] had defined the function $\theta(\zeta, \omega)$ by

$$\theta(\zeta, \omega) = \sum_\nu e^{\frac{1}{4}(\nu^2\omega+4\nu\zeta-2\nu)\pi i},\ \nu=\pm 1,\ \pm 3,\ \pm 5, \ldots,$$

and the function Λ by

$$\Lambda(\sigma, \tau, \omega_1, \omega_2) = (4\pi^2)^{\frac{1}{3}} e^{\tau^2(\omega_1+\omega_2)\pi i}\cdot\frac{\theta(\sigma+\tau\omega_1, \omega_1)\theta(\sigma-\tau\omega_2, \omega_2)}{[\theta'(0, \omega_1)\theta'(0, \omega_2)]^{\frac{1}{3}}},$$

in which σ, τ are arbitrary complex numbers; ω_1, ω_2 are any complex numbers such that $\omega_1 i$ and $\omega_2 i$ have negative real parts. He[176] found that if ω_1 and $-\omega_2$ are the roots of $a+bw+cw^2=0$, where $b^2-4ac=-\Delta$ is a negative discriminant, then

$$(6) \qquad \log\Lambda(\sigma, \tau, \omega_1, \omega_2) = \frac{-\sqrt{\Delta}}{2\pi}\lim_{\rho=0}\sum_{m,n}\frac{e^{2(m\sigma+n\tau)\pi i}}{(am^2+bmn+cn^2)^{1+\rho}}$$

and therefore Λ is a class invariant. Relation (6) was afterward developed by Kronecker[177] into what J. de Séguier[178] has called Kronecker's second fundamental formula.

For D_1, D_2 two arbitrary conjugate divisors of $D=D_1\cdot D_2=D_0\cdot Q^2$ (1) is found to imply what J. de Séguier[179] has called Kronecker's first fundamental formula, namely,[180]

$$\tau\sum_{h=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{D_1Q^2}{h}\right)\left(\frac{D_2Q^2}{k}\right)F(hk)$$
$$=\frac{1}{2}\sum_{a,b,c}\left[\left(\frac{D_1}{A}\right)+\left(\frac{D_2}{A}\right)\right]\sum_{m,n}\left(\frac{Q^2}{m}\right)F(am^2+bmn+cn^2),$$

172 Götting. Nachr., 1893, 51–52.
173 Formes quadratiques et multiplication complexe,[226] Berlin, 1894, 32.
174 Zahlentheorie, Art. 105, ed. 4, 1894, 274–5.
175 Sitzungsber. Akad. Wiss. Berlin, 1883, I, 497–498.
176 *Ibid.*, 528.
177 Sitzungsber. Akad. Wiss. Berlin, 1889, I, 134, formula (16); 205, formula (18).[218]
178 Formes quadratiques et multiplication complexe, 1894, 218, formula (3).[226]
179 *Ibid.*, 133, formula (6).[226]
180 L. Kronecker, Sitzungsber. Akad. Wiss. Berlin, 1885, II, 779.

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with ranges of summation as in (1), while $2am/n+b \gtreqless U/T$ and $n>0$, if D is >0; A is an arbitrary number relatively prime to $2D$ and representable by (a, b, c). An elegant demonstration has been given by H. Weber.[181]

Take $Q=1, D_1<0, D_2>0, F(x)=x^{-1-\rho}$. When (6) is applied to the right member, the result, when $\rho=0$, is

$$(7)\qquad \frac{\tau\sqrt{\Delta}}{2\pi}H(D_1)H(D_2)=\sum_{a,b,c}\left(\frac{D_1}{a}\right)\log c[\theta'(0,\omega_1)\theta'(0,\omega_2)]^{-\frac{2}{3}},\quad \Delta=-D.$$

This formula refers the problem of the class-number of a positive discriminant to that of a negative discriminant. For the purposes of calculation, this formula has been improved by J. de Séguier.[182]

L. Kronecker[183] considered solutions (U, V) of $U^2+DV^2=4p$, where $p\equiv 1 \pmod{D}$, D a prime $=4n+3>0$. If $x^p=1, \alpha^D=1, x\neq 1, \alpha\neq 1$, and g is a primitive root of p, then

$$\prod_a(x+\alpha^a x^g+\alpha^{2a}x^{g^2}+\ldots+\alpha^{(p-2)a}x^{g^{p-2}})=\tfrac{1}{2}u+\tfrac{1}{2}v\sqrt{-D},$$

where a ranges over the incongruent quadratic residues of D, and u and v are integers. Whence finally he stated that U and V are determined from

$$\frac{u+v\sqrt{-D}}{u-v\sqrt{-D}}=\left(\frac{U+V\sqrt{-D}}{U-V\sqrt{-D}}\right)^{\frac{\Sigma b-\Sigma a}{D}},$$

Cf. Dirichlet's[23] formula (6).

A. Hurwitz[184] stated that his[185] modular equations of the 8th grade[184] yield those class-number relations which L. Kronecker[124] had given in Monatsber., Berlin, 1875, 230–233. He modified Gierster's[145] deduction of the class-number relation of the first grade by showing that a modular function $f(J, J')$ has as many poles as zeros in the fundamental polygon.

For genus[188] $p>0$, Hurwitz employed a system of normalized integrals $j_1(\omega)$, $j_2(\omega), \ldots, j_p(\omega)$ of the first kind on the Riemann surface formed from the fundamental polygon for the largest invariant sub-group of grade q. For arbitrary constants e_r the θ functions[186] of j_r have the property

$$\theta[j_r(T(\omega))-e_r]=\theta[j_r(\omega)-e_r]e^k,\qquad k\equiv\sum_{r=1}^{p}2t_r(j_r(\omega)-e_r)+C_t,$$

where T is an arbitrary unit substitution $\equiv\binom{1\ 0}{0\ 1}\pmod{q}$; while $t_1, t_2, t_3, \ldots, t_p, C_t$ depend only on T. Constants c_r are so chosen that

$$\theta[j_r(\omega)-j_r(\Omega)-c_r]=\theta[j_r(\Omega)-j_r(\omega)+c_r],$$

and $\theta=0$ when and only when the zero regarded as a value of ω (and Ω) is relatively

[181] Reproduced by de Séguier, Formes quadratiques, 332–334.
[182] Formes quadratiques et multiplication complexe, Berlin, 1894, 314, (25).
[183] Göttingen Gelehrte Anzeigen; Nachrichten Konigl. Gesells. Wiss., 1885, 368–370, letter to Dirichlet.
[184] Math. Annalen, 25, 1885, 157–196.
[185] Göttingen Nachr., 1883, 350.
[186] Cf. B. Riemann: Jour. für Math., 65, 1866, 120; Werke, 1892, 105; Oeuvres, 1898, Mém. XI. 207; C. Neumann, Theorie der Abel'schen Integrale, Leipzig, 1884, Chaps. XII, XIII.

equivalent to $\Omega, \omega_1, \omega_2, \ldots, \omega_{p-1}$ (and $\omega, \omega_p, \omega_{p+1}, \ldots, \omega_{2p-2}$), where $\omega_1, \omega_2, \ldots, \omega_{2p-2}$ are constants chosen almost[187] arbitrarily; moreover, that zero is of the first order.

The transformations $R_1(\omega), R_2(\omega), \ldots$ are a system of representative substitutions[188] of order n and are

$$\Omega \equiv \frac{a\omega+b}{c\omega+d} \pmod{q},$$

where a, b, c, d are fixed for all R's.

Consider the function

$$\Phi(\omega) = \prod_i \theta[j_r(\omega) - j_r(R_i(\omega)) - c_r],$$

where if n is a square, we omit the representative

$$\equiv \frac{\sqrt{n}\omega}{\sqrt{n}} \pmod{q},$$

which is relatively[184] equivalent to ω. Aside from the zero values which are due to the choice of $\omega_1, \omega_2, \ldots, \omega_{2p-2}$, and aside from the rational points ω, the theory of the zeros[189] of a θ-function shows that, since $\Phi(\omega)$ is reproduced except for a finite exponential factor under the substitution $T(\omega)$, $\Phi(\omega)$ vanishes in the fundamental polygon as many times as there are identities

$$\omega_0 = \frac{a'\omega_0+b'}{c'\omega_0+d'}, \qquad a'd'-b'c'=n, \quad \begin{pmatrix} a'b' \\ c'd' \end{pmatrix} \equiv \begin{pmatrix} ab \\ cd \end{pmatrix} \pmod{q}.$$

From this point Hurwitz treats the θ-functions as Gierster[145] had treated the factors $\eta(\omega)-\eta(\omega')$ of the modular equation and his determination of Gierster's σ differs only in details from Gierster's determination.

To complete Gierster's nine class-number relations[190] of the 7th grade for $n \not\equiv 0 \pmod{7}$ and without recourse to non-invariant subgroups, Hurwitz, after F. Klein,[191] put

$$z_1(\omega) = \Sigma(-1)^\nu q^{\frac{1}{28}[7(2\nu+1)+8]^2}, \qquad z_2(\omega) = \Sigma(-1)^\nu q^{\frac{1}{28}[7(2\nu+1)+1]^2},$$
$$z_4(\omega) = \Sigma(-1)^\nu q^{\frac{1}{28}[7(2\nu+1)+2]^2}.$$

Three normalized integrals of the first kind and of grade 7 are

$$I_r(\omega) = -\frac{1}{7}\int_0^q z_r \theta_1'(0|q)\frac{dq}{q} = \Sigma \frac{\psi_r(m)}{m} q^{2m/7}, \quad r=1,2,4;$$

summed for values of $m \equiv r \pmod{7}$, where necessarily $\psi_r(m) = \frac{1}{2}\Sigma a$, the summation extending over all positive and negative integer solutions a, β of $4m = a^2+7\beta^2$, $m \equiv r \pmod{7}$, $(a/7)=1$. Now $I_r(\omega)$ has the property

$$\sum_{i=1}^{\Phi(n)} I_r(R_i(\omega)) = \text{const., or } \psi(n) I_r(S(\omega)) + \text{const.},$$

according as $(n/7) = -1$ or $+1$, while

$$S(\omega) \equiv \frac{a\omega+b}{c\omega+d} \pmod{7},$$

[187] Cf. H. Poincaré and E. Picard, Comptes Rendus, Paris, 97, 1883, 1284.
[188] F. Klein, Math. Annalen, 14, 1879, 161.
[189] B. Riemann, Jour. für Math., 65, 1866, 161–172; Werke, 1892, 212–224.
[190] Math. Annalen, 17, 1880, 82; 22, 1883, 201–202.
[191] *Ibid.*, 17, 1880, 569.

and $\psi(n)=\frac{1}{2}\Sigma a$, the summation extending over all positive and negative integer solutions a, β of $4n=a^2+7\beta^2$, $(a/7)=1$. Let this property of the integrals I_r be possessed by the integrals j_1, j_2, j_4. Hurwitz put

$$\Phi(\omega',\omega)=\prod_{i=1}^{\Phi(n)}\theta[j_r(\omega')-j_r(R_i(\omega))-c_r],$$

$$\Psi(\omega',\omega)=\begin{cases}\theta[j_r(\omega')-j_r(S(\omega))-c_r]^{-\psi(n)}, & \text{if } (n/7)=1,\\ 1 & \text{, if } (n/7)=-1,\end{cases}$$

$$\psi(\omega',\omega)=\Phi(\omega',\omega)\cdot\Psi(\omega',\omega),\qquad F(\omega',\omega)=\frac{\psi(\omega',\omega)}{\psi(\omega_0',\omega)\cdot\psi(\omega',\omega_0)};$$

where ω_0', ω_0 are arbitrary fixed values of ω with positive imaginary parts. Then $F(\omega',\omega)$ is invariant under $T(\omega')$ and hence as a function of ω and of ω' is an algebraic function belongs to the Riemann surface of the 7th grade. $F(\omega',\omega)=0$ expresses algebraically the modular correspondence[192] of grade q and order n.

$F(\omega',\omega)$ is an algebraic function which belongs to the surface and has as many zeros as poles in the fundamental polygon. Hence

$$(1)\qquad \sigma-k\cdot\psi(n)=2\Phi(n)-2\psi(n)\ \text{if}\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\neq\begin{pmatrix}\sqrt{n}&0\\0&\sqrt{n}\end{pmatrix},$$

where k is the number of zeros of $\theta[j_r(\omega)-j_r(S(\omega))-c_r]$ in the fundamental polygon, and σ has the value given by Gierster.[145]

Similarly

$$(2)\qquad \sigma=2\Phi(n)-6\psi(n)+\eta,\ \text{if}\ \begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}\sqrt{n}&0\\0&\sqrt{n}\end{pmatrix},$$

where $\eta=4$ or 0 according as n is or is not a square.

From (1) and (2) and the relation[145]

$$2(U_\rho+U_{2\rho}+U_{4\rho})=\Phi(n)-\psi(n),\qquad \rho=\sqrt{n\left(\frac{n}{7}\right)},$$

Gierster's[193] class-number relations of grade 7 follow at once; for, Gierster's[189] $\xi(n)$ is Hurwitz's $-2\psi(n)$.

A. Hurwitz[194] generalized completely his[184] deduction of the class-number relations of grade 7 to grade q, where q is a prime >5; and showed that the right member of these relations is $2\Phi(n)$ plus a simple linear combination of coefficients $\psi(n)$ which occur in an expansion of Abelian integrals of the first kind and of grade q. That is, if $\sigma(n)$ be determined in terms of class-number as by Gierster[145] and Hurwitz,[184]

$$\sigma(n)-2\Phi(n)-\eta=h_1\psi_1(n)+h_2\psi_2(n)+\ldots+h_\mu\psi_\mu(n),$$

where $\eta=2(p-1)$ or 0 according as n is or is not a square; and $h_1, h_2, \ldots, h_\mu$ are independent of n. Klein and Fricke[217] have since shown for $q=7$, 11, how the h's may be simply evaluated when the ψ's are known.

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[192] Cf. A. Hurwitz, Göttingen Nachr., 1883, 359.
[193] Math. Annalen, 22, 1883, 199–203 (Gierster[148]).
[194] Berichte Königl. Sächs. Gesells., Leipzig, 37, 1885, 222–240.

E. Pfeiffer[195] wrote $H(n)$ for the number of classes of forms of negative determinant $-n$, and sharpened Merten's[117] asymptotic expression for the sum $\Sigma H(n)$ to the equivalent of

$$\sum_{n=1}^{x} H(n) = \frac{2}{9}\pi x^{\frac{3}{2}} - \frac{x}{2} + O(x^{\frac{3}{4}+\epsilon}),$$

where the order[117] only of the last term is indicated and ϵ is a small positive quantity. Pfeiffer, in a discussion which lacks rigor, indicated a method of proof (see Landau[330] and Hermite[204]).

L. Gegenbauer[196] denoted by $f(n)$ the number of representations of n as the sum of two squares, and deduced from four of Kronecker's formulas like[124] (1) four formulas similar to and including the following:

$$12 \sum_{x=1}^{n} E(x) = f_2(n) + 2 \sum_{x=1}^{[\sqrt{n}]} f_2(n-x^2),$$

where[54]

$$E(n) = 2F(n) - G(n), \quad f_2(r) = \sum_{x=1}^{r} f(x) = \sum_{x=0}^{[\sqrt{r}]} [\sqrt{r-x^2}].$$

His earlier result[197]

$$\sum_{x=0}^{[\sqrt{m/a}]} [\sqrt{m-ax^2}] = \frac{\pi m}{4\sqrt{a}} + O(\sqrt{m})$$

transforms this into

$$\sum_{x=1}^{n} E(x) = \tfrac{1}{9}\pi n^{3/2} + O(n).$$

(For the notation O, see F. Mertens.[117]) The other analogous results are

$$\lim_{n=\infty} \sum_{x=0}^{n} F(4x+a)/n^{3/2} = \tfrac{1}{3}\pi, \qquad \lim_{n=\infty} \sum_{x=0}^{n} F(8x+3)/n^{3/2} = \tfrac{1}{3}\pi\sqrt{2},$$

where $a=1$ or 2. Hence the asymptotic median number in the three cases is $\frac{1}{2}\pi\sqrt{n}$, $\frac{1}{2}\pi\sqrt{n}$, $\pi\sqrt{n/2}$. These four results combined with those of Gauss[198] and Mertens[117] give the asymptotic median number of odd classes as

$$\pi\sqrt{n}\left\{\frac{1}{12} + \frac{1}{7\zeta(3)}\right\}, \qquad \zeta(3) \equiv 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots.$$

Gegenbauer[199] derived from four of Kronecker's[200] and four of Hurwitz's[202] formulas, twelve class-number relations with more elegance than he[196] or Hermite[164] had derived three of the same formulas. For example, from the following formula of Hurwitz,[202]

$$4 \sum_{n=0}^{\infty} F(8n+1) q^{2n+\frac{1}{4}} = \theta_2(q)\theta_3^2(q^2),$$

195 Jahresbericht der Pfeiffer'schen Lehr-und Erziehungs-Anstalt, Jena, 1885-1886, 1-21.
196 Sitzungsber. Akad. Wiss. Wien, Math-Natur., 92, II, 1885, 1307-1316.
197 *Ibid.*, 384.
198 Disq. Arith.,[4] Art. 302; Werke, II, 1876, 284.
199 Sitzungsber. Akad. Wiss. Wien., Math.-Natur., 93, II, 1886, 54-61.
200 Monatsber. Akad. Wiss. Berlin,[124] 1875, 229.

it follows that $4F(8n+1)$ is the number of integral (positive, negative or zero) solutions of

$$8n+1=8x^2+8y^2+(2z-1)^2.$$

Put $x^2+y^2=k$ and solve for z. For a fixed k, the number of integer values of z as n ranges from 1 to N is therefore

$$[\tfrac{1}{2}\sqrt{8N+1-8k}+\tfrac{1}{2}].$$

Hence

$$2\sum_{x=0}^{n} F(8x+1)=\sum_{x=0}^{n} f(x)\,[\tfrac{1}{2}\sqrt{8n+1-8x}+\tfrac{1}{2}],$$

where $f(x)$ denotes the number of representations of x as the sum of two squares. The symbol $f(x)$ is decomposed so that the last formula becomes

$$\sum_{x=0}^{n} F(8x+1)=\tfrac{1}{3}\pi\sqrt{2}n^{3/2}+O(n),$$

with O as in Mertens.[117] As in the previous case,[196] Gegenbauer now finds that the asymptotic median number of odd classes of the determinant $-(8n+i)$, $i=1, 2, 3, 5$, or 6 is $\pi\sqrt{n/2}$.

Gegenbauer[201] without giving proofs supplemented his earlier list[199] of 12 class-number relations with 20 others which are easily deduced by processes analogous to those used before[199] and which include the following three types:

$$\sum_{x=0}^{n} F(8x+3)\,[\tfrac{1}{2}\sqrt{8n+1-8x}+\tfrac{1}{2}]=\sum_{x=0}^{n} \psi_1(2x+1),$$

in which [presumably] $\psi_1(n)$ denotes the number of representations of $4n$ as the sum of four uneven squares, where the order of terms is regarded, but $(-a)^2$ is regarded as the same as $(+a)^2$.

$$\sum_{x=0}^{n} F(16x+14)=2\sum_{x=0}^{n} \rho(8x+5)\,[\tfrac{1}{2}\sqrt{4n+1-4x}+\tfrac{1}{2}],$$

$\rho(m)=\Sigma(-2/d_1)$, d_1 ranging over the odd divisors of n.

$$\sum_{x=0}^{n} F(8x+6)=2\sum_{y}(-1)^{y-1}+4\sum_{y,z}(-1)^{y-1}\left[\frac{2(n-2y^2+2y)-z^2+z}{8y-4}\right],$$

$y \gtreqless 1$; $z \gtreqless 1$; $2n-4y^2+4y-z^2+z \gtreqless 0$.

A. Hurwitz[202] employed four formulas of Kronecker[203] all of the same type and including

$$(1)\qquad 4\Sigma F(4n+2)q^n=q^{-\frac{1}{2}}\theta_2^2(q)\theta_3(q),$$

$$(2)\qquad 4\Sigma F(4n+1)q^n=\tfrac{1}{2}q^{-\frac{1}{4}}\theta_2(q^{\frac{1}{4}})\cdot\theta_3^2(q).$$

He enlarged the list to 12 such formulas by simple methods, for example by replacing q by $-q$ in (1), adding the result to (1), and then using the relation

$$\theta_2^2(q)=2\theta_2(q^2)\theta_3(q^2).$$

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[201] Sitzungsber. Akad. Wiss. Wien, Math-Natur., 93, II, 1886, 288-290.
[202] Jour. für Math., 99, 1886, 165-168; letter to Kronecker, 1885.
[203] Monatsber. Akad. Wiss. Berlin,[124] 1875, 229-230.

The result in this case is

$$(5)\qquad 2\Sigma F(8n+2)q^n=q^{\frac{1}{4}}\theta_2(q)\theta_3(q)\theta_3(q^2).$$

Seven class-number relations are obtained similarly to the following. We multiply (2) by $\theta_2(iq^{\frac{1}{2}})$. The relation

$$\theta_2(iq)\theta_3^2(q^2)=\theta_1'(iq)$$

now gives

$$4\theta_2(iq^{\frac{1}{2}})\cdot\Sigma F(4n+1)\cdot q^{n+\frac{1}{4}}=\theta_2(q)\theta_1'(iq^{\frac{1}{2}});$$

and the equating of coefficients here gives

$$\sum_h(-1)^{(h^2-1)/8}F\left(\frac{m-h^2}{2}\right)=\tfrac{1}{4}\Omega_2(m),\qquad m\equiv 3 \pmod 8)$$

in which h is uneven and positive, $\Omega_2(m)=\Sigma(-2/\nu)\nu$, where ν ranges over all positive uneven numbers satisfying $m=\nu^2+2n^2$.

C. Hermite[204] represented the totality of reduced unambiguous quadratic forms of negative determinant and positive middle coefficient by $(2s+r,\ s,\ 2s+r+t)$, $r, s, t=1, 2, 3, \ldots$. Hence in

$$S=2\Sigma q^{(2s+r)(2s+r+t)-s^2}$$

the coefficient of q^N is the number of unambiguous classes of determinant $-N$. And if we put $n=2s+r$, we get

$$S=2\Sigma\frac{q^{n^2+n-s^2}}{1-q^n},\qquad \begin{array}{l} n=3, 4, 5, \ldots;\\ s=1, 2, 3, \ldots, \left[\dfrac{n-1}{2}\right].\end{array}$$

The number of ambiguous forms $(A,\ 0,\ C)$, $A\leqq C$, of determinant $-N$ is the number of factorizations $N=n(n+i)$, where n is a positive integer and $i\gtreqless 0$. This implies that the number of ambiguous forms of this type is the coefficient of q^N in the doubly infinite sum

$$S_1=\sum_{n,\,i}q^{n(n+i)}=\Sigma\frac{q^{n^2}}{1-q^n}.$$

Similarly the number of ambiguous reduced forms of the type $(2B,\ B,\ C)$ and (A, B, A) of determinant $-N$ is the coefficient of q^N in the expansion of

$$S_2=\Sigma\frac{q^{n^2+2n}}{1-q^{2n}},\qquad n=1, 2, 3, \ldots.$$

This gives[205]

$$S_1+S_2=\Sigma\frac{q^m}{1-q^{2m}}+\Sigma\frac{q^{2n}}{1-q^{2n}},\qquad \begin{array}{l} n=1, 2, 3, \ldots;\\ m=1, 3, 5, 7, \ldots.\end{array}$$

Hence, if $H(n)$ denotes the number of classes of determinant $-n$,

$$\Sigma H(n)q^n=\Sigma\frac{q^{n^2}}{1-q^n}+\Sigma\frac{q^{n^2+2n}}{1-q^{2n}}+2\Sigma\frac{q^{n^2+n-s^2}}{1-q^n}.$$

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[204] Bull. des sc. math., 10, I, 1886, 23-30; Oeuvres, IV, 1917, 215-222.

[205] Cf. C. G. J. Jacobi, Fundamenta Nova, 1829, Art. 65, p. 187; Werke, I, 1881, 239 (transformation of C. Clausen).

We divide each member by $1-q$ and expand according to increasing powers of q. Then the coefficient of q^N in the left member is $U=H(1)+H(2)+\dots+H(N)$. By the use of the identity[206]

$$(1)\qquad \frac{q^b}{(1-q)(1-q^a)}=\Sigma E\left(\frac{N+a-b}{a}\right)q^N,$$

the coefficient of q^N in the second member becomes

$$U=\Sigma E\left(\frac{N+n-n^2}{n}\right)+\Sigma E\left(\frac{N-n^2}{2n}\right)+2\Sigma E\left(\frac{N+s^2-n^2}{n}\right).$$

Neglecting quantities of the order of $E(\sqrt{N})=\nu$, we get

$$\Sigma E\left(\frac{N+n-n^2}{n}\right)=\Sigma\frac{N+n-n^2}{n}=N\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{\nu}\right)-\frac{\nu^2-\nu}{2}$$
$$=N(\tfrac{1}{2}\log N+C)-\tfrac{1}{2}N;$$

$$\Sigma E\left(\frac{N-n^2}{2n}\right)=\Sigma\frac{N-n^2}{2n}=\frac{N}{2}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{\nu}\right)-\frac{\nu^2-\nu}{4}$$
$$=\tfrac{1}{2}N(\tfrac{1}{2}\log N+C)-\tfrac{1}{4}N,$$

where C is the Euler constant.[330] In short,

$$U=\tfrac{3}{4}N\log N+2\Sigma\frac{N+s^2-n^2}{n}.$$

Geometric[207] considerations give the approximate value of the last term as

$$2\iint\frac{N+x^2-y^2}{y}\,dxdy,\quad x,y>0,$$

where the limits of integration are given by the relations $y>2x$, $N+x^2-y^2>0$. Hence for N very great, $U=\frac{2}{9}\pi N^{\frac{3}{2}}$. Cf. Pfeiffer,[195] Landau.[330]

L. Gegenbauer,[208] employing the same notation as had G. L. Dirichlet[209] and the same restrictions, obtained by new methods the results of Dirichlet, that the mean number of representations of a single positive integer by a system of representative forms of fundamental discriminant Δ is

$$\tau\sum_{x=1}^{\infty}\left(\frac{\Delta}{x}\right)\frac{1}{x},\text{ if }\Delta\gtrless 0;\qquad 2\pi K(\Delta)/\sqrt{-\Delta},\text{ if }\Delta<0,$$

where $K(\Delta)$ is the number of classes of negative discriminant Δ. For example, in the first case, the identity

$$\sum_{x=1}^{n}\left[\frac{n}{x}\right]\left(\frac{\Delta}{x}\right)=\sum_{x,\,y=1}^{n}\epsilon\left(\frac{n}{x\,y}\right)\left(\frac{\Delta}{x}\right)=\sum_{r=1}^{n}\epsilon\left(\frac{n}{r}\right)\sum_{d}\left(\frac{\Delta}{d}\right),$$

in which, presumably, $\epsilon(x)=0$ or 1 according as $x<1$ or $\geqq 1$; and the last summation extends over divisors of r, implies that

$$\sum_{x=1}^{n}\tau\sum_{d}\left(\frac{\Delta}{d}\right)=\tau\sum_{x=1}^{n}\left[\frac{n}{x}\right]\left(\frac{\Delta}{x}\right),$$

[206] C. Hermite, Acta Math., 5, 1884–5, 311; Oeuvres, IV, 1917, 152.
[207] Cf. R. Lipschitz,[102] Sitzungsber. Akad. Wiss. Berlin, 1865, 174–175.
[208] Sitzungsber. Akad. Wiss. Wien, 96, II, 1887, 476–488.
[209] Zahlentheorie, Braunchweig, 1894, 229; Dirichlet.[19]

where d ranges over the divisors of x and $\tau\Sigma(\Delta/d)$ is Dirichlet's[93] expression (2) for the number of representations of x by a system of representative forms of determinant Δ. Hence

$$\sum_{x=1}^{n}\tau\sum_{d}\left(\frac{\Delta}{d}\right)=\tau n\sum_{x=1}^{\infty}\left(\frac{\Delta}{x}\right)\frac{1}{x}-\tau n\sum_{x=[\sqrt{n}]+1}^{\infty}\left(\frac{\Delta}{x}\right)\frac{1}{x}-\tau\sum_{x=1}^{[\sqrt{n}]}\epsilon_x\left(\frac{\Delta}{x}\right)+\tau\sum_{x=[\sqrt{n}]+1}^{n}\left[\frac{n}{x}\right]\left(\frac{\Delta}{x}\right),$$

where $0\leqq\epsilon_x<1$, and each of the last three terms remains finite when n becomes infinite.

Gegenbauer[210] defined a certain function by

$$\chi_k(n)=n^k\sum_{d}\frac{\mu(d)}{d^k}\left(\frac{\Delta}{d}\right),$$

in which (Δ/d) is the Jacobi-Legendre symbol, d ranges over the divisors of n, and $\mu(x)$ is the Moebius function (this History, Vol. I, Ch. XIX). Then

$$\sum_{x=1}^{n}\left[\frac{n}{x}\right]\left(\frac{\Delta}{x}\right)\chi_k(x)=\sum_{x=1}^{n}\left(\frac{\Delta}{x}\right)x^k,$$

if Δ is prime to $1, 2, 3, \ldots, n$. This relation combined with Kronecker's[171] formulas (2) and (5) gives the number of classes of a prime discriminant Δ. That is,

$$\left.\begin{aligned}K(\Delta)&=\frac{\tau}{2\Delta}\sum_{x=1}^{|\Delta|-1}\left[\frac{|\Delta|-1}{x}\right]\left(\frac{\Delta}{x}\right)\chi_1(x),\\ K(\Delta)&=\frac{\tau}{2\left(2-\left(\frac{\Delta}{2}\right)\right)}\sum_{x=1}^{[\frac{1}{2}|\Delta|]}\left[\frac{|\Delta|}{2x}\right]\left(\frac{\Delta}{x}\right)\chi_0(x).\end{aligned}\right\}\quad\Delta<0$$

For example, if $\Delta=-7$, $\chi_0(1)=1$, $\chi_0(2)=0$, $\chi_0(3)=2$, $\chi_1(1)=1$, $\chi_1(2)=1$, $\chi_1(3)=4$, $\chi_1(4)=2$, $\chi_1(5)=6$, $\chi_1(6)=4$. Therefore $K(-7)=1$.

C. Hermite[211] employed an earlier result[69]

$$\sum\frac{q^{\frac{1}{4}(a^2+2a)-c^2}}{1-q^a}=\Sigma F(n)q^{\frac{n}{4}},\qquad n=4m-1,\qquad \begin{cases}a=1, 3, 5, \ldots\\ c=0, \pm1, \pm2, \ldots \pm\left(\frac{a-1}{2}\right),\end{cases}$$

for $a=2c'+1$, divided by $q^{\frac{3}{4}}$, then applied his[204] identity (1), and equated coefficients of q^{m-1} and obtained

$$\sum_{r=1}^{m}F(4r-1)=2\Sigma E\left(\frac{m-dd'}{d+d'+1}\right),$$

where d, d' are of the same parity; $d'\geqq d$; $m\geqq(d+1)(d'+1)$; and the coefficient 2 is to be replaced by 1 if $d=d'$. But when in mathematical induction $m-1$ is replaced by m, the right member of the last equation is increased by double the number of solutions of

$$\frac{m-dd'}{d+d'+1}=c,\text{ or }4m-1=4(c+d)(c+d')-(2c-1)^2,$$

in which $c=1, 2, \ldots, m$; $d\equiv d'\pmod 2$, $d'>d$; while, if $d'=d$, each solution is counted $\frac{1}{2}$. This gives the value of $F(4m-1)$.

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210 Sitzungsber. Akad. Wiss. Wien (Math.), 96, II, 1887, 607-613.
211 Jour. für Math., 100, 1887, 51-65; Oeuvres, IV, 1917, 223-239.

Hermite equated the coefficients of certain powers of q in two expansions of $H_1^3(0)$ and found that, for $m\equiv 3 \pmod 8$, the number of odd classes of the negative determinant $-m$ is $\Sigma\Phi(m-b^2)$, in which $b=0, \pm 2, \pm 4, \ldots$; $b^2<m$; and $\Phi(m)=\Sigma(-1)^{\frac{1}{2}(d'+1)}$, d' ranging over the divisors of m which are $>\sqrt{m}$ and $\equiv 3 \pmod 4$.

P. Nazimow[212] gave an account of the use[64, 145] of modular equations, and of Hermite's[69] method of equating coefficients in the theta-function expansions, to obtain class-number relations.

X. Stouff[128] of Ch. I extended Dirichlet's[19] determination of the class-number when the quadratic forms and the definition of equivalence both relate to a fixed set of integers called modules.

L. Kronecker[213] let $ax^2+bxy+cy^2$ be a representative form of negative discriminant $D=-\Delta=b^2-4ac$; put $a=a_0\sqrt{\Delta}$ and (Cf. Kronecker[175])

$$\Lambda'(0, 0, \omega_1, \omega_2)=\frac{4\pi^2}{c_0}\left(\frac{\theta'(0,\omega_1)}{2\pi}\cdot\frac{\theta'(0,\omega_2)}{2\pi}\right)^{\frac{2}{3}},$$
$$\omega_1=\frac{-b+i\sqrt{\Delta}}{2c},\qquad \omega_2=\frac{b+i\sqrt{\Delta}}{2c}.$$

He obtained the fundamental formula

$$\lim_{\rho=0}\left[-\frac{1}{\rho}+\frac{1}{2\pi}\sum_{m,n}\frac{(\sqrt{\Delta})^{1+\rho}}{(am^2+bmn+cn^2)^{1+\rho}}\right]=\log 4\pi^2+2C-\log\Lambda'(0,0,\omega_1,\omega_2),$$

where C is a constant independent of D, a, b, c. When each member of this identity is summed for the $K(D_0)$ representative forms of fundamental discriminant D_0, the result enables Kronecker[171] to evaluate the ratio $H'(-\Delta_0)/H(-\Delta_0)$ in terms of $K(D_0)$, where

$$H(-\Delta_0)=\sum_{k=1}^{\infty}\left(\frac{-\Delta_0}{k}\right)\frac{1}{k},\qquad H'(-\Delta_0)=\sum_{k=1}^{\infty}\left(\frac{-\Delta_0}{k}\right)\frac{\log k}{k}.$$

This is called Kronecker's limit ratio.

H. Weber[214] denoted by ω the principal root of a reduced quadratic form of determinant $-m$, and denoted by $j(\omega)$ the product of F. Klein's[184] class-invariant J by 1728. The *class equation*

$$(1)\qquad \Pi[u-j(\omega)]=0,$$

in which ω ranges over the principal roots of a representative system of primitive quadratic forms of determinant $-m$, he expressed by

$$(2)\quad H_m(u)=0,\quad \text{or}\quad (3)\quad H'_m(u)=0,$$

according as the forms are of proper or improper order. By applying transformations of the second order to ω, he set up a correspondence between the roots of (2) and (3). This correspondence is 1 to 1, if $m\equiv -1 \pmod 8$; 3 to 1, if $m\equiv 3 \pmod 8$, except when $m=3$. Whence he obtained Dirichlet's[20] ratio between $h(D)$ and $h'(D)$, $D<0$.

[212] On the applications of the theory of elliptic functions to the theory of numbers, 1885, (Russian). Summary in Annales Sc. de l'Ecole Norm. Sup., (3), 5, 1888, 23-48, 147-176 (French).

[213] Sitzungsber. Akad. Berlin, 1889, I, 199-220.

[214] Elliptische Functionen und Algebraische Zahlen, 1891, 338-344.

Weber[215] gave the name (cf. Dedekind's[128] valence equation (1)) *invariant equation* to

$$(4)\qquad \Pi\left[j(\omega)-j\left(\frac{c+d\omega}{a+b\omega}\right)\right]=0,$$

of order $ad-bc=n$, in which the g.c.d. of a, b, c, d is 1, and

$$\omega'=\frac{c+d\omega}{a+b\omega}$$

is a complete set of non-equivalent representatives. He observed that, if ω furnishes a root $j(\omega)$ of (4), then ω must be the principal root of a quadratic form

$$(5)\qquad A\omega^2+B\omega+C,\qquad B^2-4AC=D,$$

where, for a positive integer x,

$$b=Ax,\qquad c=-Cx,\qquad a-d=Bx;$$

and if we set $a+d=y$, we must have

$$(6)\qquad 4n=y^2-Dx^2.$$

Conversely, for each of the k representations of $-D$ in the form

$$-D=\frac{4n-y^2}{x^2},$$

there are $Cl(D)=h'(D)$ forms (5) each of whose principal roots furnishes one root of (4). Hence (4) can be written (cf. Weber's Algebra, III, 1908, 421)

$$(7)\qquad CH'^{k_1}_{D_1}(u)H'^{k_2}_{D_2}(u)\ldots=0,\qquad u=j(\omega).$$

If $j(\omega)$ is a root of (4), expansion of the left member in powers of $q=e^{\pi i\omega}$ shows that the degree of (4) in $j(\omega)$ is

$$2\Sigma\frac{\partial}{e}\phi(e)+\phi(\sqrt{n})\text{ or }2\Sigma\frac{\partial}{e}\phi(e),$$

according as n is or is not a square (cf. Dedekind,[128] (2)) where $\partial>\sqrt{n}$ is a divisor of n. The degree of (7) in $j(\omega)$ is $\Sigma h'(D_i)k_i$, summed for $i=1, 2, 3, \ldots$.

For brevity, (4) is written $F_n(u, u)=0$. The simplest case of deducing a class-number relation of L. Kronecker's type[48] is presented by equating two valuations of the highest degree of $u=j(\omega)$ in the reducible invariant equation

$$F_{n'_0}(u, u)\cdot F_{n'_1}(u, u)\cdot F_{n'_2}(u, u)\ldots=0,$$

where $n'_0, n'_1, n'_2, \ldots$ are derived from n in every possible way by removing square divisors including 1, but excluding n when n is square. The relation is

$$K(n)+2K(n-1)+2K(n-4)+\ldots+2K'(4n-1)+2K'(4n-9)+\ldots$$
$$=2\Sigma\partial\text{ or }2\Sigma\partial+\sqrt{n}+\tfrac{1}{6},$$

according as n is not or is a square. Here $K(m)$ denotes the number of classes of

[215] Elliptische Functionen und Algebraische Zahlen, 1891, 393–401.

determinant $-m$, and $K'(m)$ denotes the number of classes of determinant $-m$ derived from improperly primitive classes. Finally, $\Sigma\partial$ is the sum of the divisors of n which are $>\sqrt{n}$.

J. Hacks[216] considered the negative prime determinant $-q$, where $q=4n+3$; he put

$$S=\sum_{s=1}^{\frac{1}{2}(q-1)}\left[\frac{s^2}{q}\right],\quad S'=\sum_{s=1}^{\frac{1}{2}(q-1)}\left[\frac{2s^2}{q}\right],$$

and found that the number of properly primitive classes of determinant $-q$ is $h=\frac{1}{2}(q-1)-2S'+4S$. This is given the two following modified forms

$$h=\frac{q-1}{2}-2\sum_{1}^{\frac{1}{2}(q-3)}(-1)^s[\sqrt{\tfrac{1}{2}q\cdot s}],\qquad h=\frac{q-1}{2}+2\sum_{s=1}^{\frac{1}{2}(q-3)}\sum_{1}^{[2s^2/q]}(-1)^t;$$

and finally is reduced to Dirichlet's[28] formula (6).

F. Klein and R. Fricke[217] reproduced the theory of modular functions of Dedekind[128] and Klein,[134, 138] also (Vol. II, pp. 160–235, 519–666) the application by Gierster,[135, 139, 145, 147] Hurwitz,[167, 184, 194] and Weber[214] of that theory to the deduction of class-number relations of negative determinants. They gave (Vol. II, p. 234) the relations of grade 3 which come from the tetrahedron equation and (Vol. II, pp. 231–233) the relations of grade 5 that come from the icosahedron equation. Their formulas (1_1) p. 231, and (7), p. 233, should all have their right members divided by 2. They reproduced (Vol. II, pp. 165-73, 204-7) the theory of the relation between modular equations and Smith's[182] reduced forms of positive determinant.

In connection with Hurwitz's[194] general class-number relation of prime grade $q>5$ and relatively prime to n, Klein and Fricke constructed a table of values of ψ_i and χ_i for $n\leqq 43$. A sample of the table follows (p. 616):

n	ψ_1	ψ_2	ψ_3	n	χ_1	χ_2
1	-1	1	0	2	0	-1
3	1	1	-1	6	1	0
4	2	0	1	7	-1	0

For $q=11$ and $(n/q)=-1$, Hurwitz's first general formula becomes

$$6\sum_{3\sqrt{-n}} H(4n-\kappa^2)=2\Phi(n)+t_1\chi_1+t_2\chi_2,$$

where κ is positive or negative and $\kappa^2\equiv(3\sqrt{-n})^2 \pmod{q}$. Hence, by the table for $n=2$, $n=6$,

$$12H(4)=6-t_2,\qquad 12H(23)=24+t_1.$$

But it is known that $H(4)=\frac{1}{2}$; $H(23)=3$. Therefore $t_1=12$; $t_2=0$.

G. B. Mathews[218] reproduced in outline the researches of G. L. Dirichlet[93] on the number of properly primitive classes of a given determinant; and those of Lipschitz[219] on the ratio of the numbers of classes of different orders of the same determinant.

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[216] Acta Math., 14, 1890-1, 321-328.
[217] Elliptische Modulfunctionen, I, 1890, 163-416; II, 1892, 37-159.
[218] Theory of Numbers, Cambridge, 1892, 230-256.
[219] *Ibid.*, 159-170.

H. Weber,[220] by arithmetical processes, obtained L. Kronecker's[221] expression for the number h of primitive classes of forms $ax^2+bxy+cy^2$ of discriminant D. For $D=Q^2\cdot\Delta$, Δ a fundamental discriminant, he obtained by Dirichlet's[20] methods, Kronecker's[171] ratio (4) of the class-number of D and of Δ. By the use of Gauss sums, he transformed the former result for $Q=1$ into

$$h=\frac{\tau}{2\Delta}\sum_s(\Delta, s)s, \qquad \Delta<0, \tag{1}$$

$$h\log\tfrac{1}{2}(T+U\sqrt{\Delta})=-\sum_s(\Delta, s)\log\sin s\pi/\Delta, \qquad \Delta>0, \tag{2}$$

in which[222] (Δ, s) is the generalized symbol (Δ/s) of Kronecker[171]; and $0<s<\pm\Delta$.

By Dirichlet's methods, he obtained the analogue of Dirichlet's[28] formulas (5). See Lerch,[240] (4). By use of the Gauss function

$$\Psi(u)=\lim_{m=\infty}\left(\log m-\frac{1}{u+1}-\frac{1}{u+2}-\cdots\frac{1}{u+m}\right),$$

the formulas written above become

$$h=\frac{-\tau}{2\sqrt{-\Delta}}\sum_\nu(\Delta, \nu)\cot\frac{\pi\nu}{\Delta}, \quad \Delta<0; \tag{3}$$

$$h\log\tfrac{1}{2}(T+U\sqrt{\Delta})=\frac{-1}{\sqrt{\Delta}}\sum_\nu(\Delta, \nu)\left[\psi\left(\frac{-\nu}{\Delta}\right)+\psi\left(-1+\frac{\nu}{\Delta}\right)\right], \quad \Delta>0, \tag{4}$$

$0<\nu<\pm\Delta/2$ (cf. Lebesgue,[86] (1)).

For $\Delta=-m<0$ and uneven, (3) is equivalent (cf. M. Lerch,[238] (1)) to

$$h=\frac{\tau}{2\sqrt{m}}\sum_\nu\cot\frac{\pi\nu^2}{m}. \tag{5}$$

Weber transformed (p. 264) his formula (2) above by cyclotomic considerations[223] and observed that $h(\Delta)$ is odd if Δ is an odd prime or 8, and even in all other cases. (Cf. Dirichlet, Zahlentheorie, 1894, §§ 107–109.)

P. Bachmann[224] reproduced (pp. 89–145, 188–227) a great part of the class-number theory of Gauss[4, 9] Dirichlet,[98] and (pp. 228-231) Schemmel[95]; and also (pp. 437–65) the researches of Lipschitz[102] and Mertens[117] on the asymptotic value of $h(D)$.

J. de Séguier[225] showed that Kronecker's[171] formula (5_2) is valid for $D_0<0$, if in the right member, D_0 be replaced by $|D_0|$. This proof is reproduced in his[226] treatise.

J. de Séguier[226] wrote a treatise on binary quadratic forms from Kronecker's[171] later point of view making special reference[227] to two fundamental formulas of

[220] Göttingen Nachr., 1893, 138–147, 263–4.
[221] Sitzungsber.[171] Akad. Wiss. Berlin, 1885, II, 771.
[222] Cf. H. Weber, Algebra, III, 1908, § 85, pp. 322–328.
[223] Cf. Dirichlet,[98] (1); Arndt.[58]
[224] Zahlentheorie, II, Die Analytische Zahlentheorie, Leipzig, 1894.
[225] Comptes Rendus, Paris, 118, 1894, 1407–9.
[226] Formes quadratiques et multiplication complexe; deux formules fondamentales d'après Kronecker, Berlin, 1894.
[227] *Ibid.*, 133, formula (6); p. 218, formula (3).

Kronecker.[228] He extended (p. 32) Kronecker's[171] identity (4a) in Gauss sums (cf. H. Weber, Gött. Nachr., 1893, 51) to the form

$$\sum_{s=1}^{|D_0|-1}\left(\frac{D_0}{s}\right)e^{2sh\pi i/|D_0|}=\left(\frac{D_0}{h}\right)\sqrt{D_0}(-1)^{\frac{1}{4}(\operatorname{sgn} h-1)(\operatorname{sgn} D_0-1)},\quad h\neq 0,$$

where $\operatorname{sgn} x=+1$ or -1, according as x is $>$ or $x<0$, while D_0 is a fundamental discriminant.

Then, whether D is positive or negative, it follows at once in Kronecker's[171] notation that the number of primitive classes is given by

$$(1)\quad K(D_0)\log E(D_0)=\sqrt{D_0}H(D_0)=\sqrt{D_0}\sum_{n=1}^{\infty}\left(\frac{D_0}{n}\right)\frac{1}{n}$$
$$=\sum_{k=1}^{|D_0|-1}\left(\frac{D_0}{k}\right)\sum_{n=1}^{\infty}\frac{e^{2nk\pi i/|D_0|}}{n}=-\sum_{k=1}^{|D_0|-1}\left(\frac{D_0}{k}\right)\log(1-e^{2k\pi i/|D_0|}),$$

in which $E(D_0)$ is a fundamental unit; and, if $z=re^{i\theta}$, then $\log z=\log r+i\theta$, $-\pi<\theta<\pi$ (pp. 118–126). For $D_0>0$, this formula is Kronecker's[171] (5_2). Elsewhere de Séguier[225] repeated briefly his own deduction of (1).

By noting that

$$\log(1-e^{2k\pi i/|D_0|})=\log 2\sin k\pi/|D_0|+i(\tfrac{1}{2}\pi-k\pi/|D_0|),$$

he obtained from (1) two distinct formulas; one being Kronecker's[171] (5_1) and the other (p. 127) being Weber's[220] (2),

$$K(D_0)=-\frac{1}{\log E(D_0)}\sum_{k=1}^{D_0}\left(\frac{D_0}{k}\right)\log\sin\frac{k\pi}{D_0},\quad D_0>0.$$

By a study of groups of classes in respect to composition of classes, de Séguier (pp. 77–96) obtained the ratio of $Cl(D\cdot S^2)$ to $Cl(D)$. Cf. Gauss,[4] Arts. 254–256.

Denoting the Moebius function (see this History, Vol I, Ch. XIX) by ϵ_n, de Séguier found (p. 116) that for any function F which insures convergence in each member of the following formula, we have

$$\sum_{m=1}^{\infty}\left(\frac{Q^2}{m}\right)F(m)=\sum_{d|Q}\epsilon_d\sum_{n=1}^{\infty}F(nd).$$

If a, b, c are arbitrary constants (eventually integers) and F is taken such that $F(xy)=F(x)\cdot F(y)$, we have

$$\sum_{m,n}\left(\frac{Q^2}{m}\right)F(am^2+bmn+cn^2)=\sum_{d|Q}\epsilon_d\sum_{m,n}F(d)F(adm^2+bmn+\frac{c}{d}n^2),$$

$m, n=0, \pm 1, \pm 2, \ldots, \pm\infty$, except $m=n=0$. Let $F(u)$ be $\rho/u^{1+\rho}$. Since, for such a function,

$$\lim_{\rho\to 0}\sum_{m,n}F(am^2+bmn+cn^2)$$

[228] L. Kronecker,[171, 218] Sitzungsber. Akad. Wiss. Berlin, 1885, II, 779; 1889, I, 205.

depends only on b^2-4ac, we have

$$\lim_{\rho=0}\rho\sum_{m,n}\left(\frac{Q^2}{m}\right)(am^2+bmn+cn^2)^{-1-\rho}=\frac{\phi(Q)}{Q}\lim_{\rho=0}\rho\Sigma(adm^2+bmn+\frac{c}{d}n^2)^{-1-\rho}.$$

But[171]

$$\tau H(D)=K(D)\lim_{\rho=0}\rho\sum_{m,n}(am^2+bmn+cn^2)^{-1-\rho}.$$

Hence we have, for $D<0$,

$$\tau_{D_0Q^2}\frac{H(D_0Q^2)}{K(D_0Q^2)}\frac{\phi(Q)}{Q}=\Sigma\frac{\epsilon_d}{d^2}\frac{H(D_0d'^2)}{K(D_0d'^2)}\tau_{D_0d'^2}\quad(dd'=Q). \tag{4}$$

To this formula is applied the following lemma due to Kronecker[229]: Let $f(n)$, $g(n)$ be two arbitrary functions of n and let $h(n)=\Sigma f(d)g(d')$ $(dd'=n)$, and let g have the property $g(mn)=g(m)g(n)$, $g(1)=1$; then

$$f(n)=\Sigma\epsilon_d g(d)\cdot h(d')\quad(dd'=n).$$

Hence we deduce from (4) the new relation (p. 128)

$$\tau_{D_0Q^2}\frac{H(D_0Q^2)}{K(D_0Q^2)}=Q^{-1}\sum_{dd'=Q}\tau_{D_0d^2}\frac{\phi(d)}{d'}\frac{H(D_0d^2)}{K(D_0d^2)},\quad D<0.$$

For discriminants $D_1<0$, $D_2>0$, de Séguier gave the following approximation formula (p. 314):

$$K(D_1)\log E(D_1)=\frac{-\tau_{D_2}}{2K(D_2)}\Sigma\left(\frac{D_1}{A}\right)\left(\frac{\pi\sqrt{-D}}{6a}+\log a\right),$$

the summation extending over a system of primitive forms (a, b, c) of discriminant $D=D_1\cdot D_2$; while A is an arbitrary number representable by (a, b, c) and relatively prime to $2D$.

M. Lerch,[230] in the case of Kronecker's forms of negative fundamental discriminant $-\Delta\equiv 5 \pmod 8$, gave to Dirichlet's[20] equation (2) the form

$$\sum_{a,b,c}\sum_{m,n}{}' F(am^2+bmn+cn^2)=\sum_{h,k}(-\Delta/h)F(hk),$$

$m, n=0, \pm1, \pm2, \ldots$, except $m=n=0$; $h, k=1, 2, 3, \ldots$. He took

$$F(x)=(-1)^x e^{-x\pi/\sqrt{\Delta}}$$

and obtained

$$\sum_{a,b,c}\sum_{m,n}{}'(-1)^{mn+m+n}e^{-\pi(am^2+bmn+cn^2)/\sqrt{\Delta}}=\sum_{h,k}\left(\frac{-\Delta}{h}\right)(-1)^{hk}e^{-hk\pi/\sqrt{\Delta}}. \tag{1}$$

But by taking $\sigma=\tau=0$ in Kronecker's[231] fundamental formula, it is seen that the left member of (1) would vanish if it contained the terms with $m=n=0$. Hence the left member of (1) is $-Cl(-\Delta)$, and (1) can be written

$$Cl(-\Delta)=\sum_{h=1}^{\infty}\left(\frac{h}{\Delta}\right)\frac{H}{1+H},\quad H\equiv(-1)^{h-1}e^{-h\pi/\sqrt{\Delta}}.$$

[229] De Séguier's Formes quadratiques, 114; L. Kronecker, Sitzungsber. Akad. Wiss. Berlin, 1886, II, 708.
[230] Comptes Rendus, Paris, 121, 1895, 879.
[231] Sitzungsber. Akad. Wiss. Berlin, 1883, I, 505.[175]

By expressing the right member in terms of a θ-function,[175] we obtain

$$\sum_{\rho=1}^{\Delta-1}\left(\frac{\rho}{\Delta}\right)\frac{\theta_1'\left(\frac{\rho}{\Delta}\,\middle|\,\frac{-2}{\Delta+i\sqrt{\Delta}}\right)}{\theta_1\left(\frac{\rho}{\Delta}\,\middle|\,\frac{-2}{\Delta+i\sqrt{\Delta}}\right)}=\gamma Cl(-\Delta)\cdot(\Delta+i\sqrt{\Delta})\pi i,$$

$$\gamma=\tfrac{5}{8}\text{ if }\Delta=3;\qquad \gamma=1\text{ if }\Delta>3.$$

G. Osborn,[282] from Dirichlet's[23] formulas (6) and his own elementary theorems[283] on the distribution of quadratic residues, drew the immediate conclusion that the number of properly primitive classes of determinant $-N$, N a prime, is

$$\tfrac{1}{2}(N-1)-\frac{2}{N}\Sigma(R),\qquad N=8m-1>0,$$

but is 3 times that number if $N=8m+3>0$, where $\Sigma(R)$ is the sum of the quadratic residues of N between 0 and N.

*R. Götting[284] found transformations of the more complicated of Dirichlet's[23] closed expressions for class-numbers of negative determinants.

A. Hurwitz[285] denoted by $h(D)$ the number of classes of properly primitive positive forms of negative determinant $-D$. Let p be a prime $\equiv 3 \pmod 4$ and write $p'=\frac{1}{2}(p-1)$. Since $(s/p)\equiv s^{p'} \pmod p$, Dirichlet's[25] result (5_1) implies

$$h(p)\equiv 1^{p'}+2^{p'}+\ldots+p'^{p'} \pmod p.$$

The right member is the coefficient of

$$(1)\qquad (-1)^{\frac{1}{2}(p'-1)}x^{p'}/p'!$$

in the expansion of

$$\phi(x)=\sin x+\sin 2x+\ldots+\sin p'x=\frac{\cos\frac{1}{2}x-\cos\frac{1}{2}px}{2\sin\frac{1}{2}x}.$$

This numerator is congruent to $\cos\frac{1}{2}x-1$ modulo p, and by applying a theorem on the congruence of infinite series, we get

$$\phi(x)\equiv\frac{\cos\frac{1}{2}x-1}{2\sin\frac{1}{2}x}=\frac{-2\sin^2\frac{1}{4}x}{4\sin\frac{1}{4}x\cos\frac{1}{4}x}\equiv-\tfrac{1}{2}\tan\tfrac{1}{4}x \pmod p.$$

But when x is replaced by $4x$, (1) is multiplied by $4^{p'}$ or $2^{p-1}\equiv 1 \pmod p$. Hence $h(p)$ is congruent modulo p to the coefficient of (1) in the expansion of $-\frac{1}{2}\tan x$. When $p\equiv 1 \pmod 4$, we employ the expansion of $\frac{1}{2}\sec x$. Other such theorems give $h(2p)$.

The same result of Dirichlet is used to prove that if $q\equiv 1 \pmod 4$ and q has no square factor >1, and if

$$\frac{1}{\cos qx}\left\{\left(\frac{1}{q}\right)\sin x-\left(\frac{3}{q}\right)\sin 3x+\left(\frac{5}{q}\right)\sin 5x-\ldots-\left(\frac{q-2}{q}\right)\sin(q-2)x\right\}$$
$$=c_1x+c_2x^3/3!+c_3x^5/5!+\ldots,$$

[282] Messenger Math., 25, 1895, 157.
[283] *Ibid.*, 45.
[284] Program No. 257 of the Gymnasium of Turgau, 1895.
[285] Acta Math., 19, 1895, 351–384.

and if $p\equiv 3 \pmod 4$ is a prime not dividing q, then

$$h(pq)\equiv(-1)^{\frac{1}{2}(p+1)}c_{\frac{1}{2}(p+1)} \pmod p.$$

There are analogous theorems for $h(pq)$ and $h(2pq)$ for all combinations of residues 1 and 3 (mod 4) of p and q.

To obtain a lower bound for the number of times that 2 may occur as a divisor of h, the number of genera of the properly primitive order is calculated.[286] If $h_g(D)$ denote the number of classes in a properly primitive genus of determinant $-D$, the parities of $h_g(pq)$ and $h_g(2pq)$ depend only on the values of (p/q) and p (mod 8) and q (mod 8), and are shown in tables.

By combining the two theories of this memoir one obtains, for special q, results such as the following:

If $p\equiv 3 \pmod 4$, $h(5p)$ is the least positive residue modulo $2p$ of $(-1)^{\frac{1}{2}(p+1)}c_{\frac{1}{2}(p+1)}$, where $c_1, c_2, \dots$ are the coefficients in the expansion

$$\frac{\sin x+\sin 3x}{\cos 5x}=c_1x+c_2\frac{x^3}{3!}+\dots+c_n\frac{x^{2n-1}}{(2n-1)!}+\dots.$$

F. Mertens[287] completed the solution of Gauss' problem (Disq. Arith.[4], Art. 256) to find by the composition of forms the ratio of the number of the properly primitive classes of the determinant $S^2\cdot D$ to that of D. He modified Gauss' procedure by taking *schlicht* forms (Mertens[87] of Ch. III) as the representatives of classes and by means of them found for any determinant the number of primitive classes which when compounded with an arbitrary class of order S would produce an arbitrary class of order S (Mertens[87] of Ch. III).

M. Lerch[288] rediscovered Lebesgue's[86] class-number formula (1) above, and wrote it for the case $\Delta=p=4m+3$, a prime:

$$2\Sigma'\cot\frac{k\pi}{p}=\frac{4\sqrt{p}}{\tau}Cl(-p),\quad k=1,2,\dots,p-1,\ \left(\frac{k}{p}\right)=1.$$

By replacing k by $a^2-p[a^2/p]$, he obtained Weber's formula[220] (5):

$$\frac{2\sqrt{p}}{\tau}Cl(-p)=\sum_{a=1}^{\frac{1}{2}(p-1)}\cot\frac{a^2\pi}{p}. \tag{1}$$

He found for $\Delta=4p$, $p=4m+1$, a prime >1,

$$\sqrt{p}Cl(-4p)=\sum_\nu\frac{1}{\sin\nu^2\pi/(2p)}-\frac{p-1}{2}\ (\nu=1,3,5,\dots,p-2). \tag{2}$$

For $\Delta=8p$, Lerch derived more complicated formulas which are analogous to (1) and (2).

L. Gegenbauer[289] in a paper on determinants of m dimensions and order n, stated the following theorem. If for $k=1,\dots,n$ in turn in a non-vanishing determinant of even order m, we replace, in the sequence of elements which belong to any particular

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[286] C. F. Gauss, Disq. Arith., Art. 252; G. L. Dirichlet, Zahlentheorie, Supplement IV, ed. 4, 1894, 313–330.
[287] Sitzungsber. Akad. Wiss. Wien, 104, IIa, 1895, 103–137.
[288] Sitzungsber. Böhm. Gesells. Wiss., Prague, 1897, No. 43, 16 pp.
[289] Denkschrift Akad. Wiss. Wien, Math.-Natur., 57, 1890, 735–52.

τth index, the elements which belong to the σth index k, by the corresponding elements respectively which have the σth index $k^2+k+\Delta$, where $-\Delta$ is a negative fundamental discriminant and where all the indices are taken modulo n; and if we divide each of the resulting determinants by the original, the product of $\sqrt{\Delta}$ by the sum of the quotients has mean value, $G(-\Delta)$, when n becomes infinite (p. 749). Three similar theorems include a case of n finite.

M. Lerch[240] employed

$$E^*(x)=x-\tfrac{1}{2}+\sum_{\nu=1}^{\infty}\frac{\sin 2\nu x\pi}{\nu\pi}.$$

Then $E^*(x)=[x]$ if $x>0$ is fractional, but $=[x]-\frac{1}{2}$ if x is an integer. In the initial equation, x is replaced by $x+am/\Delta$, where $-\Delta$ is a negative fundamental discriminant; each member is then multiplied by $(-\Delta/a)$ and summed for $a=1, 2, 3, \ldots, \Delta-1$. Since [a misprint is corrected here],

$$(1)\qquad \sum_{a=1}^{\Delta-1}\left(\frac{-\Delta}{a}\right)=0,\quad \left(\frac{-\Delta}{\Delta-a}\right)=-\left(\frac{-\Delta}{a}\right),$$

it follows from the theory of Gauss' sums (cf. G. L. Dirichlet, Zahlentheorie, Art. 116, ed. 4, 1894, p. 303) that

$$\sum_{a=1}^{\Delta-1}\left(\frac{-\Delta}{a}\right)E^*\left(x+\frac{am}{\Delta}\right)=\frac{m}{\Delta}\sum_{a=1}^{\Delta-1}\left(\frac{-\Delta}{a}\right)a+\left(\frac{-\Delta}{m}\right)\frac{\sqrt{\Delta}}{\pi}\sum_{\nu=1}^{\infty}\left(\frac{-\Delta}{\nu}\right)\frac{\cos 2\nu x\pi}{\nu}.$$

Then by Kronecker's[171] formula (5_1) we have

$$(2)\quad \sum_{a=1}^{\Delta-1}\left(\frac{-\Delta}{a}\right)E^*\left(x+\frac{am}{\Delta}\right)+\frac{2m}{\tau}Cl(-\Delta)=\left(\frac{-\Delta}{m}\right)\frac{\sqrt{\Delta}}{\pi}\sum_{\nu=1}^{\infty}\left(\frac{-\Delta}{\nu}\right)\frac{\cos 2\nu x\pi}{\nu}.$$

By comparing this result with the case $m=1$, we have for $x=0$,

$$(3)\qquad \frac{2}{\tau}\left[m-\left(\frac{-\Delta}{m}\right)\right]Cl(-\Delta)=-\sum_{a=1}^{\Delta-1}\left(\frac{-\Delta}{a}\right)E^*\left(\frac{am}{\Delta}\right).$$

For m not divisible by Δ, $E^*(am/\Delta)$ is equal to $[am/\Delta]$. Taking $m=2$ and applying (1), we get[241]

$$(4)\qquad \frac{2}{\tau}\left[2-\left(\frac{2}{\Delta}\right)\right]Cl(-\Delta)=\sum_{a=1}^{\frac{1}{2}(\Delta-1)}\left(\frac{-\Delta}{a}\right).$$

Hereafter we take $\Delta>4$, i. e., $\tau=2$. Then, for $m=4$, we have

$$(5)\qquad \left[4-\left(\frac{4}{\Delta}\right)\right]Cl(-\Delta)=-\sum_{a=1}^{\Delta-1}\left(\frac{-\Delta}{a}\right)\cdot\left[\frac{4a}{\Delta}\right].$$

When we put $S(a,\ldots,b)$ for $\Sigma_a^b(-\Delta/a)$, formula (5) is reduced by means of (1) to

$$3S\left(0,\ldots,\frac{\Delta}{4}\right)+S\left(\frac{\Delta}{4},\ldots,\frac{\Delta}{2}\right)=\left[4-\left(\frac{4}{\Delta}\right)\right]Cl(-\Delta).$$

But (4) is equivalent to

$$S\left(0,\ldots,\frac{\Delta}{4}\right)+S\left(\frac{\Delta}{4},\ldots,\frac{\Delta}{2}\right)=\left[2-\left(\frac{2}{\Delta}\right)\right]Cl(-\Delta).$$

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[240] Bull. des sc. math. (2), 21, I, 1897, 290–304.
[241] Cf. H. Weber,[220] Göttingen Nachr., 1893, 145.

By combining the last two formulas we obtain the two serviceable ones

$$\text{(6)}\qquad \sum_{a=1}^{[\Delta/4]}\left(\frac{-\Delta}{a}\right)=\frac{1}{2}\left[2+\left(\frac{2}{\Delta}\right)-\left(\frac{4}{\Delta}\right)\right]Cl(-\Delta),$$

$$\text{(7)}\qquad \sum_{a=[\Delta/4]+1}^{\frac{1}{2}(\Delta-1)}\left(\frac{-\Delta}{a}\right)=\frac{1}{2}\left[2-3\left(\frac{2}{\Delta}\right)+\left(\frac{4}{\Delta}\right)\right]Cl(-\Delta).$$

A still more expeditious formula is obtained by taking $m=3$ in (3), whence

$$\sum_{a=1}^{[\Delta/8]}\left(\frac{-\Delta}{a}\right)=\frac{1}{2}\left[3-\left(\frac{\Delta}{3}\right)\right]Cl(-\Delta);$$

and this relation combined with (6) yields

$$\sum_{a=[\Delta/4]+1}^{[\Delta/8]}\left(\frac{-\Delta}{a}\right)=\frac{1}{2}\left[1-\left(\frac{\Delta}{2}\right)+\left(\frac{\Delta}{3}\right)+\left(\frac{\Delta}{4}\right)\right]Cl(-\Delta).$$

For $m=1$, (2) becomes

$$\text{(8)}\qquad \frac{2}{\tau}\,Cl(-\Delta)=\frac{\sqrt{\Delta}}{\pi}\sum_{\nu=1}^{\infty}\left(\frac{-\Delta}{\nu}\right)\frac{\cos 2\nu x\pi}{\nu}\quad\left(0\leqq x<\frac{1}{\Delta}\right).$$

This is a generalization of Dirichlet's[19] formula (1) and it holds for $-\Delta$ not a fundamental discriminant. Lerch showed that (8) is valid for any negative discriminant when $0\leqq x<1/\Delta$ by reducing it from Dirichlet's[19] formula (1). By simply integrating (8), he deduced

$$Cl(-\Delta)=\frac{\tau\Delta\sqrt{\Delta}}{4\pi^2}\sum_{\nu=1}^{\infty}\left(\frac{-\Delta}{\nu}\right)\frac{\sin 2\nu\pi/\Delta}{\nu^2},$$

$$Cl(-\Delta)=\frac{\tau\Delta^2\sqrt{\Delta}}{2\pi^3}\sum_{\nu=1}^{\infty}\left(\frac{-\Delta}{\nu}\right)\frac{\sin^2\nu\pi/\Delta}{\nu^3}.$$

M. Lerch[242] applied to Kronecker forms $ax^2+bxy+cy^2$ the unit substitution and for a given value of $b^2-4ac=D<0$ studied the number of principal roots ω of reduced forms which would lie in the fundamental region.[128] By arithmetical methods he obtained cumbersome formulas, involving the Legendre symbol $E(x)$, for $\Sigma F(4k)$ and $\Sigma F(4k-1)$, summed for $k=1, 2, \ldots, n$, where $F(\Delta)$ denotes the number of classes of discriminant $-\Delta$. He identified these results with the concise ones of Hermite[211] which had been obtained from elliptic functions for forms $ax^2+2bxy+cy^2$.

Lerch[243] in an expository article, deduced for negative and positive discriminants Dirichlet's[19] class-number formulas (1) in which enters $P(D)=\Sigma_1^\infty(D/h)/h$. For an arbitrary discriminant D, where $|D|=\Delta$, he found by logarithmic differentiation of the ordinary Γ-function that

$$P(D)=-\frac{1}{\Delta}\sum_{k=1}^{\infty}\left(\frac{D}{k}\right)\Gamma'\left(\frac{k}{\Delta}\right)\Big/\Gamma\left(\frac{k}{\Delta}\right).$$

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[242] Rozpravy české Akad., Prague, 7, 1898, No. 4, 16 pp. (Bohemian).

[243] Rozpravy české Akad., Prague, 7, 1898, No. 5, 51 pp. (Bohemian); resumé in French, Bull. de l'Acad. des Sc. Bohème, 5, 1898, 33–36.

To this he applied the identity:

$$\Gamma'\left(\frac{k}{\Delta}\right)\Big/\Gamma\left(\frac{k}{\Delta}\right)-\Gamma'(1)=-\log 2\Delta-\frac{\pi}{2}\cot\frac{k\pi}{\Delta}+\sum_{a=1}^{\Delta-1}\cos\frac{2ak\pi}{\Delta}\log\sin\frac{a\pi}{\Delta}.$$

For the fundamental discriminant D_0, this furnishes familiar formulas including, e. g., for $D_0>0$, Weber's[220] formula (1).

Lerch[244] repeated the deduction of his[240] formula (8) and established the validity of the formula for a non-fundamental discriminant D for the interval $0\leqq x<1/(\Delta_0 Q')$, where $D=\Delta_0\cdot Q$ and Q' is the product of the distinct factors of Q.

Lerch[245] transformed the Gauss sum

$$\sum_{a=0}^{n-1} e^{2a^2m\pi i/n}$$

as it occurs in class-number formulas (cf. G. L. Dirichlet, Zahlentheorie, Arts. 103, 115) and so obtained finally

$$(1)\qquad \sum_{a=1}^{n-1}\left\{\frac{a^2m}{n}-E\left(\frac{a^2m}{n}\right)\right\}=\frac{n-q}{2}-\sum_{d}\left(\frac{m}{d}\right)\frac{2}{\tau_d}Cl(-d),$$

where m, n are relatively prime positive integers, n is uneven and q^2 its greatest square divisor, while d ranges over the divisors of n which are $\equiv 3 \pmod 4$. Lerch has since[274] repeated the deduction in detail. From (1) follows [74]

$$(2)\ \sum_{a=1}^{n-1}\left\{\tfrac{1}{4}+\frac{a^2m}{n}-E\left(\tfrac{1}{4}+\frac{a^2m}{n}\right)\right\}$$
$$=\frac{2n-1}{4}-(-1)^{\frac{n-1}{2}}\sum_{d_1}\left(\frac{m}{d_1}\right)\frac{2}{\tau_{4d_1}}Cl(-4d_1)-\sum_{d_3}\left(\frac{m}{d_3}\right)\frac{1-(2/d_3)}{\tau_{d_3}}Cl(-d_3),$$

in which d_1 and d_3 range over the divisors of n such that $d_1\equiv 1$, $d_3\equiv 3 \pmod 4$.

J. de Séguier[246] in a paper primarily on certain infinite series and on genera simplified his results by substituting the class-number for its known value. He found, for example (p. 114), if $F(x)$ is an arbitrary function which insures convergence, then

$$\frac{1}{K(D_0Q^2)}\sum_{a,b,c}^{D_0Q^2}\left(\frac{D_1}{A}\right)F(am^2+bmn+cn^2)$$
$$=\sum_{d}^{Q}\frac{O(D_1,d)\tau(D_0d^2)}{K(D_0d^2)}\sum_{hk}\left(\frac{D_1d^2}{h}\right)\left(\frac{D_0D_1^{-1}d^4}{k}\right)F(d'^2hk),$$

where $K(m)$ is the number of properly primitive classes of discriminant m; A is representable by $am^2+bmn+cn^2$; $D=D_1D_2=D_0Q^2$, D_0 being fundamental; and $O(D_1, d)$ is the number of classes of discriminant D_1 and of order d, where $dd'=Q$.

*J. S. Aladow[247] evaluated in four separate cases the number G of classes of odd binary quadratic forms of prime negative determinant $-p$:

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[244] Rozpravy české Akad., Prague, 7, 1898, No. 6; French resumé in Bull. de l'Acad. des Sc. Bohême, 5 1898, 36–37.

[245] Rozpravy české Akad., Prague, 7, 1898 No. 7 (Bohemian). French resumé in Bull de l'Acad. des Sc. Bohême, Prague, 5, 1898, 37–38.

[246] Jour. de Math. (5), 5, 1899, 55–115.

[247] St. Petersburg Math. Gesells., 1899, 103–5 (Russian).

(i) If $p\equiv 7 \pmod 8$, G equals the difference between the number of quadratic residues and non-residues $\leqq \frac{1}{8}\{p-3-2(3/p)\}$.

(ii) If $p\equiv 3 \pmod 8$, G equals the difference between the number of quadratic residues in the sequence

$$\tfrac{1}{4}(p+1),\quad \tfrac{1}{4}(p+5),\ldots,\quad \tfrac{1}{8}\{2p-3-(3/p)\},$$

and the number in the sequence

$$\tfrac{1}{8}\{p+3-2(3/p)\},\ldots,\quad \tfrac{1}{4}(p-3).$$

(iii) If $p\equiv 5 \pmod 8$, G equals twice the difference between the number of quadratic residues and non-residues in the sequence

$$\tfrac{1}{8}\{p+3+2(3/p)\},\quad \tfrac{1}{8}\{p+9+2(3/p)\},\ldots,\quad \tfrac{1}{4}(p-1).$$

(iv) If $p\equiv 1 \pmod 8$, G equals twice the sum of the difference between the number of quadratic residues and non-residues in the sequence

$$\tfrac{1}{4}(p+3),\ldots,\quad \tfrac{1}{8}\{2p-3+(3/p)\}$$

and the corresponding difference in the sequence

$$\tfrac{1}{8}\{p+3+2(3/p)\},\quad \tfrac{1}{8}\{p+9+2(3/p)\},\ldots,\quad \tfrac{1}{4}(p-1).$$

R. Dedekind,[248] in a long investigation of ideals in a real cubic field, proved the following result. If at least one of the integers a, b, ab is divisible by no square, and if we write $k=3ab$ or $k=ab$, according as a^2-b^2 is not or is divisible by 9, then the number of all non-equivalent, positive, primitive forms $Ax^2+Bxy+Cy^2$ of discriminant $D\equiv B^2-4AC=-3k^2$ is a multiple $3K$ of 3. For primes $p\equiv 1 \pmod B$, p not dividing D, K of the forms represent all and only such primes p of which ab^2 is a cubic residue, while the remaining $2K$ forms represent all and only such primes p of which ab^2 is a cubic non-residue.

D. N. Lehmer[249] calls any point in the cartesian plane a *totient point* if its two co-ordinates are integers and relatively prime. He wrote

$$P_{(m,k)}=\prod_{i=1}^{r}\frac{p_i-1}{p_i^{\alpha_i-1}(p_i^{m+1}-1)},\qquad k=\prod_{i=1}^{r}p_i^{\alpha_i}.$$

The number of totient points[250] in the ellipse $ax^2+2bxy+cy^2=N$, $b^2-4ac=D=-\Delta$, is

$$\text{(1)}\qquad \frac{12N}{\pi}\sqrt{\Delta}P_{(1,\,2\Delta)};$$

and in the hyperbolic sector, always taken[251] in this connection, the number is

$$\text{(2)}\qquad \frac{6}{\pi^2}\sqrt{D}P_{(1,\,2D)}N\log(T+U\sqrt{D}),$$

N being very great in both cases. Noting now Dirichlet's[93] formula (2) for the

[248] Jour. für Math., 121, 1900, 95.
[249] Amer. Jour. Math., 22, 1900, 293–335. Cf. Lehmer,[218] Ch. V, Vol. I. of this History.
[250] Cf. G. L. Dirichlet,[20] Zahlentheorie, Art. 95.
[251] Cf. *ibid.*,[19] Art. 98, ed. 4, 1894, 246.

number of representations of a given number by a system of quadratic forms of determinant D, he finds the class-number, for example, for $D=-\Delta<0$,

$$h(D)=\epsilon\frac{\pi}{12}\frac{1}{\sqrt{\Delta}}P_{(1,\,2\Delta)}\lim_{N=\infty}\frac{1}{N}\sum_{x=1}^{N}2^{\nu(x)}\Theta_s(x),$$

in which ϵ is the number of solutions of $t^2-Du^2=1$; x is any positive number relatively prime to $2D$, $\nu(x)$ is the number of distinct prime factors of x; $\Theta_s(x)=1$ or 0, according as each prime divisor does or does not have D as a quadratic residue.

K. Petr,[252] by the use of five functions A (=Hermite's[69] $\mathscr{A}$), B, C, D, E, all analogous to Hermite's[69] $\mathscr{A}$, deduced all of Kronecker's[54] eight classic relations.

For example, from expansions by C. Jordan (Cours d'analyse, II, 1894, 409–411), he obtained

$$(1)\qquad \Theta_3\Theta_1^2\frac{\Theta^2(v)\Theta_1(v)}{\Theta_3^2(v)}=C\cdot\Theta_1(v)-8\sum_1^{\infty}\cos(2n+1)\pi v\cdot q^{(n+\frac{1}{2})^2}\sum_1^{n}kq^{-k^2}.$$

Also C is the coefficient[253] of $2q^{\frac{1}{4}}\cos\pi v$ in the product of the right member of

$$(2)\qquad \Theta_1\frac{\Theta(v)\Theta_1(v)}{\Theta_2(v)}=2\sum_1^{\infty}\sin 2n\pi v q^{n^2}\cdot\{2q^{-\frac{1}{4}}+2q^{-\frac{9}{4}}+\ldots+2q^{-(n-\frac{1}{2})^2}\}$$

by the right member of[254]

$$(3)\qquad \Theta_1\Theta_s\frac{\Theta(v)}{\Theta_2(v)}=\frac{4q^{\frac{1}{2}}\sin\pi v}{1-q}+\frac{4q^{3/2}\sin 3\pi v}{1-q^3}+\frac{4q^{5/2}\sin 5\pi v}{1-q^5}+\ldots.$$

But in that product, the coefficient of $\cos\pi v$ is a power series in q in which the coefficient of $q^{N+\frac{1}{4}}$ is 8 times the combined number of solutions of

$$n^2-(k+\tfrac{1}{2})^2+(n-\tfrac{1}{2})(2l+1)=N+\tfrac{1}{4},$$
$$n^2-(k+\tfrac{1}{2})^2+(n+\tfrac{1}{2})(2l+1)=N+\tfrac{1}{4},$$

where n and l are positive integers, l taking also the value zero; $k=0, 1, 2, \ldots, n-1$. But these equations can be written[255] in the forms

$$(4)\qquad \begin{cases}(n-k+1)(n+k)+(n-k-1)(l+1)+(n+k)(l+1)=N,\\ (n-k)(n+k+1)+(n-k)(l)+(n+k+1)(l)=N;\end{cases}$$

and the left members may be regarded as the discriminants $N=ab+bc+ca$ of reduced Selling[255a] quadratic forms $a(y-t)^2+b(t-x)^2+c(x-y)^2$, in which a, b, c do not agree in parity. Since there is a correspondence between such Selling forms of discriminant N and odd classes of Gauss forms of determinant $-N$, we have

$$(5)\qquad C=8\Sigma F(n)q^n.$$

The identity (Fundamenta Nova, § 41)

$$\Theta_1^2\Theta_3^2\frac{\Theta^2(v)}{\Theta_3^2(v)}=8\Sigma\frac{nq^n}{1-q^{2n}}-8\Sigma\frac{nq^n\cos 2\pi nv}{1-q^{2n}}$$

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252 Rozpravy české Akad., Prague, 9, 1900, No. 38 (Bohemian); Abstract,[261] Bull. Internat. de l'Acad. des Sc. de Bohème, Prague, 7, 1903, 180–187 (German).

253 Cf. P. Appell, Annales de l'Ecole Norm. Sup. (3), 1, 1884, 135; 2, 1885, 9.

254 Cf. C. G. J. Jacobi, Fundamenta Nova 1829, p. 101, (19); Werke I, 1881, 157.

255 Cf. J. Liouville,[88] Jour. de Math. (2), 7, 1862, 44; Bell,[370] and Mordell.[372]

255a E. Selling, Jour. für Math., 77, 1874, 143.

is multiplied member by member with Jacobi's expansion formula for $\Theta_1(v)$. In the resulting left member, the coefficient of $\cos \pi v$ is $C\cdot 2q^{\frac{1}{4}}\Theta_3$. When this coefficient is equated to the coefficient of $\cos \pi v$ in the resulting right member, a comparison with (5) yields the relation:

$$\text{(I)}\qquad F(n)+2F(n-1^2)+2F(n-2^2)+\ldots=\Sigma d_\lambda-\Sigma d_1,$$

where d_λ denotes a divisor of n which has an odd conjugate and d_1 denotes a divisor of n which is $\leqq \sqrt{n}$ and which agrees with its conjugate in parity.

He also found the classic formula[54] for the number of solutions of $x^2+y^2+z^2=n$.

To obtain a class-number relation of Liouville's[256] second type, Petr expands in powers of v each member of an identity of the same general type as (1) above. Coefficients of v^2 are equated, with the result that

$$1^2F(8n-1^2)+3^2F(8n-3^2)+5^2F(8n-5^2)+\ldots$$
$$=2n\Sigma d_\lambda-2n\Sigma(d_{1l}+d_{1\lambda})-\Sigma(d_{1l}^3+d_{1\lambda}^3),$$

where, the d's are the divisors of $2n$; $d_1<\sqrt{2n}$; d_l is odd; d_λ has an odd conjugate; and the subscripts of d retain their significance when they are compounded.

To obtain a class-number relation of Liouville's[257] first type, each member of an identity of the same general type as (1) above is expanded in the neighborhood of $v=\frac{1}{2}$. Equating coefficients of v, Petr then obtains

$$H(8n-1^2)-3H(8n-3^2)+5H(8n-5^2)-\ldots=\Sigma(-1)^{\frac{1}{2}(d'_1+d'_2+1)}d_1'^2,$$

where d'_1 is a divisor of $2n$ such that its conjugate d'_2 is of different parity, and $d'_1<\sqrt{2n}$.

K. Petr,[258] employing the same notation as[252] in 1900, multiplied member by member the identity

$$\Theta_1^2\Theta_2\Theta_3^2\frac{\Theta(v)}{\Theta_2^2(v)}=8\Sigma(-1)^n(n+k)q^{n^2-\frac{1}{4}+k(2n+1)}\sin(2k-1)\pi v,$$
$$n=0,1,2,3,\ldots;\ k=1,2,3,\ldots.$$

by the formula for transformation of order 2

$$\Theta(v)\Theta_1(v)/\Theta_2(0,2\tau)=\Theta(2v,2\tau).$$

In the resulting left member, the coefficient of $q^{\frac{1}{4}}\cos\pi v$ is $16\Sigma F(n)q^n\Theta_2(0,2\tau)$; in the right member it is 8 times the sum of

$$(-1)^{n+k-1}(n+2k)q^{(n+2k)^2-2k^2},\qquad (-1)^{n+k-1}(n+2k-1)q^{(n+2k-1)^2-2(k-1)^2},$$

for $n=0,1,2,3,\ldots;k=1,2,3,\ldots$. Hence

$$(1)\qquad \Sigma(-1)^\nu F(n-2\cdot\nu^2)=\Sigma(-1)^{x+y-1}x,$$

where x and y are the integer solution of $x^2-2y^2=n$, $x\gtreqless 2y$, $y\gtreqless 0$; while, as also in

[256] J. Liouville,[107] Jour. de Math., (2) 12, 1867, 99. Cf. G. Humbert,[298] Jour. de Math., (6), 3, 1907, 369–373, formulas (40)—(44), as numbered in the original memoir.

[257] Cf. J. Liouville,[107] Jour. de Math. (2), 14, 1866, 1; also G. Humbert,[298] *ibid.* (6), 3, 1907, 366–369, formulas (35), (36).

[258] Rozpravy české Akad., Prague, 10, 1901, No. 40 (Bohemian). Abstract, Bull. Internat. de l' Acad. des Sc. de Bohême Prague, 7, 1903 180–187, (German).

(2), ν ranges over all integers, positive, negative, or zero. In the summation x receives an extra coefficient $\frac{1}{2}$ if one of the inequalities becomes an equality. Similarly,

$$(2)\quad \Sigma(-1)^\nu F(8n-1-8\nu^2)=\Sigma(-1)^{\frac{1}{2}(x+y)}y,\quad x^2-2y^2=8n-1,\quad x>2y,\quad y>0.$$

These are the first published class-number relations which are obtained from elliptic function theory and which involve an indefinite quadratic form, e. g., x^2-2y^2.

By means of the elementary relation

$$\pi\Theta_1\Theta_2\Theta_3=2\pi\Sigma(-1)^n(2n+1)q^{(n+\frac{1}{2})^2},\quad n=0,1,2,3,\ldots$$

and the relation

$$\Theta_1^2\Theta_3=4\Sigma F(4n+2)q^{\frac{1}{4}(4n+2)}$$

the identity $\Theta_1^2\Theta_3\cdot\Theta_2=\Theta_1\Theta_2\Theta_3\cdot\Theta_1$ yields

$$F(4n+2)-2F(4n+2-4\cdot1^2)+2F(4n+2-4\cdot2^2)+\ldots=\Sigma(-1)^{\frac{1}{2}(x-1)}x,$$

$x, y>0$, $x^2+y^2=4n+2$; which is of the type of Hurwitz.[202]

A transformation formula of order 3 in a treatment similar to the above yields five such relations as

$$F(4n+3)-H(4n+3)-2[F(4n+3-3\cdot1^2)-H(4n+3-3\cdot1^2)]+\ldots=\Sigma(-1)^{\frac{1}{2}(x+y-1)}y,$$

$x^2+3y^2=4n+3$, $x\gtreqless 0$, $y>0$; and

$$(3)\qquad F(4n)-2F(4n-3\cdot1^2)+2F(4n-3\cdot2^2)-\ldots=-2\Sigma x,$$

$x^2-3y^2=2n$, $y\gtreqless 0$, $x\gtreqless 3y$.

From transformations of order 5, Petr obtained three relations including,

$$(4)\qquad F(8n)-2F(8n-5\cdot1^2)+2F(8n-5\cdot2^2)-\ldots=-4\Sigma x$$

$x^2-5y^2=2n$, $y\gtreqless 0$, $5y\leqq x$.

M. Lerch[259] wrote

$$R(\omega,s)=\sum_{\nu=0}^{\infty}\frac{1}{(\omega+\nu)^s},\quad K(a,b,c;s)=\Sigma'(am^2+bmn+cn^2)^{-s},$$

where ω is an arbitrary constant; $m, n=0, \pm1, \pm2, \ldots$, except $m=n=0$; (a, b, c), a positive form of negative discriminant $-\Delta$; a, b, c real. From Dirichlet's[20] fundamental equation (2), it follows that the relation

$$(1)\qquad \sum_{a,b,c}K(a,b,c;s)=\tau\Delta^{-s}R(1,s)\sum_{r=1}^{\Delta-1}\left(\frac{-\Delta}{r}\right)R\left(\frac{r}{\Delta},s\right)$$

is valid over the complex s-plane, if (a, b, c) ranges over a system of representative primitive positive forms of discriminant $-\Delta$, which is now supposed to be fundamental.

[259] Comptes Rendus, Paris, 135, 1902, 1314–1315.

Employ the Maclaurin developments in powers of s,

$$(2)\qquad R(\omega, s) = (\tfrac{1}{2} - \omega) + \log \frac{\Gamma(\omega)}{\sqrt{2\pi}} \cdot s + \ldots;$$

$$(3)\qquad K(a, b, c; s) = -1 - 2s \log \left[\frac{2\pi}{\sqrt{c}} H\left(\frac{-b+i\sqrt{\Delta}}{2c}\right) H\left(\frac{b+i\sqrt{\Delta}}{2c}\right)\right] + \ldots,$$

where

$$H(\omega) = e^{\omega\pi i/12} \prod_1^\infty (1 - e^{2n\omega\pi i}).$$

When substitution is made of (2) and (3) in (1), Lerch compares the terms which are independent of s and obtains Kronecker's[171] class-number formula (5).

E. Laudau[260] showed that every negative determinant < -7 has more than one properly primitive reduced form (cf. the conjecture of Gauss,[4] Disq. Arith., Art. 303) by proving that if $-\Delta = b^2 - ac$ is < -7, there is always another such form in addition to $(1, 0, \Delta)$. If there is no properly primitive reduced form $(a, 0, c)$ other than $(1, 0, \Delta)$, then Δ has no distinct factors, but must be of the form p^λ, p a prime.

(I) If $p=2$, and $\lambda \gtreqless 4$, there is the additional properly primitive reduced form $(4, 2, 2^{\lambda-2}+1)$.

(II) If p is an odd prime and if there is no reduced properly primitive form with $b=1$, then $\Delta+1$ cannot be expressed as $a \cdot c$, where one of the factors is uneven and >2. Hence $\Delta+1=2^\nu$. When $\nu \gtreqless 6$, there is an additional properly primitive reduced form $(8, 3, 2^{\nu-3}+1)$.

Landau now tested the few remaining admissible Δ's and found none which are >7 and have a single class.

K. Petr[261] gave in German an abstract of his two long Bohemian papers,[252, 258] including eleven class-number relations of the second paper. He indicated completely a method of expanding $\Theta_1^3\Theta_1(0, 5\tau)$, which leads to new expressions[340] for the number of solutions of $x^2+y^2+z^2+5w^2=n$ and hence to generalizations of Petr's[258] relation (4).

M. Lerch,[262] in order to find the negative discriminants $-\Delta$ for which $Cl(-\Delta)=1$, wrote $-\Delta = -\Delta_0 Q^2$, where Δ_0 is fundamental and q ranges over the distinct factors of $Q = Q'\Pi q$. Then the equation to be satisfied is (Kronecker,[171] (4))

$$Cl(-\Delta) = \frac{2}{\tau_0} Q' \prod_q \left\{q - \left(\frac{-\Delta_0}{q}\right)\right\} Cl(-\Delta_0) = 1.$$

If $\Delta_0 = 4$, then $\tau_0 = 4$, $Q = 1$ or 2.

If $\Delta_0 = 3$, then $\tau_0 = 6$, $Q = 1, 2$ or 3.

If $\Delta_0 > 4$, then $\tau_0 = 2$. Here $Cl(-\Delta)$ can be uneven only for $Q'=1$ and Δ_0 prime, or for $\Delta_0 = 8$. The case $\Delta_0 = 8$ is excluded if $Q \neq 1$. If Δ_0 is a prime, $Cl(-\Delta) > 1$ unless $Q = q = 2$, $(2/\Delta_0) = 1$, i. e., $\Delta_0 = 8k-1$. But if $k \gtreqless 2$, $(1, 1, 2k)$ and $(2, 1, k)$ are non-equivalent reduced forms of discriminant $-\Delta_0$.

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260 Math. Annalen, 56, 1902, 671–676.

261 Bull. Internat. de l'Acad. des Sc. de Bohème, Prague, 7, 1903, 180–187.

262 Math. Annalen, 57, 1903, 569–570.

Hence $Cl(-\Delta)=1$ for $\Delta=4, 8$; $3, 12, 27$; $8, 7, 28$. Any further solution Δ must be a prime $\equiv 3 \pmod 8$. But it is undecided whether there are such solutions other than 11, 19, 43, 67, 163.

Lerch[263] wrote $\psi(x)$ for $\Gamma'(x)/\Gamma(x)$ and observed that Dirichlet's[28] formula (7) for the number of positive classes of a positive fundamental discriminant D gives the relation

$$\sum_{h=1}^{D-1}\left(\frac{D}{h}\right)\psi\left(\frac{h}{D}\right)=-\sqrt{D}\,Cl(D)\log E(D).$$

From this ψ is eliminated by means of

$$\begin{aligned}-C-\log 4\pi+\log a-\psi(x)-\psi(1-x)\\ =\sum_{m=-\infty}^{\infty}\int_{1/a}^{\infty}a^{-(x+m)^2 z\pi}\frac{dz}{\sqrt{z}}+2\sum_{n=1}^{\infty}\cos 2nx\pi\int_a^{\infty}e^{-n^2 z\pi}\frac{dz}{z},\end{aligned}$$

where C is the Euler constant[380] and a an arbitrary positive constant. The final result is that $Cl(D)$ is determined uniquely by

$$\frac{S-2(P_r+Q_r)}{\log E(D)}-2<Cl(D)<\frac{S-2(P_r+Q_r)}{\log E(D)},$$

in which, to a close approximation,

$$S=\tfrac{1}{2}\sqrt{D}(\log D+.046181)-\tfrac{1}{2}\log D+.023090,$$

$$P_r=\sum_{\beta\leqq r}\frac{2}{\beta\sqrt{\pi/D}}\int_{\beta\sqrt{D/\pi}}^{\infty}e^{-x^2}dx,\qquad Q_r=\tfrac{1}{2}\sum_{\beta\leqq r}\int_{\beta^2\pi/D}^{\infty}e^{-x}\frac{dx}{x},$$

while r is chosen sufficiently large to insure a unique determination of $Cl(D)$. For example, if $D=9817$, $\log E(D)=222$, $S=450.5$, whence $Cl(D)<450/222$. We need not compute P_r and Q_r since $Cl(D)$ is uneven (Dirichlet[98]) and hence is 1.

J. W. L. Glaisher[264] called a number s a *positive*, a *negative* or a *non-prime* with respect to a given number P, according as the Jacobi-Legendre symbol $(s/P)=+1$, -1, or 0. He denoted by a_r, b_r, λ_r, respectively, the number of positives, negatives and non-primes in the r-th octant of P. For example, if $P=8k+1$ is without a square factor, Dirichlet's[28] formulas (5) for the number of properly primitive classes of determinant $-P$ and $-2P$, respectively,

$$h'=2(a_1-b_1+a_2-b_2),\qquad h''=2(a_1-b_1-a_4+b_4)$$

become[265]

$$h'=4(\alpha_1+\alpha_2)-\tfrac{1}{2}(P-1),\qquad h''=4(\alpha_1-\alpha_2),$$

where $\alpha_r=a_r+\frac{1}{2}\lambda_r$. Similarly for other types of P. Obvious congruencial properties (mod 8) of h' and h'' are deduced from all of these formulas.

Again h' and h'' are expressed in terms of $\beta_r=b_r+\frac{1}{2}\lambda_r$ $(r=1, 2, 3, 4)$. Next, l_r and μ_r are used to denote respectively the number of positives and non-primes $<P$

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[263] Jour. de Math. (5), 9, 1903, 377–401; Prace mat. fiz. Warsaw, 15, 1904, 91–113 (Polish).
[264] Quar. Jour. Math., 34, 1903, 1–27.
[265] Glaisher, Quar. Jour. Math., 34, 1903, 178–204.

which are of the form $8k+r$, while $L_r=l_r+\frac{1}{2}\mu_r$. A table (p. 13) transforms the preceding formulas into results such as

$$P=8k+1, \qquad h'=2(L_1-L_3), \qquad h''=4(L_1-L_7).$$

If Q_r denotes the number of uneven positives in the rth quadrant plus $\frac{1}{2}\lambda_r$, we have, for example,

$$P=8k+1, \qquad h'=2(Q_4-Q_2), \qquad h''=4(Q_4-Q_1).$$

L. C. Karpinski[266] gave details of R. Dedekind's[267] brief proofs of his theorems which state the distribution of quadratic residues of a positive uneven number P in octants and 12th intervals of P in terms of the class-number of $-P$, $-2P$ and $-3P$. He added to Dedekind's notation the symbols C_5 and C_6, which denote the number of properly primitive classes of determinant $-5P$ and $-6P$, respectively, and, by an argument precisely parallel to that of Dedekind, obtained for all positive uneven numbers P which have no square divisor, the distribution of quadratic residues in the 24th intervals of P as linear functions of C_1, C_2, C_3, C_6. He put $S_r^t=\Sigma(s_r/P)$, where t is a positive integer, and s_r ranges over the integers x for which

$$(r-1)P/t<x<rP/t.$$

He deduced such relations as the following: If $P\equiv 23 \pmod{24}$,

$$(1) \qquad S_1^6=-S_6^6=C_1, \qquad S_2^6=S_3^6=S_4^6=S_5^6=0.$$

If $P\equiv 1$, 5 or 17 (mod 24), C_3 is a multiple of 6. For $P\equiv 3 \pmod 4$,

$$C_1=S_1^{10}+S_2^{10}+S_3^{10}+S_4^{10}+S_5^{10}, \qquad C_5=2S_2^{10}+4S_3^{10}+2S_4^{10}.$$

Cf. Dirichlet,[28] (5). Three other relations among S_r^{10} which arise from familiar properties of quadratic residues lead to a complete determination of S_r^{10} as linear functions of C_1 and C_5 for $r=1, 2, 3, \ldots, 10$.

E. Landau[268] studied the identity

$$\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\frac{1}{n^s}=\frac{\Gamma(1-s)}{\pi}\left(\frac{2\pi}{\Delta}\right)^s\sqrt{\Delta}\cos\tfrac{1}{2}s\pi\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\frac{1}{n^{1-s}}, \quad D<0,$$

which is valid for a real s, $0<s<1$. The limit of the right member for $s=0$ is

$$\frac{\sqrt{\Delta}}{\pi}\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\frac{1}{n}.$$

The customary evaluation of the divergent left member for $s=0$ would give (Dirichlet,[20] (1) above) the erroneous result $h=\Sigma_1^\infty(D/n)$. A similar study is made of the limit for $s=0$ of the ratio

$$\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\frac{\log n}{n^s}\div\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\frac{1}{n^s},$$

which for $s=1$ is Kronecker's[218] limit ratio.

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[266] Thesis, Strassburg, 1903, 21 pp.; reprinted, Jour. für Math., 127, 1904, 1-19.
[267] Werke of Gauss, II, 1863, 301-3; Maser's German translation of Disq. Arith., 1889, Remarks by Dedekind, 693-695.
[268] Jour. für Math., 125, 1903, 130-132, 161-182.

*M. Lerch[269] denoted by g an arbitrary primitive root of a prime $p=2m+1$, and put

$$F_n(x)=\sum_{\nu=1}^{p=1} a^{\operatorname{ind}\nu} x^\nu,$$

where a is an integer of index $m=p-1-n$ referred to the primitive root g as base. C. G. J. Jacobi[270] had found the relation

$$F_m(1+y)\equiv -Y_m/m!\ (\text{mod } p),$$

where Y_m is the sum of the terms in $y^m, y^{m-1}, \ldots, y^{p-1}$ in the Maclaurin's expansion of $[\log(1+y)]^n$. Thus

$$\sum_{\nu=1}^{p-1}\left(\frac{\nu}{p}\right)x^\nu\equiv -\frac{1}{m!}Y_m(x-1)\quad (\text{mod } p).$$

Hence if c_j is the coefficient of y^j in $Y_m(y)$, and if we set

$$-\frac{1}{m!}\sum_{\nu=m}^{2m} c_\nu (i-1)^\nu=A+iB,$$

then $A\equiv B\equiv H$ (mod p), in which H is the number of positive quadratic forms of discriminant $-4p$.

H. Poincaré[271] wrote

$$F(q)=\sum_{m,\,n} q^{am^2+2bmn+cn^2},\quad q=e^{-t},$$

where (a, b, c) is a fixed representative properly primitive form of negative determinant $-p$ and the summation is taken over every pair of integers m, n, for which the value of (a, b, c) is prime to $2p$ except $m=n=0$. $F(q)$ is regarded as a special case of the Abelian function

$$(A)\qquad \Theta(x,y)=\sum e^{i(mx+ny)}q^{am^2+2bmn+cn^2}.$$

The theory of the flow of heat is used to show that if k, k' each range over all integral values, $\Theta(x, y)$ may be written

$$(B)\qquad \Theta(x,y)=\sum_{k,\,k'}\frac{2\pi}{Et}e^{-P},\quad -\tfrac{1}{4}E^2=b^2-ac=-p,$$

$$P=\frac{1}{E^2t}[a(y-2k'\pi)^2-2b(y-2k'\pi)(x-2k\pi)+c(x-2k\pi)^2].$$

Now for $x=y=0$ and t small, $\Theta(x, y)$ is asymptotically $F(1)$. Hence, in the neighborhood of $q=1$,

$$(1)\qquad F(q)=\frac{2\pi}{Et}$$

and is therefore independent of the choice of (a, b, c) of determinant $-p$.

But, for p a prime $\equiv 3$ (mod 4), we have (cf. Dirichlet's[20] formula (2))

$$(2)\qquad \sum_{(a,b,\,c)}\ \sum_{m,\,n} q^P=2\sum\left(\frac{n}{p}\right)q^{nn'},$$

[269] Bull. Int. de l'Acad. des Sc. de Cracovie, 1904, 57-70 (French).
[270] Monatsber. Akad. Wiss. Berlin, 1837, 127; Jour. für Math., 30, 1846, 166; Werke, Berlin, VI, 1891, 254-258.
[271] Jour. für Math., 129, 1905, 120-129.

where now $P=am^2+2bmn+cn^2$ ranges over a system of properly primitive forms of determinant $-p$, and m, n take all pairs of integral and zero values for which P is prime to $2p$; in the second member, n, n' range over every pair of odd positive integers each prime to P. By a simple transformation of each member, (2) can be written

$$(3) \qquad \sum_{(a,\,b,\,c)} F(q) - \sum_{(a,\,b,\,c)} F(-q) = 4\sum\left(\frac{n}{q}\right)\frac{q^n}{1-q^{2n}}.$$

But from (A), it follows that $F(q)=\Theta(0,0)$, $F(-q)=\Theta(\pi,\pi)$; and hence from (B), it follows that

$$F(q)=\sum\frac{2\pi}{Et}e^{-P}, \quad F(-q)=\sum\frac{2\pi}{Et}e^{-P}, \quad P=\frac{\pi^2}{E^2t}(a\mu^2-2b\mu\nu+c\nu^2),$$

where μ, ν are even integers in the case of $F(q)$, and odd integers in the case of $F(-q)$. Since for t small, all terms of the left member of (3) except those having $\mu=\nu=0$ are to be neglected, the left member becomes

$$\frac{\pi}{t\sqrt{p}}h(-p).$$

Moreover

$$\lim_{t=0}\frac{tq^n}{1-q^{2n}}=\frac{1}{2n}.$$

Hence[272] (3) becomes Dirichlet's[14] formula (2). Equation (3) is also transformed to give Dirichlet's[28] closed form (5) for $h(-p)$.

A. Hurwitz[273] by the substitution

$$u=\frac{\alpha_1x+\beta_1y+\gamma_1z}{\alpha x+\beta y+\gamma z}, \quad v=\frac{\alpha_2x+\beta_2y+\gamma_2z}{\alpha x+\beta y+\gamma z}$$

transformed the Cartesian area $\iint du\,dv$ of a plane region G into what he called the generalized area of G with respect to the form $\alpha x+\beta y+\gamma z$. Such a generalized area of the conic $xy-z^2=0$ is

$$(1) \qquad 2\pi/(\sqrt{4\alpha\gamma-\beta^2})^3.$$

For points on the conic, we put $x=r^2$, $y=rs$, $z=s^2$, and consider points $(x, y, z)=(r, s)=(-r, -s)$, r and s being relatively prime integers. An *elementary triangle* is one having as its three vertices the points

$$(2) \qquad (r, s), \quad (r_1, s_1), \quad (r+r_1, s+s_1), \quad rs_1-r_1s=\pm1.$$

All such possible triangles in the aggregate cover the conic simply six times and their total area is

$$(3) \qquad \tfrac{1}{2}\sum\{\alpha r^2+\beta rs+\gamma s^2)(\alpha r_1^2+\beta r_1s_1+\gamma s_1^2)\,[\alpha(r+r_1)^2+\beta(r+r_1)(s+s_1)+\gamma(s+s_1)^2]\}^{-1}$$

summed for the solutions r, s, r_1, s_1 of $rs_1-r_1s=\pm1$.

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[272] Cf. G. L. Dirichlet, Zahlentheorie, Art. 97.
[273] Jour. für Math., 129, 1905, 187–213.

But if the Gauss form $au^2+\beta uv+\gamma v^2$ be subjected to all the unitary substitutions, it goes over into $a'u'^2+\beta'u'v'+\gamma'v'^2$, where a', β', γ' have values such that (3) can be written as $\Sigma 8/\{a'\gamma'(a'+\beta'+\gamma')\}$, where $(a', \beta'/2, \gamma')$ ranges over all forms equivalent to $(a, \beta/2, \gamma)$. Hence by comparison of (1) and (3) we have

$$\frac{3\pi}{2D\sqrt{D}}\cdot h=\sum_{a,b,c}\frac{1}{ac(a+2b+c)},$$

where (a, b, c) ranges over all positive forms of determinant D.

By modifying his definition of generalized area Hurwitz obtained for the right member a more rapidly convergent series.

M. Lerch,[274] by use of his[240] trigonometric formula for E^*, showed by means of Gauss sums that

$$S\equiv\sum_{a=0}^{n-1}E^*\left(x+\frac{a^2m}{n}\right)=\sum_{a=0}^{n-1}\left(x+\frac{a^2m}{n}\right)-\frac{n}{2}+S_1,$$

in which S_1 is the imaginary part of

$$\sum_{\nu=1}^{\infty}\frac{1}{\nu\pi}\left(\frac{m\nu'}{d'_\nu}\right)d_\nu\sqrt{d'_\nu}\cdot i^{\frac{1}{4}(d'_\nu-1)^2}e^{2d_\nu\nu'x\pi i},$$

where d_ν is the g.c.d. of n and ν, and $d'_\nu=n/d_\nu$, $\nu'=\nu/d_\nu$. Then, if we put $d'_\nu=d$ $d_\nu=d'$, and also

$$\Phi(z,d)=\sqrt{d}\sum_{\nu=1}^{\infty}\left(\frac{\nu}{d}\right)\frac{\cos 2\nu z\pi}{\nu\pi},\text{ if } d\equiv-1\ (\text{mod }4);$$

$$=\sqrt{d}\sum_{\nu=1}^{\infty}\left(\frac{\nu}{d}\right)\frac{\sin 2\nu z\pi}{\nu\pi},\text{ if } d\equiv+1\ (\text{mod }4);$$

we find

$$S_1=\sum_{d|m}\left(\frac{m}{d}\right)\Phi(d'x,d).$$

Hence we get the chief formula of this memoir:

$$(1)\qquad \sum_{a=0}^{n-1}\left\{E^*\left(x+\frac{a^2m}{n}\right)-\left(x+\frac{a^2m}{n}\right)\right\}=-\frac{n}{2}+\sum_d\left(\frac{m}{d}\right)\Phi(d'x,d).$$

But by Kronecker,[171] (2), $\Phi(0,\Delta)=2\tau^{-1}Cl(-\Delta)$, where $Cl(-\Delta)$ denotes the number of primitive positive classes of discriminant $-\Delta$. And for $x=0$, m positive and relatively prime to n, (1) becomes[275] Lerch's formula[245] (1).

For $x=\frac{1}{2}$, (1) becomes[276]

$$\sum_{a=1}^{n-1}\left[\frac{a^2m}{n}-E\left(\frac{a^2m}{n}+\frac{1}{2}\right)\right]=\sum_d\left(\frac{m}{d}\right)\left\{1-\left(\frac{2}{d}\right)\right\}\frac{2}{\tau_d}Cl(-d),$$

d ranging over the divisors $4k+3$ of n. Similar results are obtained by taking $x=\frac{1}{4}$ (cf. Lerch[245] (2)) and $x=1$.

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[274] Annali di Mat. (3), 11, 1905, 79–91.
[275] Cf. Lerch, Rozpravy české Akad., Prague, 7, 1898, No. 7; also Bull. de l'Acad. des Sc. Bohême, Prague, 1898, 6 pp.
[276] Reproduced by Lerch in his Prize Essay,[278] Acta Math., 30, 1906, 242, formula (40).

Lerch observed that the sum A of the quadratic residues of an odd number n, which are prime to n, >0 and $<n$, is given by

$$2^\omega A=\sum_{\nu=1}^{n}\left[1+\left(\frac{\nu}{p_1}\right)\right]\left[1+\left(\frac{\nu}{p_2}\right)\right]\cdots\left[1+\left(\frac{\nu}{p_\omega}\right)\right]\left(\frac{n^2}{\nu}\right)\nu,$$

where $p_1, p_2, \ldots, p_\omega$ are the distinct prime divisors of n. Hence

$$2^\omega A=\sum_{d}\sum_{\nu=1}^{n}\left(\frac{\nu}{d}\right)\left(\frac{n^2}{\nu}\right)\nu,$$

where d ranges over those divisors of n which have no square factor. By means of the Moebius function (this History, Vol. I, Ch. XIX), he transformed this into

$$2^\omega A=\tfrac{1}{2}n\phi(n)-n\sum_{d}\frac{2}{\tau_d}Cl(-d)M_d(n),$$

where d ranges over those divisors $\equiv -1 \pmod 4$ of n which have no square factor and

$$M_d(n)=\Pi\{1-(p/d)\},$$

where p ranges over the distinct prime factors of $d'=n/d$.

M. Lerch[277] in a prize essay wrote an expository introduction on class-number from the later view-point of L. Kronecker[171]; and stated without proof that if $\mathscr{R}(x)=x-[x]$ and

$$\mathscr{R}(m, n)=\sum_{\rho=1}^{m-1}\rho\mathscr{R}\left(\frac{n\rho}{m}\right)$$

and if $-\Delta_1$ and $-\Delta_2$ are two negative fundamental discriminants, and $D=\Delta_1\Delta_2$; moreover, if for an arbitrary positive integer τ, t and u be defined by

$$\left(\frac{T+U\sqrt{D}}{2}\right)^\tau=\frac{t+u\sqrt{D}}{2},$$

then

$$\frac{2\tau u}{\tau_1\tau_2}Cl(-\Delta_1)Cl(-\Delta_2)=\sum_{a,\,b,\,c}\left(\frac{-\Delta_1}{(a, b, c)}\right)\left[\frac{1}{a}\mathscr{R}\left(au, \frac{bu-t}{2}\right)+\frac{t}{12a}-\frac{au^2}{4}\right],$$

where (a, b, c) ranges over a complete system of representative forms of discriminant D, $a>0$.

In Ch. I, use is made of Dirichlet's[20] fundamental formula (2) to make rigorous Hermite's[83] deduction of Dirichlet's[23] classic class-number formula (5). By new methods he obtained the familiar evaluations of the class-number that are due to Dirichlet,[23] Kronecker,[171] Lebesgue,[86] and Cauchy,[29] and established anew Kronecker's[171] ratio (4) of $Cl(D_0\cdot Q^2)$ to $Cl(D_0)$.

He found that if D_i are fundamental discriminants ($i=1, 2, 3, \ldots, r$), and $|D_i|=\Delta_i$, and if 2ν of the determinants are negative, then

$$(1)\quad Cl(D_1D_2\ldots D_r)\log E(D_1D_2\ldots D_r)$$
$$=(-1)^{\nu+1}\sum_{h_1, h_2, \ldots, h_r}{}'\left(\frac{D_1}{h_1}\right)\left(\frac{D_2}{h_2}\right)\cdots\left(\frac{D_r}{h_r}\right)\log\sin\left(\frac{h_1}{\Delta_1}+\frac{h_2}{\Delta_2}+\ldots+\frac{h_r}{\Delta_r}\right)\pi,$$

[277] Full notes of the Essay were published in Acta Math., 29, 1905, 334–424; 30, 1906, 203–293; Mém. sav. étr., Paris, 1906, 244 pp.

where $0<h_i<\Delta_i$, the term containing $\log 0$ is to be suppressed, and $E(D)=\frac{1}{2}(T+U\sqrt{D})$. By taking $r=2$, $D_1=-\Delta$, $D_2=-4$, we obtain one of the corollaries:

$$Cl(4\Delta)\log E(4\Delta)=2\sum_{h=1}^{\frac{1}{2}(\Delta-1)}\left(\frac{-\Delta}{h}\right)\log\frac{1+\tan h\pi/\Delta}{1-\tan h\pi/\Delta}.$$

In Ch. II, Lerch extended his[240] methods of 1897 and obtained new formulas including the following comprehensive formula, suitable for computation:

$$\sum_{a=1}^{[D/2]}\left(\frac{D}{a}\right)\sum_{\nu=1}^{[a\Delta/D]}\left(\frac{-\Delta}{\nu}\right)=-\tfrac{1}{2}Cl(\Delta D),$$

where $-\Delta$, D are fundamental discriminants, and $\Delta, D>0$. Also,

$$\frac{2}{\tau}Cl(-\Delta)=\frac{\sqrt{\Delta}}{\pi}\sum_{\nu=1}^{\infty}\left(\frac{-\Delta}{\nu}\right)\frac{1}{\nu}\frac{\Gamma(\xi)^2}{\Gamma(\xi+2\nu x/\Delta)\Gamma(\xi-2\nu x/\Delta)},\quad \xi\gtreqless\tfrac{1}{2},\quad 0<x<1.$$

In Ch.[278] III, the identity in cyclotomic theory[279]

$$\log\frac{A(x)}{B(x)}=-\sqrt{D}\,\mathrm{sgn}\,D\sum_{\mu=1}^{\infty}\left(\frac{D}{\mu}\right)\frac{x^\mu}{\mu},\quad |x|<1,$$

where $D>0$ is a fundamental discriminant, for the limiting value $x=1$, gives, by Kronecker,[171] (2) above, the formula,

$$Cl(D)\log E(D)=\log\frac{Y(1)+\sqrt{D}Z(1)}{Y(1)-\sqrt{D}Z(1)}=2\log\frac{|Y(1)+\sqrt{D}Z(1)|}{\sqrt{4F(1)}}.$$

Suppose D is prime and >3; if in the known identity $Y^2(1)-DZ^2(1)=F(1)=4D$, we put $Y(1)=Dz$, $Z(1)=y$, we get $y^2-Dz^2=-4$; and hence y and z do not satisfy the equation $t^2-Du^2=4$. Hence

$$\log\frac{|y|+|z|\sqrt{D}}{2}\div\log\frac{T+U\sqrt{D}}{2}$$

is not an integer. Therefore $Cl(D)$ is odd. Similarly, it is proved that if D is >8 and composite, $Cl(D)$ is even (cf. G. L. Dirichlet,[98] Zahlentheorie, near the end of each of the articles 108, 109, 110). Congruences (mod 2) are given for $Cl(-8m)$, m a prime.

Lerch showed (Acta Math., pp. 231–233) how to obtain $Y(x, D_1D_2)$ and $Z(x, D_1D_2)$ from the cyclotomic polynomials for D_1 and D_2, and thence found for D_1, D_2 fundamental and >0,

$$Cl(D_1D_2)\log E(D_1D_2)=\sum_{h=1}^{\Delta-1}\left(\frac{D_2}{h}\right)\log\frac{Y(g,D_1)+\sqrt{D_1}Z(g,D_1)}{Y(g,D_1)-\sqrt{D_1}Z(g,D_1)},\quad g\equiv e^{2h\pi i/\Delta_2}.$$

Lerch obtains the following as a new type of formula analogous to Gauss sums:

$$(1)\qquad \sum_{\nu=1}^{m-1}\cot\frac{\nu^2\pi}{m}=4m\sum_{\delta}\frac{1}{\tau_\delta\sqrt{\delta}}Cl(-\delta),$$

where $-m$ is a negative, fundamental, odd discriminant, and δ ranges over the divisors of m which have the form $4k+3$ (Acta Math., 1906, 248).

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[278] Chapters III, IV appear in Acta Math., 30, 1906, 203–293.
[279] Cf. G. L. Dirichlet, Zahlentheorie, Art. 105, for notation.

To express $Cl(D)$, where D is fundamental, negative, and uneven (Acta Math., 1906, pp. 260–279), as the root of congruences (mod 4, 8, 16, ...), Lerch put $D=D_1D_2D_3\dots D_m$, where the D_i are relatively prime discriminants, and put $\Delta=|D|$, $\Delta_i=|D_i|$. All possible products $D'=D_{r_1}D_{r_2}\dots D_{r_a}$ and their complementary products $Q'=D_{r_{a+1}}D_{r_{a+2}}\dots D_{r_m}$ are formed and Δ' is written for $|D'|$; also, we let

$$F(D')=\frac{2}{\tau'}Cl(D'),\ \text{if } D'<0;\ =0 \text{ if } D'>0,$$

$(D', Q')=\Pi[1-(D'/q)]$, q ranging over the distinct divisors of Q'; and $(D', 1)=1$. Then

$$\tfrac{1}{2}\phi(\Delta)-\frac{2^m}{\Delta}\Sigma^*s=\sum_{D'}(D', Q')F(D'), \tag{2}$$

where Σ^*s denotes the number of those of the integers $s=1, 2, \dots, \Delta$ which satisfy $(D_i/s)=1$ for all D_i simultaneously.

For example, when $m=2$, $D_1=-p$, $D_2=+q$, p and q being primes, $p\equiv3$, $q\equiv1$ (mod 4), then the last formula becomes

$$\tfrac{1}{2}(p-1)(q-1)-\frac{4}{pq}\sum_{1}^{pq}{}^*s=Cl(-pq)+\left[1-\left(\frac{q}{p}\right)\right]\frac{2}{\tau_p}Cl(-p).$$

Since $\frac{1}{2}(p-1)(q-1)$ is $\equiv0 \pmod 4$ and $Cl(-p)\equiv1 \pmod 2$, we have

$$Cl(-pq)\equiv1-(q/p) \pmod 4.$$

Lerch also obtained congruences for $Cl(-pqr)$ modulis 8 and 16.

In Ch. IV, a complicated Kronecker relation in exponentials applied to Lebesgue's[86] class-number formula (1) gives finally the following result:

$$\begin{aligned}\frac{4\pi}{\tau_1\tau_2}&Cl(-\Delta_1)\cdot Cl(-\Delta_2)\\ &=\sqrt{\Delta_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{-\Delta_1}{m}\right)\left(\frac{-\Delta_2}{n}\right)e^{2mn\omega\pi i}\frac{e^{2mu\pi i}+e^{-2mu\pi i}}{2m}\\ &\qquad+\sqrt{\Delta_2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{-\Delta_1}{m}\right)\left(\frac{-\Delta_2}{n}\right)e^{s}(e^{t}+e^{-t})/(2n),\end{aligned}$$

in which $s=-2mn\pi i/(\Delta_2\omega)$, $t=nu\pi i/(\Delta_2\omega)$, while u, ω are complex variables, the imaginary part of ω is real and, in the complex plane, u is in the interior of the parallelogram with vertices at $0, 1, 1+\omega, \omega$. Lerch specializes the formula in several ways. For example, for $\Delta_1=\Delta_2$, $u=0$, $\omega=i$, it becomes

$$[Cl(-\Delta)]^2=\frac{\tau^2\sqrt{\Delta}}{2\pi}\sum_{n=1}^{\infty}\left(\frac{-\Delta}{n}\right)\frac{\Theta_1(n)}{n}e^{-2n\pi/\Delta},$$

where $\Theta_1(k)$ is the sum of the divisors of k.

H. Holden,[280] in the usual notations[281] for the cyclotomic polynomial, wrote

$$4X_p=4X_1X_2=Y^2+pZ^2 \qquad H=h(-p)/[2-(2/p)].$$

[280] Messenger Math., 35 1906, 73–80 (first paper).
[281] Gauss, Disq. Arith., Art. 357.

For p a prime of the form $4n+3>0$, he found, putting $h=h(-p)$,

$$(1), (2)\quad \left(\frac{dY}{dx}\right)_{x=1}=(-1)^{\frac{1}{2}(h-1)}\left(\frac{2}{p}\right)pH,\qquad \left(\frac{d^2Y}{dx^2}\right)_{x=1}=(-1)^{\frac{1}{2}(h-1)}\left(\frac{2}{p}\right)\frac{p(p-3)}{2}\cdot H,$$

$$(3), (4)\quad \left(\frac{dZ}{dx}\right)_{x=-1}=-h,\qquad \left(\frac{d^2Z}{dx^2}\right)_{x=-1}=\frac{p-3}{2}h,$$

$$(5), (6)\quad \Sigma\frac{1}{1-r^\alpha}-\Sigma\frac{1}{1-r^\beta}=i\sqrt{p}H,\qquad \Sigma\frac{1}{1+r^\alpha}-\Sigma\frac{1}{1+r^\beta}=\left(\frac{2}{p}\right)i\sqrt{p}h,$$

$$\alpha,\ \beta>0,\ <p,\ (\alpha/p)=1,\ (\beta/p)=-1,\ r=e^{2\pi i/p}.$$

The fifth formula had been obtained in a different way by V. Schemmel.[282] The fifth and sixth are true also when $p=4n+3$ is a product of distinct primes.

Holden,[283] by a study of the quadratic residues and non-residues, transformed the Schemmel-Holden formula (5) above into

$$(7)\quad \left[q-\left(\frac{q}{p}\right)\right]H=(q-1)\sum_{a=0}^{p/q}\left(\frac{a}{p}\right)+(q-2)\sum_{p/q}^{2p/q}\left(\frac{a}{p}\right)+\ldots+\sum_{(q-2)p/q}^{(q-1)p/q}\left(\frac{a}{p}\right),$$

$$(8)\quad \left[q-\left(\frac{q}{p}\right)\right]H=(q-1)\sum_{a=0}^{p/q}\left(\frac{a}{p}\right)+(q-3)\sum_{p/q}^{2p/q}\left(\frac{a}{p}\right)+(q-5)\sum_{2p/q}^{3p/q}\left(\frac{a}{p}\right)+\ldots,$$

where q is any positive integer relatively prime to p; and the last series terminates with the last possible positive coefficient. If $q=3$ and $q=4$, (8) becomes

$$\left[3-\left(\frac{3}{p}\right)\right]H=2\sum_{a=0}^{p/3}\left(\frac{a}{p}\right),\quad 3H=3\sum_{0}^{p/4}\left(\frac{a}{p}\right)+\sum_{p/4}^{p/2}\left(\frac{a}{p}\right),$$

the latter[284] being Dirichlet's[28] formula (5_1); for, the first or second term of the second member vanishes according as $p=8n+3$ or $8n+7$.

When $q=2, 3, 6$ successively, (8) becomes three equations which yield

$$H=2\sum_{0}^{p/6}\left(\frac{a}{p}\right)\Big/\left\{1+\left(\frac{2}{p}\right)+\left(\frac{3}{p}\right)-\left(\frac{6}{p}\right)\right\},$$

and which also lead simply to expressions[285] for

$$a_r^{(6)}=\sum_{(r-1)p/6}^{rp/6}\left(\frac{n}{p}\right),\quad 1\leqq r\leqq 6,$$

in terms of H. When $q=2, 4, 8$ successively, (8) leads to linear expressions for H in terms of the distribution of quadratic residues and non-residues in the first four of the octants of p.

When $q=p-1$, (7) yields Dirichlet's[28] formula (6_1).

By taking $q=2, 3, 4, 6, 12$ in (8), a table is constructed which shows an upper bound for h when $p\equiv 7, 11, 19$ or $23 \pmod{24}$, as $h\leqq (p+5)/12$ if $p\equiv 7$.

282 Dissertation, Breslau, 1863, 15: Schemmel,[95] (6).
283 Messenger Math., 35, 1906, 102–110 (second paper).
284 Cf. Zahlentheorie, Art. 106, ed. 4, 1894, 276.
285 Cf. Remarks by R. Dedekind[127] in Maser's German translation of Disq. Arith., 693–695. Cf. L. C. Karpinski,[266] Jour. für Math., 127, 1904, 1–19.

Dirichlet's[28] formula (6_1) is transformed into

$$H=\tfrac{1}{6}(p-1)(p-2)-2\sum_{k=1}^{\frac{1}{4}(p-3)}[\sqrt{kp}], \quad p\equiv 3 \pmod 4.$$

Holden[286] multiplied each member of (7) by (p/q). The result for $q=4$ or $q=2$, $p\equiv 3 \pmod 4$, is reduced to

$$3H=3\Sigma\left(\frac{4n}{p}\right)+\Sigma\left(\frac{4n+1}{p}\right), \quad \left[2\left(\frac{2}{p}\right)-1\right]H=\Sigma\left(\frac{4n}{p}\right)-\Sigma\left(\frac{4n+1}{p}\right).$$

Hence

$$h=\Sigma(-1)^{\frac{1}{2}(n-1)}\left(\frac{n}{p}\right), \quad p\equiv 3 \pmod 4,$$

n odd, $n<p$. He found eight similar expressions for h including the cases of determinants $-D$, where $D=4m+3$, $2(4m+1)$, $2(4m+3)$ is a product of distinct primes.

Holden[287] for the case $p=4n+3$, a prime, put

$$\left(\omega^m, \frac{1}{1+r}\right)=\frac{1}{1+r}+\frac{\omega^m}{1+r^g}+\frac{\omega^{2m}}{1+r^{g^2}}+\ldots+\frac{\omega^{(p-2)m}}{1+r^{g^{p-2}}},$$

where ω is a primitive root of $x^{p-1}=1$, g is a primitive root of $x^{p-1}\equiv 1 \pmod p$, and r is a root of $x^p=1$. Then (6) becomes:

$$\left(\omega^{\frac{1}{2}(p-1)}, \frac{1}{1+r}\right)=\left(\frac{2}{p}\right)i\sqrt{p}\cdot h(-p).$$

A study of the new symbol gives

$$h^2=\sum_{\mu=1}^{\frac{1}{2}(p-1)}\left(\frac{\mu}{p}\right)[\tfrac{1}{2}(p-1)-2\lambda_\mu], \quad bh=\Sigma\left(\frac{\mu}{p}\right)\lambda_\mu,$$

where λ_μ is the number of positive integral solutions $k, l\leqq\frac{1}{2}(p-1)$ for a given μ of the congruence $k\mu+l\equiv 0 \pmod p$, and b is the number of quadratic non-residues $<\frac{1}{2}p$ of p. Similarly,

$$h^2=-\sum_{\mu=1}^{p-1}(\mu/p)\lambda_\mu.$$

Holden[288] in a treatment similar to his first paper[280] obtained from his own transformation[289]

$$\frac{x^p-1}{x-1}=S^2-(-1)^{\frac{1}{2}(p-1)}pxT^2$$

of the cyclotomic polynomial, six expressions for h. For example, if p is prime,

$$\left(\frac{d^2S}{dx^2}\right)_{x=1}=(-1)^{\frac{1}{2}(h-1)}\tfrac{1}{4}p(p-3)H\left\{1-\left(\frac{2}{p}\right)\right\}, \quad \left(\frac{dT}{dx}\right)_{x=1}=-\left(\frac{2}{p}\right)\frac{h}{2},$$

according as $p\equiv 3$ or $p\equiv 1 \pmod 4$.

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286 Messenger Math., 35, 1906, 110–117 (third paper).
287 Messenger Math., (2), 36, 1907, 37–45.
288 *Ibid.*, 36, 1907, 69–75 (fourth paper).
289 Quar. Jour. Math., 34, 1903, 235.

Holden[290] removed the restriction of his second paper[288] that q be relatively prime to p. He put $p=nP$, $q=nQ$, where P and Q are relatively prime, and found that, if $p=4m+3$ is free from square factors, then for any positive integer n,

$$\tfrac{1}{2}h\left[1+\left(\frac{n}{p}\right)\right]=a_1+a_3+a_5+\ldots,\qquad \tfrac{1}{2}h\left[1-\left(\frac{n}{p}\right)\right]=a_2+a_4+a_6+\ldots,$$

where a_r $(0<r\leqq n)$ is the sum of the quadratic characters of the integers between $(r-1)p/(2n)+1$ and $rp/(2n)$. As above,[288] he found that $\frac{1}{8}(p-3)$ is an upper bound of h for $p\equiv 3$ or $15 \pmod{24}$.

Holden,[291] by a modification of his second paper,[288] obtained, when $p=4n+1$ is a product of distinct primes, Dirichlet's[28] formula (5); also writing

$$a_r=\sum_{n=(r-1)p/q}^{rp/q}\left(\frac{n}{p}\right),$$

with q prime to p, he found in the respective cases $q=8$, $q=12$,

$$a_1+a_4=\left(\frac{2}{p}\right)\frac{h}{2},\quad a_1-a_6=\tfrac{1}{2}\left[1+\left(\frac{3}{p}\right)\right]h.$$

In particular,

$$\begin{aligned}
q&=8, & p&=\ 8n+\ 1, & h&=a_1-a_3; \qquad p=8n+5,\quad h=a_2-a_4:\\
q&=12, & p&=24n+\ 1, & h&=a_1-a_6=a_1-a_2=-2(a_3+2a_4);\\
&& p&=24n+\ 5, & h&=2a_3=-2(a_4+a_5);\\
&& p&=24n+13, & \tfrac{1}{2}h&=a_1=a_3=-a_2=-a_6;\\
&& p&=24n+17, & h&=2(a_1-a_4)=-2(a_3+2a_4).
\end{aligned}$$

E. Meissner[292] supplied the details of the arithmetical proof by Liouville[90] of a class-number relation of the Kronecker type.

G. Humbert,[293] following Hermite,[69] wrote

$$\mathscr{A}=\sum_0^\infty q^{\frac{1}{4}(4N+3)}f(4N+3),\qquad 4N+3=(2m+1)(2m+4\rho+3)-4\mu^2,$$
$$\mu=0,\ \pm1,\ \pm2,\ \ldots;\quad m,\ \rho=0,\ 1,\ 2,\ \ldots,$$

and recalled that the exponent of q has a chosen value as often as there are quadratic forms

$$\phi=(2m+1)x^2+4\mu xy+(2m+4\rho+3)y^2\equiv ax^2+2bxy+cy^2$$

satisfying the conditions $c>a$, $|b|<a$, a and c uneven, b even. By means of the modular division of the complex plane, he set up a (1, 1) correspondence between the principal roots of these forms and those of the reduced uneven forms of determinant $-(4N+3)$. Hence $f(4N+3)=F(4N+3)$.

Similarly Humbert employed $\mathscr{B}$ and $\mathscr{C}$ to mean the same as $\frac{1}{2}C$ and $\frac{1}{4}+\frac{1}{2}D$ in the notations of Petr.[252]

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[290] Messenger Math., 36, 1907, 75–77 (Addition to second paper[288]).
[291] Messenger Math., 36, 1907, 126–134 (fifth paper).
[292] Vierteljahrs. Naturfors. Gesells. Zürich, 52, 1907, 208–216.
[293] Jour. de Math., (6), 3, 1907, 337–449.

A new class-number relation analogous to Kronecker's[54] (VIII) is deduced by equating coefficients of $q^{N+\frac{1}{8}}$ in the identity

$$2e^{-i\pi/8}\eta_1(i\sqrt{q})q^{\frac{1}{8}}\sum_0 q^{\frac{1}{8}(8\nu+7)}F(8\nu+7)$$
$$=4\sum_0 \frac{mq^m}{1+q^{2m}}\left[q^{\frac{1}{8}(2m+1)^2}(-1)^{\frac{1}{2}m(m-1)}+q^{\frac{1}{8}(2m-1)^2}(-1)^{\frac{1}{2}(m-1)(m-2)}\right].$$

The result is

$$\sum_{x\geqq 0}\left(\frac{2}{2x+1}\right)F[8N-(2x+1)^2]=-\sum\delta\left(\frac{2}{\delta_1+\delta}\right),$$

where $2N=\delta\delta_1$, $\delta<\delta_1$, δ and δ_1 positive and of different parity.

Similar treatment leads to relations of the Kronecker-Hurwitz type[294] such as

$$\sum_{m\geqq 0}(-1)^{\frac{1}{2}m(m+1)}F[8N+4-(2m+1)^2]=\sum(-1)^{\frac{1}{2}(a-1)}a,$$

a ranging over the solutions of $2N+1=a^2+2b^2$, $a>0$.

Four class-number relations of Liouville's[107] first type are obtained, including two of Petr,[295] and also

$$8(-1)^N\left[\sum_{m\geqq 0}(-1)^m(2m+1)F(4N+1-(2m+1)^2\right]=-2\sum\left(\frac{-1}{d}\right)d^2+\sum(a^2-4b^2),$$

in which $4N+1=a^2+4b^2$; $a>0$; $4N+1=dd'$, $d\leqq d'$; the term in which $d=d'$ is divided by 2.

New deductions of five of Petr's[296] class-number relations of Liouville's[297] second type are given (pp. 369–371).

Like Petr,[258] by recourse to transformations of order 2 of theta functions, but independently, Humbert obtained class-number relations involving the forms x^2-2y^2, including Humbert's (57), which is a slight modification of Petr's[258] (1) above, and including Humbert's (52), which is Petr's[258] (2) above.

A geometric discussion, analogous to the one above in which Humbert evaluated $\mathscr{A}$, now shows (pp. 385–8) that for a negative determinant $-M$, $M\equiv 3 \pmod 8$, there is a (3, 1) correspondence between the proper and improper reduced forms. The corresponding well-known relation (Dirichlet[20]) is similarly established for $M\equiv 7 \pmod 8$.

To prove a theorem of Liouville,[105] Humbert finds (pp. 391–2) in Liouville's notation that, for a determinant $-(8M+3)$,

$$\sum a(a'-a)=2\sum(2m_1m_1'+2m_1m_1''+2m_1'm_1''-m_1^2-m_1'^2-m_1''^2),$$

where a and $a'\gtreqless a$ are the two odd minima of any odd class, while m_1, m', m_1'' denote the first uneven minima of the three odd classes corresponding to a single even class, and where summation on the right is taken over the even classes. But the right summand equals 8M+3, whence

$$\sum a(a'-a)=\tfrac{2}{3}(8M+3)F(8M+3).$$

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[294] L. Kronecker,[124] Monatsber. Akad. Wiss. Berlin, 1875, 230; A. Hurwitz,[202] Jour. für Math., 99, 1886, 167–168.

[295] Cf.[252] Rozpravy české Akad., Prague, 9, 1900, Mem. 38. In Humbert's memoir the two are (35), (36).

[296] Rozpravy české Akad.,[256] Prague, 9, 1900, Mem. 38.

[297] Jour. de Math.,[107] (2), 12, 1867, 99. The five are numbered (40)–(44) by Humbert.

To obtain class-number relations in terms of minima of classes, Humbert equated the coefficients of $q^{N+\frac{3}{4}}$ in the identity

$$\mathscr{A}\theta = \sum_{1} q^{m^2} a_m [1 - 2q^{2m} + \ldots + 2(-1)^\rho q^{2\rho m} + \ldots],$$

where

$$a_m = q^{-1/4} - 3q^{-9/4} + \ldots + (-1)^{m-1}(2m-1)q^{-(2m-1)^2/4}, \quad \theta = \sum_{-\infty}^{+\infty} (-1)^m q^{m^2}.$$

The coefficient in the first member is

$$\sum_{x,\,y} (-1)^{x+y} F(4N+3-4x^2-4y^2).$$

In the second member, $4N+3 = 4m^2 - (2\mu-1)^2 + 8m\rho$, $(m \gtreqless 1,\ \rho > 0,\ 1 \leqq \mu \leqq m)$; and the coefficient is $\sum(-1)^{\mu+\rho-1} 2(2\mu-1)$. When

$$4N+3 = (2m+2\rho-2\mu+1)(2m+2\rho+2\mu-1) - 4\rho^2$$

is identified with $ac-b^2$ the negative of the discriminant of form (a, b, c); a and c uneven; $c > a$; $a > b$; $b \gtreqless 0$; the latter coefficient is

$$2\sum(-1)^{\frac{1}{4}(c-a+2b-2)} \tfrac{1}{2}(c-a) = \tfrac{1}{2}(-1)^N \sum(\mu_2 - \mu_1)(-1)^{\frac{1}{2}(\mu_1-1)},$$

where the summation on the right is over the proper classes of determinant $-(4N+3)$, and μ_1, μ_2 $(\mu_1 \leqq \mu_2)$ are the two uneven minima of a class.

Similarly, from $\mathscr{A}\eta_1\theta$ Humbert obtained

$$4\sum(-1)^x F[4N-4x^2-(2y+1)^2] = 2\sum \mu(-1)^{\frac{1}{4}(\mu_1+\mu_2+2)},$$

summed over all pairs of integers x, y, where μ is the even minimum, μ_1, μ_2 the odd minima of an odd class of determinant $-4N$.

By equating the coefficients of q^N in the identity

$$-4\mathscr{A}(-q)e^{i\pi/4}\eta_1\theta_1^2 = 8\sum \frac{m^2 q^2}{1-q^{2m}} + 8\sum m^2 \frac{q^{m^2+m}}{1-q^{2m}}[1+2q^{-1}+\ldots+2q^{-(m-1)^2}],$$

we obtain the class-number relation

$$(-1)^{N+1} \sum_{h \geqq 0} (-1)^h F(4N-4h-1)\Phi(4h+1) = \tfrac{1}{16}\sum\mu^2,$$

where $\Phi(n)$ is the sum of the divisors of n, and μ is the even minimum of an odd class of determinant $-4N$.

Similarly, from the expansion of $\mathscr{C}\eta_1^2\theta$, it is stated that

$$\sum_{h \geqq 0} I\left[\frac{4N-1-(4h+3)}{4}\right]\Phi(4h+3) = \tfrac{1}{16}\sum\mu^2,$$

where $I(D) = F(D) - 3F_1(D)$, and $F_1(D)$ denotes the number of even classes of determinant $-D$.

Five more new class-number relations involving minima include

$$8\sum_{h \geqq 0} F[8M+3-(4h+3)]\psi(4h+3) = \sum \nu_2(\nu_3 - \nu_1),$$

in which $\psi(n)$ denotes the sum of the divisors $< \sqrt{n}$ of n; the summation on the right extends over the even classes of $-(8M+3)$; and ν_1, ν_2, ν_3 are the three minima of a class, $\nu_1 \leqq \nu_2 \leqq \nu_3$.

To obtain class-number relations of grade 3 of the Gierster[145]-Hurwitz[167, 184] type,[298] Humbert employed the fundamental formula of Petr[299] and Humbert,[300]

$$(1)\quad \eta_1\theta_1 H_1 \Theta_1 H^2/\Theta^2 = 2H_1(x, \sqrt{q})\sum_0 q^{(8\nu+7)/8}F(8\nu+7)$$
$$-4\sum_1 q^{\frac{1}{4}(2m+1)^2}\cdot[(2m-1)q^{-\frac{1}{4}(2m-1)^2}+(2m-5)q^{-\frac{1}{4}(2m-5)^2}+\ldots]\cos(2m+1)x.$$

By setting $x=0$, and equating coefficients of q^N, we obtain

$$(2)\qquad 0=\sum_m F[8N-(2m+1)^2]-2\sum(\delta_1-\delta),$$

$2N=\delta\delta_1$, δ_1 even, δ odd, $\delta<\delta_1$, m arbitrary.

In (1), we put $x=\pi/3$ and use the formula for $\Theta(3x, q^9)$. In the resulting identity we equate the coefficients of q^N and use the fact that the number of solutions of

$$8N=(2x+1)^2+(2y+1)^2+3(2z+1)^2+3(2t+1)^2$$

is $16\sum d'$, where d' ranges over the divisors of N which have uneven conjugates and which are not multiples of 3. Whence, for $N\equiv -1 \pmod 3$, the final result is

$$-3\sum d' = 2\sum F[8N-(2m+1)^2]\cos(2m+1)\pi/3 - 4\sum(\delta_1-\delta)\cos(\delta_1+\delta)\pi/3$$

in which $2N=\delta\delta_1$, so that $\cos(\delta_1+\delta)\pi/3=\frac{1}{2}$. This result combined with (2) gives, for $N\equiv -1 \pmod 3$, the relations[301] (p. 418):

$$\sum F[8N-9(2\mu+1)^2]=\sum d',\qquad \sum F[8N-(6\rho\pm1)^2]=-\sum d'+2\sum(\delta'-\delta),$$

summed over all integers μ, ρ, where d' is a divisor of N which has an odd conjugate and $\delta'\delta=2N$, $\delta_1>\delta$, δ_1 is even and δ uneven. Corresponding results[301] are obtained for $N\equiv 0, 1 \pmod 3$.

Transformations of the third order yield also, for $N=6l+1$ (p. 431),

$$2\sum G(N-9\nu^2)=\tfrac{1}{18}\mathcal{N}_3+\tfrac{1}{3}\sum d,$$

summed over all integers ν, and all divisors d of N, where $G(m)$ is the number of classes of determinant $-m$, and $\mathcal{N}_3$ is the number of decompositions of N into the sum of 4 squares in which 3 of the squares are multiples of 3.

Humbert evaluated such sums[301] as $\sum F(N-9\nu^2)$, with N arbitrary; but it is done with less directness than by Petr.[345] New expansions lead to such relations[301] as

$$\sum(-1)^m F(24N-1-24m^2)=-\sum y(-1/y)(-1)^{\frac{1}{4}(x-2)},$$
$$48N-2=x^2-6y^2,\quad y>0,\quad x>3y;$$
$$6\sum(-1)^m F_1\left[\frac{24N+1-(6m+1)^2}{6}\right]=-\sum\left(\frac{3}{x}\right)(x-3y),$$

$24N+1=x^2-6y^2$, $y \gtreqless 0$, $x>3y$, each summed over all integers m. Terms in which $y=0$ are divided by 2.

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[298] Cf. Klein-Fricke,[217] Elliptische Modulfunctionen, II, 1892, 231–234.
[299] Cf. Petr,[252] formula (1).
[300] Numbered (10) in Humbert's memoir.
[301] Humbert gave the results also in Comptes Rendus, Paris, 145, 1907, 5–10.

Humbert[302] gave five new class-number relations involving minima[303] of the classes.

H. Teege[304] partly by induction concluded that, when $P=8n+3$ is a product of distinct primes,

$$5\sum_{1}^{(P-3)/4}\left(\frac{a}{P}\right)a+\sum_{(P+1)/4}^{(P-1)/2}\left(\frac{a}{P}\right)a>0.16124P\sqrt{P},\quad \sum_{(P+1)/4}^{(P-1)/2}\left(\frac{a}{P}\right)a-\sum_{1}^{(P-3)/4}\left(\frac{a}{P}\right)a>0.$$

These combined confirm, in view of Dirichlet's[23] formula (6), Gauss' conjecture (Disq. Arith.,[4] Art. 303) that the number of negative determinants which have a class number h is finite for every h.

K. Petr[305] recalled that the number of representations of any number N by the representatives (Dirichlet,[93] (2)) of all the classes of positive forms $ax^2+bxy+cy^2$ of negative fundamental discriminant D is $\tau\Sigma(D, d)$, summed for the divisors d of N, where the symbol (D, d) is that of Weber[306] for the generalized quadratic character of D. Hence,[307] if $D<-4$,

$$(1)\qquad \sum_{\text{class}}\sum_{x,y} q^{ax^2+bxy+cy^2}=2\sum_{N} q^N\left(\sum_{d}(D, d)\right)+h,\quad |q|<1,$$
$$x, y=0, \pm1, \pm2, \ldots;\ N=1, 2, 3, \ldots,$$

where h is the number of positive classes of D.

By methods of L. Kronecker[308] he obtained

$$(2)\qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\pi i\tau(am^2+bmn+cn^2)}=\frac{i}{\tau\sqrt{-\frac{1}{4}D}}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} e^{-4\pi i\tau_1(am^2+bmn+cn^2)/D},$$

where $\tau_1=-1/\tau$. Next, by the use of theta functions, he found

$$(3)\qquad \sum_{k=1}^{-D-1}(D, k)\frac{\Theta'(k\tau/D, \tau)}{\Theta(k\tau/D, \tau)}=4\pi i\sum_{N=1}^{\infty} q^{2N/|D|}\left(\sum_{d}(D, d)\right),$$

d is any divisor of N. Now (1), (2), (3) imply

$$(4)\qquad \Sigma(D, k)\frac{\Theta'(k\tau/D, \tau)}{\Theta(k\tau/D, \tau)}=2\pi i\sum_{Cl}\sum_{x,y} q^{-2(ax^2+bxy+cy^2)/D}-2h\pi i$$
$$=-2\pi\frac{\sqrt{-D}}{\tau}\sum_{Cl}\sum_{x,y} q_1^{2(ax^2+bxy+cy^2)}-2h\pi i,$$

where $\tau_1=-1/\tau$ and $q_1=e^{\pi i\tau_1}$. For the same transformation $\tau_1=-1/\tau$,

$$(5)\qquad \sum_{k}(D, k)\left[2\frac{k}{D}\pi i+\frac{\Theta'(k\tau/D, \tau)}{\Theta(k\tau/D, \tau)}\right]=\frac{1}{\tau}\sum_{k}(D, k)\frac{\Theta'(k/D, \tau_1)}{\Theta(k/D, \tau_1)},$$

$k=1, 2, \ldots, -D-1$. By use of (5), we get

$$(6)\qquad 2\pi i\left[-h+\sum_{k}\frac{k}{D}(D, k)\right]-2\pi\frac{\sqrt{-D}}{\tau}\sum_{Cl}\sum_{x,y} q_1^{2(ax^2+bxy+cy^2)}$$
$$=\frac{1}{\tau}\Sigma(D, k)\,\Theta'\left(\frac{k}{D}, \tau_1\right)\Big/\Theta\left(\frac{k}{D}, \tau_1\right).$$

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302 Comptes Rendus, Paris, 145, 1907, 654–658.
303 Jour. de Math., (6), 3, 1907, 393–410.
304 Mitt. Math. Gesell. Hamburg, 4, 1907, 304–314.
305 Sitzungsber. Böhmische Gesells. Wiss. (Math.-Natur.), Prague, 1907, No. 18, 8 pp.
306 Algebra, III, 1908, § 85.
307 Cf. H. Poincaré, Jour. für Math., 129, 1905, 126.
308 Sitzungsber. Akad. Wiss. Berlin, 1885, II, 761.

Now the right member of (6) is the product of $1/\tau$ by a power series in q_1. Hence the quantity in brackets in (6) is zero (Dirichlet,[19] (1)). For, otherwise (6) would imply that $\tau=-\pi i/\log q_1$ could be expressed as a power series in q_1 which converges for all q_1 such that $|q_1|<1$. Moreover, the comparison now of the two members of (6) in the light of (1) gives Lebesgue's[86] formula (1):

$$h=\frac{-1}{2\sqrt{-D}}\Sigma(D,k)\cot\frac{\pi k}{D}.$$

An alternative form of (6) is the following:

$$2\pi i\left\{-h+\Sigma\frac{k}{D}(D,k)\right\}=\frac{1}{\tau}\{2\pi h\sqrt{-D}+\pi\Sigma(D,k)\cot\pi k/D\}.$$

The last two class-number formulas above follow now elegantly when τ is regarded as a variable occurring in an identity.

H. Holden[309] applied the method of his first paper[280] to a product $p=4n+3$ of distinct primes, and stated the four possible results including

$$\left(\frac{d^2Z}{dx^2}\right)_{x=1}=-(-1)^{\frac{1}{2}h}\{\tfrac{1}{2}\phi(p)-1\}H.$$

He generalized the method of his fourth paper[288] from primes $p=4n+3$ and $4n+1$ to products of primes, and gave the four possible formulas including

$$p=4n+1,\qquad \left(\frac{dT}{dx}\right)_{x=-1}=-(-1)^m\cdot\tfrac{1}{2}h,$$

where m is the number of integers between $\frac{1}{4}p$ and $\frac{1}{2}p$, and prime to p.

H. Weber[310] in a revised edition of his book on Elliptic Functions modified his earlier discussion[214] of class-number to apply to Kronecker forms,[171] in which the middle coefficient is indifferently even or uneven. He also (§ 85) replaced the Legendre-Jacobi-Kronecker symbol[220] (D/n) by (D,n) which he redefined and gave details (§§96-100) of Dedekind's[127a] solution of the Gauss[4] Problem.

M. Plancherel[311] extended certain researches of A. Hurwitz[312] and M. Lerch[313] by finding the residue of $Cl(D)$ modulo 2^m, where $D=D_1D_2\ldots D_m$ and $D, D_1, D_2, \ldots, D_m$ are fundamental discriminants. He deduced Lerch's formula[314]

$$(1)\quad \tfrac{1}{2}\phi(\Delta)-\frac{2^m}{\Delta}\sum_{1}^{\Delta}{}^*s=\sum_{a=1}^{m}\sum_{r_1,\ldots,r_a}(D_{r_1}D_{r_2}\ldots D_{r_a},\Delta_{r_{a+1}}\ldots\Delta_{r_m})P(D_{r_1}D_{r_2}\ldots D_{r_a}),$$

where $\Delta=|D|$, $\Delta_i=|D_i|$; $(D,Q)=\Pi(1-(D/q)$, q ranging over the different prime factors of Q, and $(D,1)=1$; $P(D)=\frac{2}{\tau}Cl(D)$ if D is <0, $P(D)=0$ if D is >0; and Σ^* denotes that those values s only are taken which satisfy

$$(D_1/s)=(D_2/s)=\ldots=(D_m/s)=1.$$

309 Messenger Math., 37, 1908, 13-16.
310 Lehrbuch der Algebra, Braunschweig, III, 1908, 413-427.
311 Thesis, Pavia, 1908, 94 pp. Revista di fisica, matematica, Pavia, 17, 1908, 265-280, 505-515, 585-596; 18, 1908, 77-93, 179-196, 243-257.
312 Acta Math.,[285] 19, 1895, 378-379.
313 Acta Math., 30, 1906, 260-279; Mém. présentés par divers savants à l'Académie des sc., 33, 1906, Chapter III of the Prize Essay.[278]
314 Acta Math.,[278] 30, 1906, 261. Lerch,[278] (2).

Hereafter Δ_i are assumed to be primes. Then $\frac{1}{2}\phi(\Delta)\equiv 0 \pmod{2^{m-a}}$. But

$$(D_{r_1}D_{r_2}\dots D_{r_a},\ \Delta_{r_{a+1}}\dots\Delta_{r_m})\equiv 0 \pmod{2^{m-a}}.$$

It follows that

$$P(D_{r_1}D_{r_2}\dots D_{r_a})\equiv 0 \pmod{2^{a-1}},\quad P(D_1D_2\dots D_m)\equiv 0 \pmod{2^{m-1}}.$$

The latter for $D<0$ is the rule derived from genera (cf. C. F. Gauss, Disq. Arith., Arts. 252, 231; L. Kronecker, Monatsber. Akad. Wiss. Berlin, 1864, 297; reports of both in Ch. IV). Thus (1) implies

$$P(D_1\dots D_m)\equiv\tfrac{1}{2}\phi(\Delta_1\dots\Delta_m)$$
$$+\sum_{a=1}^{m-1}\sum_{r_1,\dots,r_a}(D_{r_1}\dots D_{r_a},\ \Delta_{r_{a+1}}\dots\Delta_{r_m})P(D_{r_1}\dots D_{r_a}) \pmod{2^m}.$$

For a negative determinant $D=-p_1p_2\dots p_{2m+1}q_1q_2\dots q_n$, where p, q are primes >0 and $-p\equiv q\equiv 1 \pmod 4$, this leads to

$$(2)\quad Cl(-p_1p_2\dots p_{2m+1}q_1q_2\dots q_n)$$
$$\equiv\tfrac{1}{2}\sum_{\mu=1}^{2m+1}\sum_{r}((-1)^{\mu}p_{r_1}p_{r_2}\dots p_{r_\mu}\,|\,-p_{r_{\mu+1}}\dots -p_{2m+1}q_1q_2\dots q_n)$$
$$\phi(p_{r_1}\dots p_{r_\mu}) \pmod{2^{2m+n+1}},$$

where the symbol $(\ |\ )$ is defined by the recurrence relation

$$(D_1D_2\dots D_a|D_{a+1}\dots D_m)$$
$$=\sum_{\mu=0}^{m-a-1}\sum_{\rho}(D_1\dots D_aD_{\rho_{a+1}}\dots D_{\rho_{a+\mu}},\ \Delta_{\rho_{a+\mu+1}}\dots\Delta_{\rho_m})\cdot(D_1\dots D_a|D_{\rho_{a+1}}\dots D_{\rho_{a+\mu}}),$$

and by the formula

$$(D_a|D_\beta D_\gamma)=(D_aD_\beta,\ \Delta_\gamma)(D_a,\ \Delta_\beta)+(D_aD_\gamma,\ \Delta_\beta)(D_a,\ D_\gamma)+(D_a,\ \Delta_\beta\Delta_\gamma).$$

He disposed completely of the new special case $m=5$ by (2) as in the following particular example:

$$m=5,\ D_1=-p_1,\ D_2=-p_2,\ D_3=-p_3,\ D_4=q_1,\ D_5=q_2,$$
$$\left(\frac{p_1}{q_1}\right)=\left(\frac{p_2}{q_1}\right)=-\left(\frac{p_3}{q_1}\right)=\epsilon_1,\quad \left(\frac{p_1}{q_2}\right)=\left(\frac{p_2}{q_2}\right)=-\left(\frac{p_3}{q_2}\right)=\epsilon_2.$$

The result in this case is

$$Cl(D)\equiv 2(1-\epsilon_1)(1-\epsilon_2)[1-(-4/\sigma)]$$
$$+2(1-\epsilon_1\epsilon_2)[2(1-\eta_1\eta_2)+(1-\mu)(1-(-4/\sigma))] \pmod{32},$$

where

$$\eta_1=(p_2/p_3),\quad \eta_2=(p_3/p_1),\quad \eta_3=(p_1/p_2),\quad \mu=(q_1/q_2),\quad \sigma=\eta_1+\eta_2+\eta_3.$$

For $D=D_0D_1D_2\dots D_m$, $|D_0|=8$ or 4, he obtained analogues of (1) and finally congruences (mod 2^{m+1}) for $Cl(D)$. He noted that $Cl(-4q_1q_2\dots q_m)\equiv 0$ (mod 2^{m+1}) if each $q_i\equiv 1 \pmod 8$.

G. Humbert[815] obtained formulas which express new relations between the minima of odd classes of a negative determinant $-n$ and those functions of the type $\psi(n)$,

[815] Jour. de Math., (6), 4, 1908, 379–393. Abstract, Comptes Rendus, Paris, 146, 1908, 905–908.

$\chi(n)$ of the divisors of n which occur in the right member of Kronecker's[54] class-number relations. Thus were obtained alternate forms for the right members of old class-number relations.

E. Chatelain[316] obtained the ratio (see the Gauss[4] Problem) between the number of properly primitive classes of forms of determinant p^2D, p a prime, and the number of determinant D. As the representatives of the first, he chose the type (ap^2, bp, c) with c prime to p; as representatives of the second, he chose the type (a', b', c') with c' prime to p. Then between the $h(p^2\cdot D)$ forms (a, b, c) and the $h(D)$ forms $(a', b' c')$ he set up a $(k, 1)$ correspondence by means of a relative equivalence given by the unit substitution $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, $\beta\equiv 0 \pmod p$. Similarly he found the ratio of the number of classes of the two primitive orders of a given determinant. His proofs are similar to those of Lipschitz.[41]

M. Lerch[317] gave two deductions of

$$\sum_{k=2}^{\Delta-1}\left(\frac{-\Delta}{k}\right)\sum_{a=1}^{k-1}(-1)^{a\Delta/k}=4(1-2\epsilon)K^2-K,$$

where $-\Delta$ is a negative uneven fundamental discriminant, $K=2\tau^{-1}Cl(-\Delta)$, $\epsilon=(2/\Delta)$. Here, if $\Delta=3$, $K=\frac{1}{3}$. The second and more elementary deduction rests on Lerch's[240] formula (3).

He deduced several formulas which he had published earlier,[318] including

$$\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\tan\frac{h\pi}{h}=(-1)^{\frac{1}{8}\Delta-1}\frac{4\sqrt{\Delta}}{\tau}K,\quad \Delta\equiv 0 \pmod 8).$$

K. Petr[319] reproduced his[305] discussion of 1907; and, by equating coefficients of q^n in the expansion of doubly periodic functions of the third kind, obtained Schemmel's[95] formula (4); also the number h of primitive classes of the negative fundamental discriminant $-D=D_1D_2$ for $D_2>0$ and $=4n+1$:

$$h=-\sum_{k_1,\,k_2}(D_1,k_1)(D_2,k_2)\epsilon_{k_1}\epsilon_{k_2},$$

where $k_i=0, 1, 2, \ldots, |D_i|-1$; $i=1, 2$, and where $\epsilon_{k_1}\epsilon_{k_2}=1$ or 0 according as $k_1/D_1+k_2/D_2>0$ or <0, and (D, k) is the Weber symbol.[220]

Similarly, for $-D=-D_1D_2D_3$, a negative fundamental discriminant,

$$h=-\Sigma(D_1,k_1)(D_2,k_2)(D_3,k_3)\cdot E\left(\frac{k_1}{D_1}+\frac{k_2}{D_2}+\frac{k_3}{D_3}\right),$$

where $k_i=1, 2, \ldots, |D_i|-1$; $i=1, 2, 3$; and $E(-a)=-E(a)-1$ if $a>0$. These two formulas are special cases of a formula of M. Lerch on p. 41 of his prize essay.[277] See Acta Math., 29, 1905, p. 372, formula (16). Cf. Lerch,[277] (1).

J. V. Pexider[320] for the case of a prime $p=8\mu+3$, wrote r and ρ respectively for a quadratic residue and non-residue of p, and combined the obvious identity

$$\Sigma r+\Sigma\rho=\tfrac{1}{2}p(p-1) \tag{1}$$

with Dirichlet's[14] formula (3), viz.,

$$\Sigma\rho-\Sigma r=\lambda p, \tag{2}$$

[316] Thesis, University of Zürich, 1908. Published at Paris, 1908, 79 pp.
[317] Rozpravy české Akad., Prague, 17, 1908, No. 6, 20 pp. (Bohemian).
[318] Lerch, Acta Math., 30, 1906, 237, formulas (36)-(39). Chapter III of Prize Essay.[278]
[319] Casopis, Prague, 37, 1908, 24-41 (Bohemian).
[320] Archiv Math. Phys. (3), 14, 1908-9, 84-88.

where 3λ is the number of properly primitive classes of determinant $-p$. The result is

$$\lambda = \frac{p-1}{2} - \frac{2}{p}\Sigma r = \frac{2}{p}\Sigma\rho - \frac{p-1}{2}.$$

According to M. A. Stern, if p is a prime $4\mu+3$, there exists an integer σ such that

$$(3) \qquad 2\Sigma r - \Sigma\rho = \sigma p.$$

From (1), (2), (3), we get $3\lambda = \frac{1}{2}(p-1) - 2\sigma$. This result compared with Dirichlet's[14] class-number formula (3) shows that σ is the number of quadratic non-residues of p which are $<\frac{1}{2}p$.

For a prime $p=8\mu+7$, (2) holds provided now $\lambda = h(-p)$. Hence by (3), $h(-p) = \frac{1}{4}(p-3) - 2\kappa$, where κ denotes the number of the quadratic non-residues of p between 0 and $\frac{1}{4}p$. Dirichlet's[14] formula (3) combined with the last result shows that

$$B = \kappa + \tfrac{1}{8}(p+1), \qquad A = R + \tfrac{1}{8}(p+1),$$

where A and R are respectively the number of positive quadratic residues of p less than $\frac{1}{2}p$ and $\frac{1}{4}p$, and B is the number of quadratic non-residues $<\frac{1}{2}p$.

A. Friedmann and J. Tamarkine,[321] in a study of quadratic residues and Bermoullian numbers, replaced $\Sigma b - \Sigma a$ in Dirichlet's[14] formula (3) so that for p a prime $\equiv 3 \pmod 4$, the latter becomes Cauchy's[28] class-number congruence (1) in the form[322]

$$h(-p) \equiv \left[2 - \left(\frac{2}{p}\right)\right] (-1)^{\frac{1}{4}(p+1)} \cdot 2B_{(p+1)/4} \pmod p.$$

M. Lerch[323] found that, for P a prime,

$$\prod_a \cot\frac{2\pi a}{P} = (-1)^{\frac{1}{2}G(-P)+\frac{1}{2}}\frac{1}{\sqrt{P}},$$

where a ranges over all positive integers $<P$ prime to P such that $(a/P)=1$. Cf. Stern.[31])

G. Humbert[324] introduced a parameter a in the θ-function, and considered $H(x+a)$ and $\Theta(a)$. Then, by Hermite's[69] method, he found that

$$\sum_{k=-\infty}^{+\infty}(-1)^k\cos 2ka \sum_{Cl(4N+3-4k^2)} \cos\tfrac{1}{2}(m_2-m_1)a = (-1)^N\sum_d d\cos da,$$

where m_1 and m_2 are odd minima ($m_1 \leqq m_2$) of a reduced form of negative determinant $-(4N+3-4k^2)$, and d is a divisor of $4N+3$ not exceeding its square root. For $a=0$, this becomes Hermite's[69] relation (5). For $a=\frac{1}{4}\pi$, it becomes

$$\sum_{k'=-\infty}^{+\infty}(-1)^{k'} \sum_{Cl(4N+3-16k'^2)} \left(\frac{2}{\frac{1}{2}(m_2-m_1)}\right) = (-1)^N\Sigma d\left(\frac{2}{d}\right),$$

where $(2/d)$ is the Jacobi-Legendre symbol. If N is uneven, this is Kronecker's[54] relation (VII).

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321 Jour. für Math., 135, 1908-9, 146-156.
322 Mém. Institut de France, 17, 1840, 445; Oeuvres (1), III, 172.
323 Encyclopédie des sc. math., 1910, tome 1, vol. 3, p. 300.
324 Comptes Rendus, Paris, 150, 1910, 431-433.

P. Bachmann[325] supplied the details of Liouville's[90] arithmetical deduction of a class-number relation of the Kronecker type (cf. Meissner[292]).

M. Lerch,[326] by a study of Kronecker's[171] generalized symbol (D/n), transformed the left member of Lerch's[240] formula (3), for a negative fundamental discriminant $-\Delta$, and m not divisible by Δ, and found that

$$(1)\quad \sum_{a=1}^{\frac{1}{2}(\Delta-1)}\left(\frac{-\Delta}{a}\right)E\left(\frac{am}{\Delta}\right)=\tfrac{1}{2}\left[(m-1)\left\{1-\left(\frac{2}{\Delta}\right)\right\}-\left\{1-\left(\frac{-\Delta}{m}\right)\right\}\right]K,\quad K=\frac{2}{\tau}Cl(-\Delta).$$

Put

$$i_a=\sum_{h=1}^{n}\left\{E\left(\frac{2ha}{\Delta}\right)-2E\left(\frac{ha}{\Delta}\right)\right\}.$$

Then by formula[240] (4), we have

$$\sum_{a=1}^{\frac{1}{2}(\Delta-1)}\left(\frac{-\Delta}{a}\right)i_a=\left[n\frac{1-\epsilon}{2}+\frac{n}{2}+\frac{\epsilon-2}{2}(2-\epsilon)K\right]K,\quad \epsilon=\left(\frac{2}{\Delta}\right),\quad n=\left[\frac{\Delta-1}{2}\right],$$

Since $h(-\Delta)=(2-\epsilon)K$, we have, for $\Delta=2n+1$,

$$\sum_{a=1}^{\frac{1}{2}(\Delta-1)}\left(\frac{-\Delta}{a}\right)i_a=\left(\frac{\Delta-1}{2}-h\right)\frac{h}{2}.$$

Similarly for $\Delta=4P$,

$$\sum_{a=1}^{2P-1}(-1)^{\frac{1}{2}(a-1)}\left(\frac{a}{P}\right)i'_a=K(2P-1-2K),\quad i'_a=\sum_{m=1}^{2P-1}\left\{E\left(\frac{ma}{2P}\right)-2E\left(\frac{ma}{4P}\right)\right\}.$$

For a negative prime discriminant $-\Delta$, $\Delta=4k+3$, (1) implies:

$$h(-\Delta)=\sum_{1}^{\frac{1}{2}(\Delta-1)}(-1)^{i_a}.$$

L. E. Dickson,[327] by a method similar to the method of Landau[260] in the case of Gauss forms, showed that for $P>28$ no negative discriminant $-P\equiv 0 \pmod 4$ could have a single primitive class.

For $P\equiv 3 \pmod 4$, P with distinct factors, there are obviously two or more reduced forms. Hence, if there is only the one reduced form $[1, 1, \frac{1}{4}(1+P)]$, then $P=p^e$, where p is a prime $\equiv 3 \pmod 4$ and e is odd. But for $p>3$ and $e \gtreqless 3$, a second primitive reduced form is $[\frac{1}{4}(p+1), 1, (p^e+1)/(p+1)]$. For $P=3^e$, $e \gtreqless 5$, a second primitive reduced form is $(7, 3, 9)$ or $[9, 3, \frac{1}{4}(3^{e-2}+1)]$. Hence beyond 27 we need consider only primes P. We set

$$T_j=\tfrac{1}{4}[(2j+1)^2+P]=T_0+j(j+1).$$

For any integer m and any T_j, there is some T_r, $0\leqq r\leqq \frac{1}{2}(m-1)$, such that $T_j\equiv T_r \pmod m$. From this lemma and by indirect proof it is found[328] that there is a single reduced form of discriminant $-P$ if and only if $T_0, T_1, T_2, \ldots, T_g$ are all prime numbers, where $2g+1$ denotes the greatest odd integers $\leqq \sqrt{P/3}$.

[325] Niedere Zahlentheorie, Leipzig, II, 1910, 423–433.
[326] Annaes scient. da Acad. Polyt., Porto, 6, 1911, 72–76.
[327] Bull. Amer. Math. Soc., (2), 17, 1911, 534–537.
[328] Cf. M. Lerch,[262] Math. Annalen, 57, 1903, 570.

When $P\equiv 7 \pmod 8$ and >7, T_0 is even and >2, and hence composite. A detailed study of $P\equiv 3 \pmod 8)=8k-5$ shows that for all $P>163$ some T_t is composite except perhaps for $k=3t$ and $t=5l+12$ or $5l+13$. With this result and by a stencil device Dickson showed that no P under 1,500,000 except $P=3$, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163 could have a single primitive class.

M. Lerch[329] obtained the chief results of Dirichlet by simple arithmetical methods and reproduced the deduction of several of his[240, 277] own labor-saving formulas.

E. Landau[330] established Pfeiffer's[195] asymptotic expression for $K(x)=\Sigma_{n=1}^{n=x} H_n$ where H_n denotes the number of classes of forms $ax^2+2\beta xy+\gamma y^2$ of negative determinant $-n$. Let $H_{n\nu}$ be the number of non-equivalent reduced forms of determinant $-n$ and with $|\beta|=\nu$. Then for a given n, in each reduced form $\gamma \gtreqless a \gtreqless 2\nu$, and $\nu \leqq \sqrt{n/3}$. Thus

$$K(x)=\sum_{n=1}^{x}\sum_{\nu=0}^{\sqrt{n/3}} H_{n\nu}=\sum_{n=1}^{x}(H_{n0}+\sum_{\nu=1}^{\sqrt{n/3}} H_{n\nu})$$
$$=\sum_{n=1}^{x} H_{n0}+\sum_{\nu=1}^{\sqrt{x/3}}\sum_{n=3\nu^2}^{x} H_{n\nu}=\sum_{n=1}^{x} H_{n0}+\sum_{\nu=1}^{\sqrt{x/3}} R(x,\nu).$$

But H_{n0} is the number of solutions of $a\gamma=n$, $\gamma \gtreqless a$. That is, if $T(n)$ is the number of divisors of n, $H_{n0}=\frac{1}{2}T(n)$, if n is not a square; but $H_{n0}=\frac{1}{2}\{T(n)+1\}$, if n is a square. Hence

$$\text{(1)}\qquad \sum_{n=1}^{x} H_{n0}=\tfrac{1}{2}\sum_{n=1}^{x} T(n)+\tfrac{1}{2}[\sqrt{x}]=\tfrac{1}{2}x\log x+(C-\tfrac{1}{2})x+O(\sqrt{x}),$$

where C is Euler's constant $(=0.57721\ldots)$ and $O(k)$ is of the order[117] of k.

For a given $\nu>0$, Landau evaluated

$$R(x,\nu)=\sum_{n=3\nu^2}^{x} H_{n\nu},$$

by noting that $R(x,\nu)$ is the number of solutions of

$$a\gamma \leqq \nu^2+x,\qquad \gamma \gtreqless a \gtreqless 2\nu,$$

each solution being counted twice when $\gamma>a>2\nu$. Hence $R(x,\nu)$ is the number of lattice points in the finite area defined by these inequalities in the $a\gamma$-plane, lattice points in the interior and on the hyperbolic arc exclusive of its extremities being counted twice. The resulting value of

$$\sum_{\nu=1}^{\sqrt{x/3}} R(x,\nu)$$

combined with (1) now gives

$$K(x)=\frac{2}{9}\pi x^{\frac{3}{2}}-\frac{x}{2}+O(x^{\frac{3}{4}}\log x).$$

If $\mathscr{K}$ corresponds to K, but refers to classes having a and γ both even, the result obtained is

$$\mathscr{K}(x)=\frac{\pi}{18}x^{\frac{3}{2}}-\frac{x}{4}+O(x^{\frac{3}{4}}\log x).$$

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[329] Casopis, Prague, 40, 1911, 425-446 (Bohemian).
[330] Sitzungsber. Akad. Wiss. Wien (Math.-Phys.), 121, II, a, 1912, 2246-2283.

Landau,[331] by a study of the number of lattice points in a sphere, found that if C_n is the number of solutions of $u^2+v^2+w^2=n$,

$$\sum_{n=1}^{x} C_n=\tfrac{4}{3}\pi x^{\frac{3}{2}}+O(x^{\frac{1}{2}+\epsilon}),$$

where[117] only the order of the last term is indicated and ϵ is a small arbitrary positive quantity. But by Kronecker[54] (XI) above, if $F(n)$ denotes the number of uneven classes of forms $ax^2+2bxy+cy^2$ of determinant $-n$, then $C_n=8F(n)$, if $n\equiv 3 \pmod 8$; $C_n=12F(n)$, in all other cases except $n\equiv 7 \pmod 8$.

In $u^2+v^2+w^2\equiv 1 \pmod 4$ evidently

$$u:v:w\equiv 1:1:0,\ 1:0:1,\ 0:1:1 \pmod 2.$$

Hence

$$\sum_{n=1}^{x} F(n)=\frac{\pi}{24}x^{\frac{3}{2}}+O(x^{\frac{1}{2}+\epsilon}),\quad n\equiv 1 \pmod 4.$$

This holds also for $n\equiv 2 \pmod 4$; but

$$\sum_{n=1}^{x} F(n)=\frac{\pi}{48}x^{\frac{3}{2}}+O(x^{\frac{1}{2}+\epsilon}),\quad n\equiv 3 \pmod 8.$$

J. V. Uspensky,[332] by means of lemmas of the types of Liouville's,[90] gave a complete arithmetical demonstration of each of Kronecker's[54] classic eight class-number relations. See Cresse.[374].

J. Chapelon[333] obtained a new identity derived from transformation of the 5th order of elliptic functions and with it followed the procedure of Humbert.[334] He added to Gierster's[185] list of class-number relations of the 5th grade two new ones and gave relations also for

$$\Sigma F(4N-x^2),\ x\equiv 5 \pmod{10};\qquad \Sigma F(4N-x^2),\ x\equiv \pm 1 \pmod{10};$$

and for $\Sigma F(N-25x^2)$ summed over all integers x, where $N=5^{\mu}N'\equiv 0 \pmod{10}$, and N' is not divisible by 5. He gave[335] 24 class-number relations for $\Sigma F(N-x^2)$ and $H(N-x^2)$ which are characterized by various combinations of the congruences $N\equiv \pm 2, \pm 4 \pmod{10}$ with $x\equiv 0, \pm 1, \pm 2 \pmod 5$. These 24 relations include Gierster's relations of the 5th grade.[184] The right hand members of Chapelon's class-number relations in these two memoirs are all illustrated by the following example for $N\equiv 2 \pmod{10}$:

$$\sum_{x} F(N-x^2)=\tfrac{3}{8}\Sigma d'-\tfrac{5}{8}\Sigma(-1)^{d'}d'+\tfrac{1}{2}\Sigma(-1)^{d_1}(d_1-d),$$

$x\equiv \pm 1 \pmod 5$, where d' is any divisor of N and $N=d_1d$ with $d_1 \gtreqless d$ (see Chapelon's thesis[340]).

G. Humbert,[336] after giving an account (Humbert[185] of Ch. I) of his principal reduced forms of positive determinant D, proved that for $D=8M+3$

$$\Sigma(-1/\beta)f(a-|b|)=2\Sigma f(2k+1),\quad \beta\equiv|b|-\tfrac{1}{2}(a+c),$$

[331] Göttingen Nachr., 1912, 764–769.
[332] Math. Sbornik, Moscow, 29, 1913, 26–52 (Russian).
[333] Comptes Rendus, Paris, 156, 1913, 675–677.
[334] Jour. de Math., (6), 8, 1907, 431.
[335] *Ibid.*, 1661–1663.
[336] Comptes Rendus, Paris, 157, 1913, 1361–1362.

where $f(x)$ is an arbitrary even function; the summation on the left extends over all principal reduced forms (a, b, c) of determinant D; and the summation on the right extends over all decompositions,

$$8M+3=(2k+1)^2+(2k'+1)^2+(2k''+1)^2, \quad k, k', k'' \text{ each } \gtreqless 0,$$

of $8M+3$ into the sum of three squares. When $f(x)=1$ and we employ the known value (cf. Kronecker's[54] formula (XI)) for the number of decompositions, we have

$$F(8M+3)=\tfrac{1}{2}\Sigma(-1/\beta).$$

If $f(x)=x^2$, we have

$$\tfrac{3}{8}(8M+3)F(8M+3)=\Sigma(-1/\beta)2\beta(a+c).$$

G. Rabinovitch[336a] proved that the class-number of the field defined by $\sqrt{-d}$, where $d=4m-1$, is unity if and only if $x^2-x+m\,(x=1, \ldots, m-1)$ are all primes. Fewer conditions are given by T. Nagel.[336b]

G. Humbert,[337] by Hermite's method of equating coefficients in theta-function expansions, found that, for all the negative determinants $-(8M+4-4k^2)$, in which M is fixed, the number of odd classes for which the even minimum is not a multiple of 8 is the sum of the divisors of $2M+1$. Similarly for determinants $-(8M-4k^2)$, the number of these classes is $\Sigma(\delta+\delta_1)$ the summation being extended over all the decompositions $2M=\delta\delta_1$, δ odd, δ_1 even, $\delta<\delta_1$. Also, by Hermite's method combined with the use of an even function (cf. Humbert[336]), he[338] obtained the following formula for the number F of odd classes having the minimum and the sum of the two odd minima $\equiv 0 \pmod p$, p arbitrary:

$$\Sigma(-1)^r F(4N+3-4r^2)=(-1)^N\Sigma' d,$$

$r<, =, >0$, and for h arbitrary, r is $\equiv h \pmod p$, $4N+3=pdd_1$, with $pd<d_1$, and $d_1\equiv \pm 4h \pmod p$.

*F. Lévy[339] discussed the determination of the number of classes of a negative determinant by means of elliptic functions.

J. Chapelon[340] gave an outline of the history of class-number relations of the general Kronecker[54] type and listed Gierster's[185] relations of the 5th grade. Examples will be given here merely to characterize each of the six exhaustive chapters of the thesis.

Chapter I contains theorems on the divisors of a number. Let $N=2^\mu N'=5^\nu N''=2^\mu 5^\nu N'''$, N' and N'' prime to 2 and 5 respectively; $N=d_1 d$, $d_1 \gtreqless d$; d' any divisor of N; and, let $\mathscr{B}=\Sigma d'(d'/5)$, $\mathscr{B}_1=\Sigma(-1)^{d'}d'(d'/5)$, $\mathscr{Q}=\Sigma(d_1-d)(d_1+d/5)$. Also let $N=da$, $d>\sqrt{N}$, $a<\sqrt{N}$. Then

$$\Sigma\left[\left(\frac{a}{5}\right)+\left(\frac{d}{5}\right)\right]a=\tfrac{1}{2}\left[1+5^\nu\left(\frac{N''}{5}\right)\right]\mathscr{B}-\tfrac{1}{2}\mathscr{Q}.$$

[336a] Jour. für Math., 142, 1913, 153–164; abstr. in Proc. Fifth Internat. Congress Math., Cambridge, I, 1913, 418–421.
[336b] Abh. Math. Seminar Hamburgischen Universität, 1, 1922, 140–150.
[337] Comptes Rendus, Paris, 158, 1914, 297.
[338] *Ibid.*, 1841–1845.
[339] Thesis, Zürich, published A. Kündig, Geneva, 1914, 48 pp.
[340] Thèse, Sur les relations entre les nombres des classes de formes quadratiques binaires, Paris, 1914, 197 pp.; Jour. de l'Ecole Polytechnique, Paris, 19, 1915.

Chapter II gives, in Hermite[80]-Humbert[293] notation, lists of standard transformation formulas for the Θ-function and expansions of Θ-functions and θ-functions.

Chapter III presents fundamental formulas for the transformation of the fifth order of Θ-functions. In

$$\Theta(5u, 5\tau) = C_1 \Pi\, \Theta(u \pm \pi i/5), \quad i=0, 1, 2,$$

C_1 is found to be $\eta^5/\bar{\eta}$ where $\bar{\eta}=\eta(q^5)$ and $\eta=\eta(q)=\Sigma_{-\infty}^{+\infty}(-1)^m q^{\frac{1}{24}(6m+1)^2}\eta^5$.

Chapter IV deals with the representation of a number by certain quaternary quadratic forms. In (p. 90)

$$\eta_1^2\theta_1^2 \frac{H^2}{\Theta^2} = 8\sum_1 \frac{mq^m}{1-q^{2m}} - 8\Sigma \frac{mq^m}{1-q^{2m}} \cos 2mx,$$

put $x=\pi/5$ and $x=2\pi/5$ and subtract. Equating coefficients of q^M, we get

(1) $$[5\mathscr{M}_3(2M) - \mathscr{M}_1(2M)] = 10\mathscr{B} + 2\mathscr{B}_1,$$

where $\mathscr{M}_1(N)$ and $\mathscr{M}_3$ (N) are respectively the number of decompositions (in which the order is regarded)

$$4N = (2x+1)^2 + (2y+1)^2 + (2z+1)^2 + 5(2t+1)^2,$$
$$4N = (2x+1)^2 + 5(2y+1)^2 + 5(2z+1)^2 + 5(2t+1)^2.$$

From another expansion it is similarly found that

(2) $$-\tfrac{1}{2}\mathscr{M}_3 + \tfrac{1}{2}\mathscr{M}_1 = 5^\nu(N'''/5)2^{\mu+2}\Sigma\theta'(\theta'/5),$$

summed over all divisors θ' of N'''. Then by (1) and (2),

$$\mathscr{M}_1(N) = \tfrac{1}{2}[1+5^{\nu+1}(N''/5)][5\mathscr{B}+\mathscr{B}_1].$$

Suppose that N is even. Since for a fixed value of x, $F[4N-5(2x+1)^2]$ is the number of positive solutions t, u, v of $t^2+u^2+v^2=4N-5(2x+1)^2$ (p. 118),

(3) $$\sum_{x=0}^{\infty} F(4N-5(2x+1)^2) = \tfrac{1}{16}\mathscr{M}_1(N) = \tfrac{1}{32}[1+5^{\nu+1}(N''/5)](5\mathscr{B}+\mathscr{B}_1).$$

This is a special case of Liouville's[107] (4).

In terms of functions like $\mathscr{M}_1$ and $\mathscr{M}_3$ above, Chapelon found in Ch. V expressions for

$$\Sigma F(8M-4-x^2),\ \Sigma F\left(\frac{8M\pm4-x^2}{25}\right),\ \Sigma F\left(\frac{8M-x^2}{25}\right),\ \Sigma F(N-x^2),$$
$$\Sigma E\left(\frac{N-x^2}{25}\right),\ J(N-x^2),\ J\left(\frac{N-x^2}{25}\right);$$

where $x=5\sigma\pm k$ or $10\sigma\pm k$, k constant; $E(N)=F(N)-H(N)$, $J(N)=F(N)+3H(N)$.

In Ch. VI, Chapelon found sets of relations equivalent to each one of Gierster's[185] relations of grade 5; and added large sets of new relations, the sets being distinguished by the residue of N modulo 10. He (p. 171) proved Liouville[114] (1).

H. N. Wright[341] tabulated the reduced forms $ax^2+2bxy+cy^2$ of negative determinant $-\Delta=D$ for $\Delta=1$ to 150 and 800 to 848. The values of b, c occur at the inter-

[341] University of California Publications, Berkeley, 1, 1914, No. 5, 97–114.

section of the columns giving a and the row giving Δ. For a given a, the reduced form occurs in periods, each period covering a values of Δ; and each period having the same sequence of b's. For a given D, the a's are found among those for which there is a solution of $x^2 \equiv D \pmod{a}$. For the case of Δ without square divisors, he wrote

$$\Delta = \prod_0^r k_i \quad a = \prod_0^u h_i^{\beta_i} \prod_1^v k_i'^{\delta_i},$$

where h and k are primes; $h_0 = 2$, $\delta_i > 1$ and the k''s are those odd k_i's which in a have exponents >1. Let ν be the number of distinct factors k'^n of a; let λ be the greatest value of ν for any a. Then for the given D, the number of reduced forms with $a \leqq \sqrt{\Delta}$ is found to be

$$\sum_{\nu=0}^{\lambda} (-1)^\nu \sum_i \left\{ \sum_{P_0} \left[\frac{[\sqrt{\Delta}]}{l_i^{(\nu)} P_0} \right] \left(\frac{D}{P_0} \right) + \sum_{P_\epsilon} \left[\frac{[\sqrt{\Delta}] + l_i^{(\nu)} P_\epsilon}{2 l_i^{(\nu)} P_\epsilon} \right] \left(\frac{D}{P_\epsilon} \right) \right\},$$

where $l_i^{(\nu)}$ is the i^{th} product formed by taking ν factors k'^n; P_0 is a positive odd integer, P_ϵ a positive even integer, both $\leqq \sqrt{\Delta}$; (D/P_0) is a modified Jacobi symbol and if $P_\epsilon = P_0' 2^r$, P_0' odd, then $(D/P_\epsilon) = (D/P_0')(D/2^r)$, where $(D/2^r)$ is defined so that $1 + (D/2^r)$ is the number of solutions of $x^2 \equiv D \pmod{2^r}$.

The few remaining possible values of a which are $> \sqrt{\Delta}$ and $\leqq \sqrt{4\Delta/3}$ or $\leqq \frac{1}{3}[-1 + 2\sqrt{1+3\Delta}]$, according as a is even or odd, are to be tested by the most elementary methods. Examples show the advantage of this whole process over the classic one of Dirichlet,[23] (5).

E. Landau[342] investigated the asymptotic sum of Dirichlet's series[19]

$$\sum (ax^2 + 2bxy + cy^2)^{-s},$$

in the neighborhood of $s=1$ for a form of positive determinant D. (For $D<0$, see Ch. de la Vallée Poussin, Annales Soc. Sc. Brussells, 20, 1895-6, 372-4).

L. J. Mordell[343] announced the equivalent of two serviceable identities of Petr[344] in theta-functions. For, Mordell's Q and R are respectively Petr's C and $4 \cdot D$. By specializing the arguments in the identities and equating coefficients of like powers of q, Mordell found new representatives of five types of class-number relations such as Petr[252, 258] and Humbert[293] had deduced.

K. Petr[345] combined C. Biehler's[346] generalized Hermetian theta-function expansions, which Petr had used twice[252, 258] before, now with W. Göring's[347] formulas given by the transformation of the third order of the theta-functions. He obtained six expansions similar to the following[348]:

$$B_3 \equiv \Theta_3 \Theta_1^2 \frac{\Theta_3^2(\frac{1}{3}) \Theta_3(\frac{1}{3})}{\Theta^2(\frac{1}{3})} = \frac{1}{\sin^2 \pi/3} + B\Theta_3(\tfrac{1}{3}) - 4 \sum_{m=2}^{\infty} \sum_{k=1}^{m-1} 2kq^{m^2-k^2} \cos 2m\pi/3,$$

in which $q = e^{\pi i \tau}$, and B is found[69] to be $8\Sigma_1^\infty q^N F(N)$, where as usual $F(n)$ is the

[342] Jahresbericht d. Deutschen Math.-Vereinigung, 24, 1915, 250–278.
[343] Messenger Math., 45, 1915, 76–80.
[344] Rozpravy české Akad., Prague, 9, 1900, No. 38 (Petr [252]).
[345] Memorial Volume for the 70th birthday of Court Councilor Dr. K. Vrby, 1915; Rozpravy české Acad., Prague, 24, 1915, No. 22, 10 pp.
[346] Thesis, Paris, 1879.
[347] Math. Annalen, 7, 1874, 311–386.
[348] Cf. G. Humbert,[293] Jour. de Math., (6), 3, 1907, 348.

number of odd Gauss forms of determinant $-n$. On expanding $\Theta_3(\frac{1}{3})$, it is found that the coefficient of q^N in B_3 is

$$-4(\Sigma d_\lambda - \Sigma d_1) + 2\Sigma(d_2 - d_1) - 6\Sigma(d_2^{(0)} - d_1^{(0)}) + 12[F(N) + 2F(N-9\cdot 1^2) + 2F(N-9\cdot 2^2) + \dots],$$

summed for the divisors d of N; the subscript λ on d denotes that the conjugate divisor is odd; the subscript 1 denotes that the divisor agrees in parity with the conjugate and is $\leqq \sqrt{N}$; but, if it $=\sqrt{N}$, it is replaced in the sum by $\frac{1}{2}\sqrt{N}$. Also, $N=d_1d_2$; $N=d_1^0\, d_2^0$, $d_1^{(0)}+d_2^{(0)}\equiv 0 \pmod 3$. This includes the case N odd and $\equiv 1 \pmod 3$ which G. Humbert[349] had failed to provide for in a direct way.

Similarly in

$$B_3' = q^{\frac{1}{3}}\Theta_3\Theta_1^2\Theta_2^2\left(\frac{\tau}{3}\right)\Theta_3\left(\frac{\tau}{3}\right) \Big/ \Theta^2\left(\frac{\tau}{3}\right),$$

the coefficient of $q^{N/9}$ is

$$8\Sigma F\left(\frac{N-(9k\pm i)^2}{9}\right) - \tfrac{2}{3}\Sigma(d_2^{(\delta)} - d_1^{(\delta)}),$$

where the subscript 1 has the same meaning as before, $d_2^{(\delta)} - d_1^{(\delta)} \equiv 0 \pmod 3$ and where $i=1, 2, 4$, according as $N\equiv 1, 4, 7 \pmod 9$. Alternative expansions of B_3 and B_3' were obtained by Petr with indication of a method of determining in them the coefficients of q^N and $q^{N/9}$ respectively in terms of divisors of N and the number of integer solutions of $x^2+y^2+z^2+9u^2=N$ and $x^2+9y^2+9z^2+9u^2=N$, respectively. The class-number relations thus resulting were given by Petr in the next paper.

K. Petr[850] completed[845] the deduction of the following class-number relations. For N arbitrary,

$$F(N) + 2F(N-9\cdot 1^2) + 2F(N-9\cdot 2^2) + \dots = \tfrac{1}{4}[\Sigma d_\lambda + \Sigma d_\lambda^{(0)} - 4\Sigma d_1^{(0)} + \Sigma(-1)^{\frac{1}{2}(x-1)+\frac{1}{6}(y\mp 1)}x],$$

summed over the positive odd numbers x, y satisfying $3x^2+y^2=4N$, such that y is not divisible by 3. [Petr in this and all the following formulas of the paper erroneously imposed the latter condition also on x.] The upper index $^{(0)}$ indicates that the sum of the corresponding divisor and its conjugate is $\equiv 0 \pmod 3$.

Again for N arbitrary,

$$F(N) - 3H(N) + 2[F(N-9\cdot 1^2) - 3H(N-9\cdot 1^2)] + 2[F(N-9\cdot 2^2) - 3H(N-9\cdot 2^2)]\dots = -\tfrac{1}{2}(\Sigma\bar{d} - \Sigma d_e) - \tfrac{1}{2}(\Sigma\bar{d}^{(0)} - \Sigma d_e^{(0)}) + 2\Sigma d_1^{(0)} + \tfrac{1}{2}(-1)^{\frac{1}{2}(x-1)+\frac{1}{6}(x+1)}x,$$

where $\bar{d}$ agrees in parity with its conjugate divisor of N, and d_e is odd.

For $N\equiv 1$, 4 or 7 (mod 9), the two following relations are given:

$$\sum_k F\left(\frac{N-(9k\pm a)^2}{9}\right) = \tfrac{1}{24}\Sigma d_\lambda - \tfrac{1}{6}\Sigma d_1 + \tfrac{1}{24}(-1)^{\frac{1}{2}(x-1)+\frac{1}{6}(y\mp 1)}x,$$

$$\sum_k \left[F\left(\frac{N-(9k\pm a)^2}{9}\right) - 3H\left(\frac{N-(9k\pm a)}{9}\right)^2\right] = -\tfrac{1}{12}(\Sigma\bar{d} - \Sigma d_e) + \tfrac{1}{3}\Sigma d_1 + \tfrac{1}{12}\Sigma(-1)^{\frac{1}{2}(x-1)+\frac{1}{6}(y\mp 1)}x,$$

in which $a=1$, 2 or 4 according as $N\equiv 1$, 4 or 7 (mod 9).

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[849] Jour. de Math., (6), 3, 1907, 431.
[850] Rozpravy české Acad., Prague, 25, 1916, No. 23, 7 pp.

In equating coefficients of q^N in the identity[345] B_8, Petr on his page 2 of the present paper employed the identity

$$2\Theta^4(\tfrac{1}{3})=18\Sigma xq^{\frac{1}{4}(3x^2+y^2)}(-1)^{\frac{1}{2}(x-1)+\frac{1}{6}(y\pm 1)},$$

and failed to observe that x may be $\equiv 0 \pmod 3$. So he introduced an error in the denotation of all the resulting class-number relations of the paper.

L. J. Mordell[851] deduced arithmetically the first class-number relation of his preceding paper[848] in the form

$$(1)\qquad F(m)-2F(m-1^2)+2F(m-2^2)-\ldots=\Sigma(-1)^{\frac{1}{2}(a+d)+1}d,$$

where d is a divisor $\leqq\sqrt{m}$ of m and of the same parity as its conjugate divisor a; but when $d=\sqrt{m}$, the coefficient d is replaced by $\frac{1}{2}d$. Mordell considered the number of representations of an arbitrary positive integer m by the two forms

$$(2)\qquad s^2+n^2+n(2t+1)-r^2=m,$$

$$(3)\qquad d(d+2\delta)=m,$$

$$n>0,\quad -(n-1)\leqq r\leqq n,\quad t\gtreqless 0,\quad d>0,\quad \delta\gtreqless 0.$$

Then, if $f(x)$ is an arbitrary even function of x,

$$(4)\qquad \Sigma(-1)^r f(r+s)=-2\Sigma(-1)^\delta df(d)$$

where the summation on the left extends over all solutions of (2), and the summation on the right extends over all solutions of (3); but, when $\delta=0$, the coefficient 2 is replaced by unity. Now take $f(x)=(-1)^x$. Then (4) becomes $\Sigma(-1)^s=-2\Sigma(-1)^{\delta+d}d$. But for a given s, Mordell[852] found that the number of solutions of (2) is $2F(m-s^2)$. Hence we get at once the above class-number relation (1).

Mordell[852] illustrated his[848] method by writing

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2}e^{2n\pi ix}}{1+q^{2n}}$$

and proving that

$$(1)\quad -\frac{f'(0)}{2\pi i\theta_{00}}=\sum_{n=1}^{\infty}q^{n^2+n}\frac{(1-2q^{-1}+\ldots\pm 2q^{-(n-1)^2}\mp q^{-n^2})}{1-q^{2n}}=\sum_{n=1}^{\infty}(-1)^r q^{n^2-r^2+n(2t+1)},$$

where $r=0,\pm 1,\pm 2,\ldots,\pm(n-1),n$; $t=0,1,2,\ldots$. But corresponding to each set of values n,r,t, there is a reduced quadratic form[853]:

$$nx^2+2rxy+(n+2t+1)y^2$$

of determinant, say, $-M$. Conversely to each reduced form $(a, 0, a)$ of determinant $-M$, there corresponds one solution, and to every other reduced form of determinant $-M$, there correspond two solutions, of the equation $M=n^2-r^2+n(2t+1)$. Hence the right member of (1) is $2\Sigma_1^\infty(-1)^M F(M)q_M$. When $f'(0)$ is given its true value, and q is replaced by $-q$, and θ_{01} by $1-2q+2q^4-2q^9+\ldots$, the

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[851] Messenger Math., 45, 1916, 177-180. See a similar arithmetical deduction by Liouville.[90]

[852] Messenger Math., 46, 1916, 113-128.

[853] And so this expansion (1) suggested to Mordell his[851] arithmetical deduction.

equating of coefficients of q^n yields his[351] relation (1); which is equivalent to Kronecker's[54] (III), (VI), and is identically Petr's[252] relation (II).

Replacing $f(x)$ by $\phi(x)=f(x)\theta_{gh}(x+\xi)/\theta_{00}(x)$, where ξ is an arbitrary constant, Mordell obtained the equivalent of Kronecker's (I), (II), (V). By the use of $\chi(x)=f(x)\theta_{01}(2x, 2\omega)/\theta_{00}(x)$, he obtained a class-number relation involving an indefinite form[354] in the equation $x^2-2y^2=m$. By the use of $f(x)\theta_{00}(3x, 3\omega)/\theta_{00}(x)$, he found (cf. Petr's[258] formula (3) above) that

$$F(2m)-2F(2m-3\cdot 1^2)+2F(2m-3\cdot 2^2)-\ldots=(-1)^{m+1}\Sigma x,$$
$$x^2-3y^2=m,\quad x>0,\quad -\tfrac{1}{3}(x-1)\leqq y\leqq \tfrac{1}{3}x.$$

Replacing $f(x)$, as initially used, by

$$F(x)=\sum_{n=-\infty}^{\infty} q^{\frac{1}{4}n^2}e^{n\pi i x}/(1-q^n),\quad n \text{ odd},$$

he obtained

$$3[G(m)+2G(m-1^2)+2G(m-2^2)+\ldots]=-6\Sigma a+4\Sigma b+2\Sigma(-1)^c c,$$

where a denotes a divisor of m which is $\leqq \sqrt{m}$ and agrees with its conjugate in parity, but if $a=\sqrt{m}$ it is replaced by $a/2$; b denotes a divisor of m whose conjugate is odd, and c a divisor of m whose conjugate is even. Kronecker's[54] (IV) is the special case of this formula for m odd.

G. Humbert,[355] in a principal reduced form (Humbert[185, 186] of Ch. I), (a, b, c) of positive determinant with $b>0$, put $\beta=b-\frac{1}{2}|a+c|$, and, by Hermite's method of equating coefficients in θ-function expansions,[69] found that

$$\sum_n{}^1\left(\frac{-1}{\beta}\right)=2F(4n+2),\quad \sum_n{}^2\left(\frac{-1}{\beta}\right)=2F(8n+5),\quad \sum_n\left(\frac{-1}{\beta}\right)=2F(8n+1),$$

where $\sum_n^1$ extends over all the principal reduced forms of determinant $4n+2$ with a and c odd; $\sum_n^2$ extends over all the principal reduced forms of determinant $8n+5$ with a and c even; $\sum_n$ extends over all the principal reduced forms of determinant $8n+1$ with $\frac{1}{2}(a+c)$ even.

From the first of the three formulas is deduced the following: Among the principal reduced forms (a, b, c) of positive determinant $4n+2$, the number of those in which $b-\frac{1}{2}|a+c|$ is of the form $4k+1$ diminished by the number of those in which it is of the form $4k-1$ is double the number of positive classes of determinant $-(4n+2)$.

By denoting by $H_1(n)$ the left member of the first of these three formulas, for example, and summing as to the argument $4M+2-(2s)^2$, Kronecker's classic formulas[54] give

$$H_1(4M+2)+2H_1(4M+2-2^2)+2H_1(4M+2-4^2)+\ldots=2\phi_i(4M+2),$$

where $\phi_i(n)$ is the sum of the odd divisors of n.

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[354] Cf. K. Petr, Rozpravy české Akad., Prag, 10, 1901, No. 40, formula (1) of the report [258]; also G. Humbert,[298] Jour. de Math., (6), 3, 1907, 381, formula (57).

[355] Comptes Rendus, Paris, 165, 1917, 321–327.

L. J. Mordell[356] recalled Dirichlet's[20] formula (2). Whence[357] if $|r|<1$,

$$(1) \qquad \sum_{a,\,b,\,c}\ \sum_{x,\,y} r^{ax^2+bxy+cy^2}=A+\tau\sum_{k=1}^{\infty}\left(\frac{D}{k}\right)\frac{r^k}{1-r^k},$$

summed for all pairs of integers $x, y>0$, $=0$ or <0, and for representative forms of negative discriminant D; while (D/k) is the generalized symbol of Kronecker[171] and A is the number of classes of discriminant D. We set $r=e^{2\pi i\omega}$ and write (1) as

$$(2) \qquad \phi(\omega)=A+\chi(\omega).$$

When $\chi(\omega)$ is evaluated in terms of θ-functions, (2) becomes:

$$(3) \qquad \phi(\omega)=A+\frac{\tau}{4\pi\sqrt{n}}\left\{f(\omega)+\frac{2\pi}{\sqrt{n}}\sum_{v=1}^{n-1}v\left(\frac{D}{v}\right)\right\},\quad f(\omega)=\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\cdot\frac{\theta'\left(\frac{v}{n}\right)}{\theta\left(\frac{v}{n}\right)},$$

where $\theta(v)=\theta_{11}(v)$. Now[358]

$$\phi\left(-\frac{1}{\omega}\right)=\frac{\omega}{\sqrt{-n}}\phi\left(\frac{\omega}{n}\right),\quad f\left(-\frac{1}{\omega}\right)=\frac{\omega}{\sqrt{-n}}f\left(\frac{\omega}{n}\right).$$

Hence when ω is replaced by $-1/\omega$, (2) gives Kronecker's[171] formula (5_1) for the class-number.

E. Landau[358a] wrote ϵ for the fundamental unit $\frac{1}{2}(T+\sqrt{D}U)$ and by means of Kronecker's[171] class-number formula (3), obtained an upper bound of $\log\epsilon/\sqrt{D}\log D$ for very great D by noting that $K(D)\gtreqless 1$ and finding an upper bound of the sum of the Dirichlet series in that formula.

E. Landau[359] wrote $h(k)$ for the number of classes of ideals of the imaginary field defined by $\sqrt{-k}$. Let δ be any positive number. If there are infinitely many negative values $-k^{(\nu)}$ of $-k(k^{(1)}<k^{(2)}<\dots)$ such that

$$h(k)<k^{\frac{1}{2}-\delta},$$

then, for every real $\omega>1$, $k^{(\nu+1)}>k^{(\nu)^{\omega}}$ for every ν exceeding a value depending on δ and ω. Given any $\omega>1$, if we can assign c, depending on ω, such that,

$$h(k)<c\sqrt{k}/\log k$$

holds for an infinitude of negative values $-k^{(\nu)}$ of $-k$, then $k^{(\nu+1)}>k^{(\nu)^{\omega}}$ for every $\nu\gtreqless 1$. Known facts are proved about limits to $h(k)$. He[360] derived inequalities relating to $h(k)$.

G. Humbert[361] let m_1 and m_2 be the odd minima of an odd Gaussian form (a, b, c), and $H(M)$ be the number of odd reduced forms of determinant $-M$ for which m_1

356 Messenger Math., 47, 1918, 138–142.
357 Obtained independently by Petr,[305] (1).
358 Cf. Mordell, Quar. Jour. Math., 46, 1915, 105.
358a Göttingen Nachr., 1918, 86–7.
359 Göttingen Nachr., 1918, 277–284, 285–295 (95–97).
360 Math. Annalen, 79, 1919, 388–401.
361 Unpublished letter to E. T. Bell, October 15, 1919.

or m_2 is $\equiv 0 \pmod{p}$, p being a given odd prime; and if simultaneously $m_1 \equiv m_2 \equiv 0 \pmod{p}$, he let the class count 2 units in $H(M)$; then, when $N \equiv 0 \pmod{p}$,

$$\Sigma(-1)^n H(N-n^2) + \Sigma(-1)^n H(N-n^2) = -2\Sigma(d/p)(-1)^{\frac{1}{2}(d'+d)},$$

where the first summation extends over all integers $n \equiv 0 \pmod{p}$, the second over the positive integers n not $\equiv 0 \pmod{p}$, and the third over all decompositions $N = dd'$, with $d \equiv 0 \pmod{p}$, $d < d'$ and d, d' of the same parity. The class $a(x^2+y^2)$, when $a \equiv 0 \pmod{p}$, counts here as one unit in $H(a^2)$.

Let $\phi_h(N)$ be the number of classes of positive odd Gaussian forms of determinant $-N$, for which the minimum μ is $\gtreqless 2h$; if $\mu = 2h$, the class counts for $\frac{1}{2}$ in $\phi_h(N)$. Then for N odd, positive, and prime to 3,

$$\phi_0(N) + 2\phi_1(N+3\cdot 1^2) + \ldots + \phi_h(N+3\cdot h^2) + \ldots$$
$$= \frac{3}{4}\left(\frac{-3}{N}\right)\left[1+\left(\frac{-N}{3}\right)\frac{1}{3}\right]\Sigma d\left(\frac{-3}{d}\right) + \frac{1}{2}\Sigma\left(\frac{3}{d}\right),$$

where in the second member, the summations extend over all divisors d of N. In the first member, ϕ_h certainly equals zero when h is $> \frac{1}{4}(N+1)$.

Similarly, N being odd, let $\phi'_h(N)$ be the number of classes of positive even forms for which the minimum μ is $\gtreqless 2h$; if $\mu = 2h$, the class counts $\frac{1}{2}$ in $\phi'_h(N)$. Then we have

$$\phi'_0(N) + 2\phi'_1(N+2\cdot 1^2) + 2\phi'_2(N+2\cdot 2^2) + \ldots + 2\phi'_h(N+2h^2) + \ldots$$
$$= \frac{1}{6}\left[\left(\frac{-2}{N}\right) - \frac{1}{2}\right]\Sigma d\left(\frac{-2}{d}\right) + \frac{1}{4}\Sigma\left(\frac{2}{d}\right) + \frac{1}{3}\left[1+\left(\frac{N'}{3}\right)\right]\Sigma\left(\frac{6}{d}\right),$$

where, in the second member, the summations extend over all divisors d of N; $(6/d) = 0$ if $d \equiv 0 \pmod{3}$, and N' is the quotient of N by the highest power of 3 that divides N.

And similarly,[362] let $\psi_h(M)$ be the number of reduced odd Gaussian forms (a, b, c) of determinant $-M$ for which simultaneously $a \gtreqless 2h$, $a+c-|b| \gtreqless 5h$; if in these relations, there is a single equality sign, the form counts $\frac{1}{2}$ in ψ_h; if there are two equality signs, the form counts $\frac{1}{4}$. Then, if $N \equiv 7, 17, 23$, or $33 \pmod{40}$,

$$\psi_0(N) + 2\psi_1(N+5\cdot 1^2) + \ldots + 2\psi_h(N+5\cdot h^2) + \ldots = \tfrac{1}{2}(-5/N)\Sigma d(-5/d),$$

the summation extending over all divisors d of N.

Class-number relations occur incidentally in Humbert's papers 18, 23, 24 of Ch. XV.

L. L. Mordell[363] deduced his[364] formula (1) from the identity

$$\omega\theta_{11}(x,\omega)\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2-2\pi tx}}{e^{2\pi t}-1}\,dt = f\left(\frac{x}{\omega}, -\frac{1}{\omega}\right) + i\omega f(x, \omega),$$

where the path of integration may be a straight line parallel to the real axis and below it a distance less than unity, and where

$$if(x) = \sum_{n \text{ odd}}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(n-1)} q^{\frac{1}{4}n^2} e^{n\pi ix}}{1+q^n}.$$

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362 Deduced by Humbert from his own formula (7), Comptes Rendus, Paris, 169, 1919, 410.
363 Messenger Math., 49, 1919, 65–72.

By applying Kronecker's[54] formula (XI) to the right member of formula[364] (1) and integrating the left member, Mordell obtained the relation[364] (3). But by applying the identity

$$\theta_{00}(0, -1/\omega)=\sqrt{-i\omega}\,\theta_{00}(0,\omega)$$

to the right member, he found

$$\frac{1}{2a^2}+\sum_1^\infty\frac{1}{(n+a)^2}=2\pi^2\sum_{M=0}^\infty[4F(M)-3G(M)]e^{-2a\pi\sqrt{M}}-4a\sum_{M=1}^\infty\frac{F(M)}{(a^2+M^2)^2}.$$

L. J. Mordell[364] announced without proof the formulas:

$$(1)\qquad \int_{-\infty}^{\infty}\frac{te^{\pi i\omega t^2}}{e^{2\pi t}-1}dt=-2\sum_1^\infty F(n)q^n+\frac{2}{\omega^2}\sqrt{-i\omega}\sum_1^\infty F(n)q_1^n+\tfrac{1}{4}\theta_{00}^3(0,\omega),$$

$$(2)\qquad \int_{-\infty}^{\infty}\frac{te^{\pi i\omega t^2}}{e^{2\pi t}+1}dt=\sum_1^\infty(-1)^nF(4n-1)q^{\frac{1}{4}(4n-1)}+\frac{2}{\omega^2}\sqrt{-i\omega}\sum_1^\infty(-1)^{n-1}F(n)q_1^n,$$

where $R(i\omega)<0$, $q=e^{\pi i\omega}$, $q_1=e^{-\pi i/\omega}$. Proofs were given elsewhere.[363] By integrating, he deduced from (1) the relation,

$$(3)\qquad \frac{1}{2a^2}+\sum_1^\infty\frac{1}{(n+a)^2}=-4\pi^2\sum_1^\infty F(M)e^{-2a\pi\sqrt{M}}+2a\sum_0^\infty\frac{4F(M)-3G(M)}{(a^2+M)^2},$$

where $R(a)>0$, a arbitrary.

E. T. Bell[365] proved that

$$\left.\begin{array}{lll}(1) & m=4k+1, & N_3(m)=6[\epsilon(m)+4\Sigma\xi\{\frac{1}{2}(m-\mu^2)\}]\\ (2) & m=4k+3, & N_3(m)=8\Sigma\xi\{\frac{1}{2}(m-\mu^2)\}\end{array}\right\}\ \mu \text{ odd } <\sqrt{m},$$

where $N_3(m)$ is the number of representations of n as the sum of 3 squares; $\epsilon(n)=1$ or 0, according as n is or not a square; and $\xi(n)$ is the excess of the number of divisors $4k+1$ of n over the number of divisors $4k+3$. He[366] then stated that elementary considerations yield

$$\begin{array}{lll}(3) & m=4k+1, & N_3(m)=6[\xi(m)+2\Sigma\xi(m-4a^2)],\\ (4) & m \text{ odd}, & N_3(2m)=12[\xi(m)+2\Sigma\xi(m-2a^2)],\\ (5) & m \text{ odd}, & N_3(2m)=12\Sigma\xi(2m-\mu^2),\\ (6) & n \text{ arbitrary}, & N_3(n)=2[\epsilon(n)+2\xi(n)+4\Sigma\xi(n-a^2)],\end{array}$$

where m, n, a are positive integers, μ is any positive odd integer, and where x is as always >0 in $\Sigma\xi(x)$. A comparison of (1), (2), (3), (4), (5), (6), with the well-known relations (Kronecker,[54] (XI); Hermite,[69] (7))

$$(7)\qquad \begin{cases} m=4k+1, & N_3(m)=12F(m);\quad N_3(n)=12[2F(n)-G(n)],\\ m=8k+3, & N_3(m)=\ 8F(m);\quad N_3(2m)=12F(2m),\end{cases}$$

[364] Quar. Jour. Math., 48, 1920, 329–334.
[365] Quar. Jour. Math., 49, 1920, 45–51.
[366] Quar. Jour. Math., 49, 1920, 46–49.

where $G(n)$ denotes the total number of classes and $F(n)$ the number of uneven classes of determinant $-n$, gives immediately

$$\begin{array}{ll} m=4k+1, & 2G(m)=\xi(m)+2\Sigma\xi(m-4a^2), \\ m=4k+1, & 2G(m)=\epsilon(m)+4\Sigma\xi\{\tfrac{1}{2}(m-\mu^2)\}, \\ m \text{ odd}, & G(2m)=\xi(m)+2\Sigma\xi(m-2a^2), \\ m \text{ odd}, & G(2m)=\Sigma\xi(2m-\mu^2), \\ m=8k+3, & F(m)=\Sigma\xi\{\tfrac{1}{2}(m-\mu^2)\}, \\ & 12F(n)-6G(n)=\epsilon(n)+2\xi(n)+4\Sigma\xi(n-a^2). \end{array}$$

Similarly by comparing (7) with seven recursion formulas[367] such as

$$m=4k+1, \quad N_8(m)=6\zeta_1\{\tfrac{1}{2}(m+1)\}-\Sigma N_8(m-8t),$$

in which $\zeta_1(n)$ denotes the sum of all the divisors of n, and $t>0$ an arbitrary triangular number, he obtained the seven following recursion formulas for class-number:

$$\begin{array}{ll} m=4k+1, & 2G(m)+2\Sigma G(m-8t)=\zeta_1\{\tfrac{1}{2}(m+1)\}, \\ m=4k+1, & 2G(m)+4\Sigma G(m-4a^2)=\zeta_1(m), \\ m \text{ odd}, & 4G(2m)+4\Sigma G(2m-8t)=\zeta_1(2m+1), \\ m \text{ odd}, & G(2m)+2\Sigma G(2m-4a^2)=\zeta_1(m), \\ m=8k+3, & F(m)+\Sigma F(m-8t)=\zeta_1\{\tfrac{1}{4}(m+1)\}, \\ m=8k+3, & 4F(m)+8\Sigma F(m-4a^2)=\zeta_1(m), \\ & 6E(n)+12\Sigma E(n-4a^2)=4(-1)^n\lambda_1(n)-\epsilon(n), \end{array}$$

in which $\lambda_1(n)=[2(-1)^n+1]\zeta_1'(n)$, where $\zeta_1'(n)$ is the sum of the odd divisors of n; and in which $E(n)=2F(n)-G(n)$. The last of these relations is equivalent to Kronecker's[54] formula (X).

L. J. Mordell,[368] starting from Dirichlet's[20] formula (1)

$$\frac{\pi}{2}h(-n)=\sqrt{n}\sum_{r \text{ odd}}\frac{1}{r}\left(\frac{-n}{r}\right),$$

and allowing for the improper classes, proved that

$$8\sum_1^\infty F(n)q^n=\Sigma\frac{i^{-\frac{1}{2}(b-1)}(a/b)}{[\sqrt{-i(a+bi)}]^3}=\Sigma\left[\frac{\theta_{00}(\omega)}{\theta_{00}\left(\dfrac{c+d\omega}{a+b\omega}\right)}\right]^3, \quad q=e^{\pi i\omega},$$

where the real part of $i\omega$ is <0; the radical is taken with positive real part; the summation is carried out first for $a=0, \pm2, \pm4, \ldots$, and then for $b=1, 3, 5, \ldots$, in this order; (a/b) is the Legendre symbol; but if $a=0$, $b=1$, we replace (a/b) by 1. Also d is any even integer, c any odd integer, satisfying $ad-bc=1$. He also proved that $F(M)/\sqrt{M}=f(1)-\frac{1}{3}f(3)+\frac{1}{5}f(5)-\ldots$, where $f(n)$ denotes the number of solutions of $\xi^2\equiv M \pmod{n}$. Formulas of the same type are also given in which $F(n)$ is replaced by $G(n)$.

E. T. Bell,[369] by equating like powers q in the expansions of functions of elliptic theta constants, showed that the class-number relations of Kronecker, Hermite and

[367] Bell, Amer. Jour. Math., 42, 1920, 185–187.
[368] Messenger Math., 50, 1920, 113–128.
[369] Annals of Math., 23, 1921, 56–67; abstract in Bull. Amer. Math. Soc., 27, 1921, 151.

others may be reversed so as to give the class-number of a negative determinant explicitly in terms of the total number of representations of certain integers each as a sum of squares or triangular numbers.

Bell,[370] by paraphrasing identities between doubly periodic functions of the first and third kinds, obtained three class-number relations involving a wholly arbitrary even function $f(u)=f(-u)$. Let $\epsilon(n)=1$ or 0 according as n is or is not the square of an integer; let $F(n)$ and $F_1(n)$ denote the number of odd and even classes respectively for the determinant $-n$, $n \gtreqless 0$. The first and simplest of the three similar relations is

$$\Sigma\epsilon(a')[f(\sqrt{a'}-d'')-f(\sqrt{a'}+d'')]+2\Sigma'\left[f\left(\frac{d'+\delta'}{2}-d''\right)-f\left(\frac{d'+\delta'}{2}+d''\right)\right]$$
$$=\Sigma F(\beta-4r^2)f(2r)-\Sigma'\left(\frac{\delta-d}{2}\right)\ f\left(\frac{\delta+d}{2}\right),$$

the Σ, Σ' extending over all indicated positive integers $a', \ldots, d, \delta$ such that, for β fixed,

$$\beta\equiv 3 \pmod 4),\ \beta=a'+2m''\equiv d'\delta'+2d''\delta''$$
$$(a'=d'\delta',\ m''=d''\delta''),\ \text{and}\ \beta=d\delta,\ d<\sqrt{\beta};$$
$$a'\equiv 1 \pmod 4),\ d'<\sqrt{a'};\ \beta-4r^2>0.$$

Interpreting results obtained by putting $f(x)=0$, $|x|>0$, $f(0)=1$ in the three relations, it follows that the total number of representations of any prime p by $xy+yz+zx$, with x, y, z all >0, is $3[G(p)-1]$ where $G(n)=F(n)+F_1(n)$; that the like is true only when p is prime; that there are more quadratic residues than non-residues of the prime $p\equiv 3 \pmod 4$ in the series $1, 2, \ldots, \frac{1}{2}(p-1)$; and so for $p\equiv 1 \pmod 4$ in the series $1, 2, \ldots, \frac{1}{4}(p-1)$.

If $f(x)=1$ for all values of x, the first relation gives Hermite's[69] (3): $\Sigma F(\beta-4r^2)=\frac{1}{2}\Psi_1(\beta)$, where $\Psi_s(n)$ is the sum of the sth powers of all the divisors $>\sqrt{n}$ of n diminished by the sum of the sth powers of all the divisors $<\sqrt{n}$ of n. For $f(x)=x^2$, the first relation gives:

$$32\Sigma r^2F(\beta-4r^2)=\Psi_3(\beta)+\beta\Psi_1(\beta)-32N(4\beta),$$

the Σ extending over all integers r such that $\beta-4r^2>0$, and $N(4\beta)$ is the number of representations of 4β in the form

$$m_1^2+m_2^2+m_3^2+m_4^2+2m_5^2+2m_6^2+2m_7^2+2m_8^2$$

for which the $m_i (i=1, 2, \ldots, 8)$ are odd and $\neq 0$, and precisely 0, 2 or 4 of m_1, m_2, m_3, m_4 in each representation are included among the forms $8k\pm 1$. The paper contains a table of the value of $F(n)$, $n=1, \ldots, 100$.

E. T. Bell[371] obtained 18 class-number relations which are similar to his[370] three above and which form a complete set in the sense that no more results of the same general sort are explicit in the analysis. By specializing the arbitrary even functions which occur in these formulas, he stated that all the class-number relations of

[370] Tôhoku Math. Jour., 19, 1921, 105–116.
[371] Quar. Jour. Math., 1923(?); abstract in Bull. Amer. Math. Soc., 27, 1921, 152.

Kronecker and Hermite and certain of those of Liouville and Humbert are obtained as special cases.

L. J. Mordell[872] showed that the number of solutions in positive integers of $yz+zx+xy=u$ is $3G(n)$. It is shown essentially by Hermite's[69] classical method that $x+y\equiv 1 \pmod 2$ for $2F(n)$ of the solutions; $x+y\equiv 2 \pmod 4$ for $F(n)$ of the solutions; and $x+y\equiv 0 \pmod 4$ for $3G(n)-3F(n)$ of the solutions, where always a solution is counted $\frac{1}{2}$ if one of the unknowns is 0. In particular, if n is not a perfect square, $x+y\equiv 1 \pmod 4$ for $F(n)$ of the solutions, $x+y\equiv 3 \pmod 4$ for $F(n)$ of the solutions. Particular cases had been given by Liouville[88] and Bell.[870]

G. H. Cresse[874] reproduced J. V. Uspensky's[882] arithmetical deduction of Kronecker's[54] class-number relations I, II, V and supplied some details of the proof.

R. Fricke[875] (p. 134) obtained and (p. 148) translated[873] a result of Dedekind[127a] in ideals into a solution of the Gauss Problem[4] (Cf. Weber[810]). He reproduced and amplified (pp. 269-541) Klein's theory of the modular function.[184] He denoted (p. 360) by W the substitution $\omega'=-\omega/n$ and by $\Gamma\psi(n)$ that sub-group of the modular group $\omega'=(\alpha\omega+\beta)/(\gamma\omega+\delta)$ for which $\gamma\equiv 0 \pmod n$. The fundamental polygon[184] for the group $\Gamma\psi(n)$ is called the transformation polygon T_n. Fricke found (p. 363) that in T_n, the number of fixed points for elliptic substitutions of period 2 among the substitutions of $\Gamma\psi(n)\cdot W$ is $Cl(-4n)$ if $n\equiv 0, 1, 2 \pmod 4$ and is $Cl(-4n)+Cl(-n)$ if $n\equiv 3 \pmod 4$.

Finally it should be noted that the class-number may be deduced[873] from the number of classes of ideals in an algebraic field since there is a (1, 1) correspondence between the classes of binary quadratic forms of discriminant D and the narrow classes of ideals in a quadratic field of discriminant D (Dedekind[29] of Ch. III). For the class-number of forms with complex integral coefficients, see Ch. VIII.

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872 Amer. Jour. Math., Jan., 1923. Abstract in Records of Proceedings of London Math. Soc., Nov. 17, 1921.

873 Dedekind in Dirichlet's Zahlentheorie, ed. 4, 1894, 639.

874 Annals of Math., 23, March, 1922.

875 Die Elliptischen Functionen und ihre Anwendungen, II, 1922.

AUTHOR INDEX.

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CH. VI. NUMBER OF CLASSES OF BINARY QUADRATIC FORMS WITH INTEGRAL COEFFICIENTS.