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ON CERTAIN LOCI PROJECTIVELY CONNECTED WITH A GIVEN PLANE CURVE

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DEPARTMENT OF MATHEMATICS

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ON CERTAIN LOCI PROJECTIVELY CONNECTED WITH A GIVEN PLANE CURVE

BY

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1. *Introduction.* The projective theory of plane curves may be based on the following interpretations due to H a l p h e n, and systematically developed by W i l c z y n s k i [1]. Let y_1, y_2, y_3 constitute a fundamental system of solutions of a linear homogeneous differential equation of the third order,

$$y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0, \tag{1}$$

and let us interpret y_1, y_2, y_3 as the homogeneous coordinates of a point P_y.

As the independent variable t assumes different values this point will describe a curve C_y:

$$y_i = y_i(t), \qquad (i = 1, 2, 3),$$

which shall be called an *integral curve* of (1). Each of the two semicovariants $z = y' + p_1 y$ and $\rho = y'' + 2p_1 y' + p_2 y$ of (1) assumes three values,

$$\begin{aligned} z_i &= y'_i + p_1 y_i, \\ & \qquad\qquad\qquad (i = 1, 2, 3), \\ \rho_i &= y''_i + 2p_1 y'_i + p_2 y_i, \end{aligned} \tag{2}$$

[1] W i l c z y n s k i, *Projective Differential Geometry of Curves and Ruled Surfaces.* Leipzig, Teubner, 1906, p. 60, Hereafter referred to as W.

according as we substitute in it for y the functions y_1 , y_2 , y_3. Thus the expressions (2) determine two further points P_z and P_ρ. If P_y is not a point of inflection the three points P_y , P_z , P_ρ will form a non-degenerate triangle, which may be used as a local triangle of reference for the purpose of studying the projective properties of the integral curve in the vicinity of the point P_y. The unit point of our local system of homogeneous coordinates may be chosen in such a way that the point determined by the expression

$$\tau = x_1 y + x_2 z + x_3 \rho$$

shall have the homogeneous coordinates x_1 , x_2 , x_3 in the local system.

W i l c z y n s k i has shown how to determine the coordinates of certain points and lines which are connected in invariant fashion with the given point P_y of the curve C_y. Most of these are connected with the theory of the so-called eight-pointic nodal cubic. But W i l c z y n s k i used a specialized system of local coordinates for this purpose, a fact which makes it rather difficult to make specific applications of his formulae to special curves. We shall, in this paper, determine the coordinates of these same points without specializing the local coordinate system. We shall, moreover, consider certain other points and lines, primarily those which are connected with the theory of the osculating anharmonic curve. These have been mentioned before, but they have nover received adequate attention. We shall then study the loci and envelopes generated by these various points and lines as the point P_y moves along the curve C_y.

At this point I wish to express my indeptedness to Professor E. J. W i l c z y n s k i, under whose direction this paper was written. His patience, kindness and encouragement have been important factors in its preparation.

2. *The specialized coordinate system.* The specialized local coordinate system used by W i l c z y n s k i is obtained as a result of the following considerations. It is always possible, by means of a trasformation of the type $y = \lambda(t)\bar{y}$, to reduce equation (1) to the semi-canonical form [1]

$$\bar{y}''' + 3P_2 \bar{y}' + P_2 \bar{y} = 0 , \tag{3}$$

characterized by the absence of the second order derivative, where

$$\begin{aligned} P_2 &= p_2 - p^2_1 - p'_1 , \\ P_3 &= p_3 - 3p_1 p_2 + 2p^3_1 - p''_1 . \end{aligned} \tag{4}$$

(1) W., p. 16.

If now we make the transformation [1]

(5) $$\bar{t} = \xi(t) ,$$

this equation reduces to

(6) $$\frac{d^3\bar{y}}{d\bar{t}^3} + 3\bar{P}_2 \frac{d\bar{y}}{d\bar{t}} + \bar{P}_3 \bar{y} = 0 ,$$

where

$$\bar{P}_2 = \frac{1}{(\xi')^2}\left[P_2 - \frac{2}{3}\mu\right],$$

$$\bar{P}_3 = \frac{1}{(\xi')^3}\left[P_3 - 3\eta P_2 - \mu' + 2\mu\eta\right],$$

and

$$\xi' = \frac{d\xi}{dt} , \text{ etc.}, \ \eta = \frac{\xi''}{\xi'} , \ \mu = \eta' - \frac{1}{2}\eta^2.$$

There are certain combinations of P_2, P_3, and of their derivatives which remain unchanged, except for a factor $\frac{1}{(\xi')^k}$, when we make such a transformation of the independent variable. Such combinations are called ***relative invariants***. The simplest of these are

$$\theta_3 = P_3 - \frac{3}{2}P'_2 ,$$

(7)

$$\theta_8 = 6\theta_3 \theta''_3 - 7(\theta'_3)^2 - 27 P_2 \theta^2_3 .$$

All other relative invariants are functions of θ_3, θ_8, and their successive Jacobians [2]. The following Jacobians will be used in this paper:

(8)

$$\theta_{12} = 3\theta_3 \theta'_8 - 8\theta'_3 \theta_8 , \qquad \theta_{20} = 3\theta_3 \theta'_{16} - 16\theta'_3 \theta_{16} ,$$

$$\theta_{16} = \theta_3 \theta'_{12} - 4\theta'_3 \theta_{12} , \qquad \theta_{24} = 3\theta_3 \theta'_{20} - 20\, \theta'_3 \theta_{20} .$$

(1) W., p. 20.
(2) W., p. 33.

It is frequently of advantage to have explicit formulae for these, in terms of θ_3 and its derivatives. For this reason the following formulae are here set down:

$$\theta'_8 = 6\theta_3\,\theta'''_3 - 8\theta'_3\,\theta''_3 - 54P_2\,\theta_3\,\theta'_3 - 27P'_2\,\theta^2_3,$$

$$\theta_{12} = 18\theta^2_3\,\theta'''_3 - 72\theta_3\,\theta'_3\,\theta''_3 + 56(\theta'_3)^3 + 54P_2\,\theta^2_3\,\theta'_3 - 81P'_2\,\theta^3_3,$$

$$\theta'_{12} = 18\theta^2_3\,\theta^{(4)}_3 - 36\theta_3\,\theta'_3\,\theta'''_3 + 96(\theta'_3)^2\theta''_3 - 72\theta_3(\theta''_3)^2 + 54P_2\,\theta^2_3\,\theta''_3 +$$

$$+ 108P_2\,\theta_3(\theta'_3)^2 - 189P'_2\,\theta^2_3\,\theta'_3 - 81P''_2\,\theta^3_3,$$

$$\theta_{16} = 18\theta^3_3\,\theta^{(4)}_3 - 108\theta^2_3\,\theta'_3\,\theta'''_3 + 384\theta_3(\theta'_3)^2\theta''_3 - 72\theta^2_3(\theta''_3)^2 - 224(\theta'_3)^4 +$$

$$+ 54P_2\,\theta^3_3\,\theta''_3 - 108P_2\,\theta^2_3(\theta'_3)^2 + 135P'_2\,\theta^3_3\,\theta'_3 - 81P''_2\,\theta^4_3.$$

We shall also need the following *absolute invariants*:

$$\lambda = \frac{\theta^3_8}{\theta^8_3}\,; \quad \mu = \frac{\theta_{12}}{\theta^4_3}\,; \quad \nu = \frac{\theta_8\,\theta_{16}}{\theta^8_3}\,; \quad \sigma = \frac{\theta^2_8\,\theta_{20}}{\theta^{12}_3}. \tag{9}$$

We shall assume throughout that θ_3 is not identically equal to zero, that is, that the curve C_y is not a conic ([4]).

Since z and ρ, as defined by (2), are changed by every non-linear transformation of the independent variable, we may think of (5) as a transformation of coordinates from one local triangle of reference to another. In particular, we can so choose $\xi(t)$ that $\overline{P}_2 = 0$. Then equation (6) becomes

$$\frac{d^3\bar{y}}{d\bar{t}^3} + \overline{P}_3\,\bar{y} = 0, \tag{10}$$

which is known as the Laguerre-Forsyth canonical form of (1). If we assume that the original equation (1) is in the Laguerre-Forsyth form, we have the following values for the relative invariants:

$$\begin{aligned} &\theta_3 = P_3, && \theta_{12} = 18P^2_2P'''_3 - 72P_3P'_3P''_3 + 56(P'_3)^3, \\ &\theta_8 = 6P_3P''_3 - 7(P'_3)^2, && \text{etc.} \\ &\theta'_8 = 6P_3P'''_3 - 8P'_3P''_3. && \end{aligned} \tag{11}$$

([1]) W., p. 61.

3. *The osculating conic and cubic.* By makins use of the special triangle of reference which corresponds to the Laguerre-Forsyth form of the equation. Wilczynski has shown that the equation of the conic and cubic which osculate the curve C_y at the point P_y are, respectively, (4)

$$x^2_2 - 2x_1x_3 = 0\,,$$

$$7[15P_3P'''_3 - 20P'_3P''_3 - 567P^3_{,2}]\,\Omega_1(x) + 20[6P_3P''_3 - 7(P'_3)^2]\,\Omega_2(x) = 0.$$

where

$$\Omega_4(x) = 5(x^2_2 - 2x_1x_3)(P'_3x_3 - 3P_3x_2) + 12P^2_3x^3_3\,,$$

$$\Omega_2(x) = 5(x^2_2 - 2x_1x_3)(21P_3x_1 - P''_3x_3) - 42P^2_3x_2x^2_3 - 14P_3P'_3x^3_3.$$

These equation may be written in the form

$$x^2_2 - 2x_1x_3 \equiv 0\,,$$

$$7\left[567P^3_3 - \frac{5}{2}(6P_3P'''_3 - 8P'_3P''_3)\right]\Omega_1(z) + 20\left[6P_3P''_3 - 7(P'_3)^2\right]\Omega_2(x) = 0\,,$$

where

$$\Omega_1(x) = 5(x^2_2 - 2x_1x_3)(P'_3x_3 - 3P_3x_2) + 12P^2_3x^3_3\,,$$

$$6P_3\Omega_2(x) = 5(x^2_2 - 2x_1x_3)\left[\}6P_3P''_3 - 7(P'_3)^2\{\,x_3 + 7(P'_3)^2x_3 - 126P^2_3x_1\right] +$$

$$+ 252P^3_3x_2x^2_3 + 84P^2_3P'_3x^3_3.$$

Making use of (11), we obtain the following equations for the osculating conic and cubic:

$$x^2_2 - 2x_1x_3 = 0\,,$$

(12)

$$7\left[567\theta^3_3 - \frac{5}{2}\,\theta'_8\right]\Omega_1(x) + 20\theta_8\,\Omega_2(x) = 0,$$

where

$$\Omega_1(x) = 5(x^2_2 - 2x_1x_3)(\theta'_3x_3 - 3\theta_3x_2) + 12\theta^2_3x^3_3\,,$$

$$6\theta_3\Omega_2(x) = 5(x^2_2 - 2x_1x_3)\,[\theta_8x_3 + 7(\theta'_3)^2x_3 - 126\theta^2_3x_1] +$$

$$+ 252\theta^3_3x_2x^2_3 + 84\theta^2_3\theta'_3x^3_3.$$

---

(4) W., p. 64.

We now raise the question: what do these equations become when the local coordinate system is not specialized? Wilczynski has given us the answer only in the case of the conic. Let us assume that the coordinates of a covariant point, referred to the special local coordinate system, are

$$
(13) \qquad
\begin{aligned}
x_1 &= f_1(\theta_3, \theta'_3, \theta_8, \theta'_8, \ldots), \\
x_2 &= f_2(\theta_3, \theta'_3, \theta_8, \theta'_8, \ldots), \\
x_2 &= f_3(\theta_3, \theta'_3, \theta_8, \theta'_8, \ldots),
\end{aligned}
$$

where θ_3, θ'_3, θ_8, etc., are defined by (11), and see if we can find the coordinates $\overline{x_1}$, $\overline{x_2}$, $\overline{x_3}$, of the same point referred to the general local system.

After making the transformation (5) it is found that [1]

$$
(14) \qquad
\begin{aligned}
\bar{y} &= y, \\
\bar{z} &= \frac{1}{\xi'}(z + \eta y), \\
\bar{\rho} &= \frac{1}{(\xi')^2}\left[\rho + \eta z + \left(\frac{1}{3}\mu + \frac{1}{2}\eta^2\right)y\right].
\end{aligned}
$$

Whence

$$
\bar{x}_1\bar{y} + \bar{x}_2\bar{z} + \bar{x}_3\bar{\rho} = \left[\bar{x}_1 + \frac{1}{\xi'}\eta\bar{x}_2 + \frac{1}{(\xi')^2}\left(\frac{1}{3}\mu + \frac{1}{2}\eta^2\right)\bar{x}_3\right]y + \left[\frac{1}{\xi'}\bar{x}_2 + \frac{1}{(\xi')^2}\eta\bar{x}_3\right]z +
+ \frac{1}{(\xi')^2}\bar{x}_3\rho,
$$

$$
= \frac{\omega}{(\xi')^2}[x_1 y + x_2 z + x_3\rho],
$$

where

$$
\begin{aligned}
\omega x_1 &= (\xi')^2\overline{x_1} + \xi'\eta\overline{x_2} + \left(\frac{1}{3}\mu + \frac{1}{2}\eta^2\right)\overline{x_3}, \\
\omega x_2 &= \qquad\qquad \xi'\overline{x_2} \qquad\qquad + \eta\,\overline{x_3}, \\
\omega x_3 &= \qquad\qquad\qquad\qquad\qquad\qquad \overline{x_3}.
\end{aligned}
$$

---

[1] W., p. 59.

Since x_1, x_2, x_3 refer to the special system, we have $P_2 = 0$ and $(\xi')^2 \overline{P}_2 = -\frac{2}{3}\mu$.

Hence the equations of transformation from the specialized to the general local system of coordinates are

$$\omega x_1 = (\xi')^2 \overline{x}_1 + \xi'\eta \overline{x}_2 + \frac{1}{2}[\eta^2 - (\xi')^2 \overline{P}_2]\overline{x}_3,$$

$$\omega x_2 = \xi' \overline{x}_2 + \eta \overline{x}_3,$$

$$\omega x_3 = \overline{x}_3.$$

For a covariant point the terms involving η must disappear and a power of ξ' come out as a factor. We are thus led to the theorem: *If the coordinates of a covariant point, referred to the special local coordinate system, are given by* (13), *the coordinates of the same point, referred to a general local coordinate system, are*

$$\overline{x}_1 = f_1(\overline{\theta}_3, \overline{\theta}'_3, \overline{\theta}_8, \overline{\theta}'_8, \ldots) + \frac{1}{2}\overline{P}_2 f_3(\overline{\theta}_3, \overline{\theta}'_3, \overline{\theta}_8, \overline{\theta}'_8, \ldots),$$

$$\overline{x}_2 = f_2(\overline{\theta}_3, \overline{\theta}'_3, \overline{\theta}_8, \overline{\theta}'_8, \ldots),$$

$$\overline{x}_3 = f_3(\overline{\theta}_3, \overline{\theta}'_3, \overline{\theta}_8, \overline{\theta}'_8, \ldots),$$

where

$$\overline{\theta}_3 = \overline{P}_3 - \frac{3}{2}\overline{P}'_2,$$

$$\overline{\theta}_8 = 6\overline{\theta}_3 \overline{\theta}''_3 - 7(\overline{\theta}'_3)^2 - 27\overline{P}_2 \overline{\theta}^2_3.$$

Hence, in order to obtain the general equations corresponding to (12) it is only necessary to take the general values of the relative invariants as defined by (7) and (8), and then replace x_1 by $x_1 - \frac{1}{2}P_2 x_3$. The equation of the osculating conic and cubic then become

$$x^2_2 - 2x_1x_3 + P_2 x^2_3 = 0,$$

(15)

$$7\left[567\theta^3_3 - \frac{5}{2}\theta'_8\right]\Omega_1(x) + 20\theta_8\,\Omega_2(x) = 0,$$

where

$$\Omega_1(x) = 5(x^2_2 - 2x_1x_3 + P_2x^2_3)(\theta'_3x_3 - 3\theta_3x_2) + 12x^2_2x^3_3 ,$$

$$\Omega_2(x) = 5(a^2_2 - 2x_1x_3 + P_2x^2_3)(\theta''_3x_3 + 6P_2\theta_3x_3 - 21\theta_3x_1) + 42\theta^2_3x_2x^2_3 + 21\theta_3\theta'_3x^3_3.$$

If $\theta_8 = 0$ the osculating conic has a double point at P_y, and its equation is

$$5(x^2_2 - 2x_1x_3 + P_2x^2_3)(\theta'_3x_3 - 3\theta_3x_2) + 12\theta^2_3x^3_3 = 0. \tag{16}$$

However, if $\theta_8 \neq 0$, this last cubic is still of special interest and has been called by Wilczynski the eight-pointic nodal cubic, or, in his more recent terminology, the *penosculating* nodal cubic (1).

4. *The canonical triangle.* By means of the osculating conic and the eight-pointic nodal cubic the canonical triangle has been defined as follows (2): *One vertex is at* P_y *and one side is the tangent at this point. The second side is the inflectional line of the eight-pointic nodal cubic. The third side is the polar of the point of intersection of the other two with respect to the osculating conic.*

It is easily shown that the three points of inflection of the eight-pointic nodal cubic are given by the three irrational invariants

$$\left[(\theta'_3)^2 + 9P_2\theta^2_3 + 6\theta_3\theta'_3\omega_k\sqrt[3]{\frac{4\theta_3}{5}}\right] y + \left[6\theta_3\theta'_3 + 18\theta^2_3\omega_k\sqrt[3]{\frac{4\theta_3}{5}}\right] z + 18\theta^2_3\rho ;$$

$$(k = 1,\ 2,\ 3),$$

so that their coordinates, referred to the triangle $P_yP_zP_\rho$, are

$$x_1^{(k)} = (\theta'_3)^2 + 9P_2\theta^2_3 + 6\theta_3\theta'_3\omega_k\sqrt[3]{\frac{4\theta_3}{5}} ,$$

$$x_2^{(k)} = 6\theta_3\theta'_3 + 18\theta^2_3\omega_k\sqrt[3]{\frac{4\theta_3}{5}} \qquad (k = 1,\ 2,\ 3),$$

$$x_3^{(k)} = 18\theta^2_3 ,$$

(1) Bull. Amer. Math. Soc., April 1916.
(2) W., p 86.

where ω_1 , ω_2 , ω_3 are the three cube roots of unity. These three points lie on the inflectional line whose equation is

$$(17) \qquad 18\theta^2_3 x_1 - 6\theta_3\theta'_3 x_2 + [(\theta'_3)^2 - 9P_2\theta^2_3]x_3 = 0.$$

The equation of the tangent to the integral curve at P_y is $x_3 = 0$. This line meets the inflectional line in the point B whose coordinates are

$$x_1 = \theta'_3 \quad , \quad x_2 = 3\theta_3 \quad , \quad x_3 = 0.$$

The polar of any point (x'_1 , x'_2 , x'_3) with respect to tge osculating conic is

$$-x'_3 x_1 + x'_2 x_2 + (P_2 x'_3 - x'_1)x_3 = 0 ,$$

so that the polar of the point B is

$$3\theta_3 x_2 - \theta'_3 x_3 = 0.$$

This line meets the inflectional line in the point C whose coordinates are

$$x_1 = (\theta'_3)^2 + 9P_2\theta^2_3 \quad , \quad x_2 = 6\theta_3\theta'_3 \quad , \quad x_3 = 18\theta^2_3.$$

Hence the vertices of the canonical triangle are defined by the covariants

$$(18) \qquad \begin{aligned} &A, && C_0 \equiv y , \\ &B, && C_4 \equiv \theta'_3 y + 3\theta_3 z , \\ &C, && C_8 \equiv [(\theta'_3)^2 + 9P_2\theta^2_3]y + 6\theta_3\theta'_3 z + 18\theta^2_3 \rho , \end{aligned}$$

and the equations of the sides are

$$(19) \qquad \begin{aligned} &BC, && 18\theta^2_3 x_1 - 6\theta_3\theta'_3 x_2 + [(\theta'_3)^2 - 9P_2\theta^2_3]x_3 = 0 , \\ &CA, && 3\theta_3 x_2 - \theta'_3 x_3 = 0 , \\ &AB, && x_3 = 0 , \end{aligned}$$

where the point A is the same as P_y.

From the form of equation (16) we see that the double point tangent of the eight-pointic nodal cubic are defined by the last two equations of (19).

Hence we may define the canonical triangle in terms of this cubic alone. Two of its sides are its double point tangents and the third is its inflectional lines.

It will be found that the inflectional line of the eight-pointic nodal cubic is tangent to the osculating conic at the point C. That is, two sides of the canonical triangle are the tangents from B to the osculating conic and the third is the polar of B with respect to this conic.

If C_y is not a conic the independent variable may be chosen in such a way that $\theta_3 = 1$. The coordinates of the point B then reduce to $(0 \, , 1 \, , 0)$. That is, P_z coincides with B. Then P_ρ is at some point D on the line CA. Hence there must be some point on CA whose coordinates, under the assumption $\theta_3 = 1$, reduce to $(0 \, , 0 \, , 1)$. It can be shown that this point is defined by the covariant

$$\left[(\theta'_3)^2 + 9P_2\theta^2_3 + \frac{1}{3}\,\theta_8\right]y + 6\theta_3\theta'_3 z + 18\theta^2_3\rho.$$

For some purposes it is convenient to use the canonical triangle as triangle of reference [1], but it must not be confused with the triangle $P_y P_z P_\rho$. In general, the two triangles have only one vertex, P_y, and one side, the tangent at P_y, in common. The canonical triangle, being defined by covariants, is not changed by any transformation of the independent variable in (1). Its position depends only on the properties of the curve at P_y. The latter, being defined by semi covariants, is changed by every non-linear transformation of the independent variable in (1). In particular, if the given equation be transformed to the H a l p h e n canonical form, i. e. $P_1 = 0$, $\theta_3 = 1$, P_z will coincide with B and P_ρ with D. The triangles then have two vertices and two sides in common.

However, if all the points of C_y happen to be *coincidence points*, i. e. if $\theta_8 = 0$, and equation (1) is written in the H a l p h e n canonical form, the points C and D will coincide and the two triangles will be identical for all values of t.

5. *The envelopes of the sides of the canonical triangle.* As the point P_y moves along the curve C_y, the sides of the canonical triangle will envelope certain curves. We now propose to find the point where each side touches its envelope. Of course we know beforehand that the envelope of the side AB is the curve C_y.

---

(1) W., p. 82.

Differentiating (2) and reducing by means of (1), we find

$$
\begin{aligned}
&y' = -p_1 y + z\,, \\
(20) \qquad &z' = -P_2 y - p_1 z + \rho\,, \\
&\rho' = -\left(\theta_2 + \frac{1}{2}P'_2\right)y - 2P_2 z - p_1\rho.
\end{aligned}
$$

Suppose t is increased by an infinitesimal δt. The points A , B , C will take the positions A_1 , B_1 , C_1 defined by the expressions

$$y + y'\delta t = (1 - p_1\delta t)y + \delta t\cdot z\,,$$

$$(\theta'_3 + \theta''_3\delta t)(y + y'\delta t) + 3(\theta_3 + \theta'_3\delta t)(z + z'\delta t) = [\theta'_3 + (\theta''_3 - \theta'_3 p_1 - 3P_2\theta_3)\delta t]y +$$

$$+ [3\theta_3 + (4\theta'_3 - 3\theta_3 p_1)\delta t]z + 3\theta_3\delta t\cdot\rho\,,$$

$$[(\theta'_3)^2 + 9P_2\theta^2_3 + (2\theta'_3\theta''_3 + 9P'_2\theta^2_3 + 18P_2\theta_3\theta'_3)\delta t](y + y'\delta t) +$$

$$+ [6\theta_3\theta'_3 + \{6(\theta'_3)^2 + 6\theta_3\theta''_3\}\delta t]\,[z + z'\delta t) + (18\theta^2_3 + 36\theta_3\theta'_3\delta t)(\rho + \rho'\delta t) =$$

$$= [(\theta'_3)^2 + 9P_2\theta^2_3 + \{2\theta'_3\theta''_3 + 12P_2\theta_3\theta'_3 - p_1(\theta'_3)^2 - 9P_2\theta^2_3 k - 18\theta^3_3\}\delta t]y +$$

$$+ [6\theta_3\theta'_3 + \{\theta_8 + 14(\theta'_3)^2 - 6\theta_3\theta'_3 p_1\}\delta t]z + [18\theta^2_3 + (42\theta_3\theta'_3 - 18\theta^2_3 p_1)\delta t]\rho\,,$$

where all terms of order higher than the first have been neglected.

We are now in a position to write the equations of the sides of the triangle $A_1B_1C_1$. In order to find the point where the line AC touches its envelope we must find the point of intersection of AC and A_1C_1 , and then find the limiting position of this point as δt approaches zero. We find, for the line AC, the point E_a defined by the covariant

$$(21) \qquad [(\theta'_3)^2 + 9P_2\theta^2_3 - \theta_8]y + 6\theta_3\theta'_3 z + 18\theta^2_3\rho\,,$$

and for the line BC, the point E_a defined by the covariant

$$(22) \qquad [(\theta'_3)^2\theta_8 + 9P_2\theta^2_3\theta_8 + 108\theta''_3\theta'_3]y + [6\theta_3\theta'_3\theta_8 + 324\theta^5_3]z + 18\theta^2_3\theta_8\rho.$$

It is easy to verify analytically that each of these expression is a covariant.

If the coordinates of our collinear points are written in the form

$$P_i = (h_i x_1 + k_i y_1 \quad , \quad h_i x_2 + k_i y_2 \quad , \quad h_i x_3 + k_i y_3), \qquad (i = 1,\ 2,\ 3),$$

the anharmonic ratio of the four points is given by

$$(23) \qquad (P_1 P_2 P_3 P_4) = \left[\begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} \cdot \begin{vmatrix} h_3 & k_3 \\ h_4 & k_4 \end{vmatrix} \right] \div \left[\begin{vmatrix} h_1 & k_1 \\ h_4 & k_4 \end{vmatrix} \cdot \begin{vmatrix} h_3 & k_3 \\ h_2 & k_2 \end{vmatrix} \right].$$

If now we take the points A , C as (x_1 , x_2 , x_3) , (y_1 , y_2 , y_2) we find (1)

$$(C \ , \ A \ , \ E_b \ , \ D) = 4.$$

It will be seen that the points E_a and E_b coincide with B and C, respectively,

if, and only if, $\theta_8 = 0$. Hence (2), a necessary and sufficient condition that the locus of each of the vertices of the canonical triangle should be at the same time the envelope of one the sides which ends there, is $\theta_8 = 0$.

6. *The differential equation of an associated curve.* As the point P_y moves along the curve C_y every point, whose coordinates x_1 , x_2 , x_3 , referred to the moving triangle of reference, are given as functions of t, will describe a certain locus. We shall speak of this locus as being *associated with* the given curve C_y. Its points correspond to those of C_y in a definite fashion, corresponding points being determined by the same value of t. The associated curve may be considered as an integral curve of a certain linear homogeneous differential equation of the third order. We propose now to find this equation. The computations involved might be simplified by taking $P_2 = 0$. However we prefer to consider the general case.

Under certain conditions the point associated with P_y may remain fixed, or it may move on a straight line. These exceptional cases will be excluded for the present. Let the required differential equation be

$$(24) \qquad \tau''' + 3r_1 \tau'' + 3r_2 \tau' + r_3 \tau = 0.$$

We wish to determine r_1 , r_2 , r_3 as functions of t so that a set of fundamental solutions of (24) will be given by

$$\tau_1 = x_1 y_i + x_2 z_i + x_3 \rho_i \, , \qquad (i = 2,\ 2,\ 3) \, ,$$

(1) W., p. 66.
(2) W., p. 70.

where y_i is a set of fundamental solutions of (1) and z_i and ρ_i are given by (2).

The point P_τ which is associated with P_y is defined by the expression

$$\tau = x_1 y + x_2 z + x_3 \rho.$$

If we differentiate τ and make use of (20) we find

$$\tau' + p_1 \tau = u_1 y + u_2 z + u_3 \rho , \tag{25}$$

where

$$\begin{aligned} u_1 &= x'_1 - P_2 x_2 - \left(\theta_3 + \frac{1}{2} P'_2\right) x_3 , \\ u_2 &= x'_2 + x_1 - 2 P_2 x_3 , \\ u_3 &= x'_3 + x_2 . \end{aligned} \tag{26}$$

Differentiating again and reducing by (20), we find

$$\tau'' + 2 p_1 \tau' + p_2 \tau = v_1 y + v_2 z + v_3 \rho , \tag{27}$$

where

$$\begin{aligned} v_1 &= x''_1 - 2 P_2 x'_2 - 2\left(\theta_3 + \frac{1}{2} P'_2\right) x'_3 - \left(\theta_3 + \frac{3}{2} P'_2\right) x_2 - \\ &\qquad - \left(\theta'_3 + \frac{1}{2} P''_2 - 2 P^2_2\right) x_3 , \\ v_2 &= x''_2 + 2 x'_1 - 4 P_2 x'_3 - 2 P_2 x_2 - \left(\theta_3 + \frac{5}{2} P'_2\right) x_3 , \\ v_3 &= x''_3 + 2 x'_2 + x_1 - P_2 x_3 . \end{aligned} \tag{28}$$

A third differentiation gives

$$\tau''' + 3 p_1 \tau'' + 3 p_2 \tau' + p_3 \tau = w_1 y + w_2 z + w_3 \rho ,$$

where

$$w_1 = x'''_1 - 3P_2x''_2 - 3\left(\theta_3 + \frac{1}{2}P'_2\right)x''_3 - 3\left(\theta_3 + \frac{3}{2}P'_2\right)x'_2 -$$

$$- 3\left(\theta'_3 + \frac{1}{2}P''_3 - 2P^2_2\right)x'_3 - \left(\theta'_3 + \frac{3}{2}P''_2\right)x_2 - \left(\theta''_3 + \frac{1}{2}P'''_2 - 6P_2P'_2\right)x_3,$$

$$(29)\quad w_2 = x'''_2 + 3x''_1 - 6P_2x''_3 - 6P_2x'_2 - 3\left(\theta_3 + \frac{5}{2}P'_2\right)x'_3 -$$

$$- 3P'_2x_2 - 2\left(\theta'_3 + \frac{3}{2}P''_2\right)x_3,$$

$$w_3 = x'''_3 + 3x''_2 + 3x'_1 - 3P_2x'_3 - 3P'_2x_3.$$

There is one important property of the quantities u_i, v_i, w_i of which we shall make use later. They are linear in x_i, x'_i, x''_i, x'''_i. Hence, after they have been found for the points $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)})$ and $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$, it is very easy to calculate their values for the point (x_1, x_2, x_3) where $x_i = ax_i^{(1)} + bx_i^{(2)}$, if a and b are constants.

We now have a system of four linear equations in y, z, ρ. Eliminating y, z, ρ, we obtain the desired differential equation

$$(30)\quad \delta_0[\tau''' + 3p_1\tau'' + 3p_2\tau' + p_3\tau] - \delta_1[\tau'' + 2p_1\tau' + p_2\tau] +$$

$$+ \delta_2[\tau' + p_1\tau] - \delta_3\tau = 0,$$

where

$$(31)\qquad \delta_0 = \begin{vmatrix} x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \quad \delta_1 = \begin{vmatrix} x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

$$\delta_2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

It is interesting to note that the coefficients of δ_0, δ_1, δ_2, δ_3, if written in terms of y instead of τ, would be semi-covariants of weight $3, 2, 1, 0$, respectively.

It will be shown later that δ_0 venishes identically only when the locus of P_τ is a straight line or a point. Since these cases have been excluded, we have, as the solution of our problem,

$$
(32)\qquad
\begin{aligned}
3r_1 &= 3p_1 - \frac{\delta_1}{\delta_0}\,,\\
3r_2 &= 3p_2 - 2p_1\frac{\delta_1}{\delta_0} + \frac{\delta_2}{\delta_0}\,,\\
r_3 &= p_3 - p_2\frac{\delta_1}{\delta_0} + p_1\frac{\delta_2}{\delta_2} - \frac{\delta_3}{\delta_0}\,,
\end{aligned}
$$

where the right members depend only on the independent variable t.

7. *The calculation of* δ_0 , δ_1 , δ_2 , δ_3. In most cases the calculation of δ_0 is fairly simple, but when wy try to find the other three the work becomes rather complicated. Hence it is desirable to have a method for determining the last three determinants from the first one.

From (26), (28), (29), we find by differentiation

$$
(33)\qquad
\begin{aligned}
x'_1 &= u_1 + P_2x_2 + \left(\theta_3 + \frac{1}{2}P'_2\right)x_3\,,\\
x'_2 &= u_2 - x_1 + 2P_2x_3\,,\\
x'_3 &= u_3 - x_2\,,\\
u'_1 &= v_1 + P_2u_2 + \left(\theta_3 + \frac{1}{2}P'_2\right)u_3 - P_2x_1\,,\\
u'_2 &= v_2 - u_1 + 2P_2u_3 \qquad\qquad - P_2x_2\,,\\
u'_3 &= v_3 - u_2 \qquad\qquad\qquad\quad - P_2x_3\,,\\
v'_1 &= w_1 + P_2v_2 + \left(\theta_3 + \frac{1}{2}P'_2\right)v_3 - 2P_2u_1 + \left(\theta_3 + \frac{1}{2}P'_2\right)x_1\,,\\
v'_2 &= w_2 - v_1 + 2P_2v_3 \qquad\qquad - 2P_2u_2 - \left(\theta_3 + \frac{1}{2}P'_2\right)x_2\,,\\
v'_3 &= w_3 - v_3 \qquad\qquad\qquad\quad - 2P_2u_3 - \left(\theta_3 + \frac{1}{2}P'_2\right)x_3\,.
\end{aligned}
$$

Differentiating δ_0 with respect to t, we obtain

$$\delta'_0 = \begin{vmatrix} x'_1 & x'_2 & x'_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ u'_1 & u'_2 & u'_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \\ v'_1 & v'_2 & v'_3 \end{vmatrix}.$$

After making use of (33) the right member reduces to

$$\delta_1 + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ f_1(u) & f_2(u) & f_3(u) \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \\ f_1(v) & f_2(v) & f_3(v) \end{vmatrix},$$

where

$$f_1(x) = P_2 x_2 + \left(\theta_3 + \frac{1}{2}P'_2\right)x_3,$$

$$f_2(x) = - x_1 + 2P_2 x_3,$$

$$f_3(x) = - x_2 .$$

If the last three determinants are expanded ad added it is found that all the terms cancel. Hence δ_0 *is the derivative of* δ_0 *with respect to the independent variable* t.

It is evident that δ_0, δ_1, δ_2, δ_3 are connected by the following relations:

$$w_1\delta_0 - v_1\delta_1 + u_1\delta_2 - x_1\delta_3 = 0,$$

$$w_2\delta_0 - v_2\delta_1 + u_2\delta_2 - x_2\delta_3 = 0,$$

$$w_3\delta_0 - v_3\delta_1 + u_3\delta_2 - x_2\delta_3 = 0.$$

From the first two we obtain

$$(u_1x_2 - u_2x_1)\,\delta_2 = (v_1x_2 - v_2x_1)\,\delta_1 - (w_1x_2 - w_2x_1)\,\delta_0,$$

(34)

$$(u_1x_2 - u_2x_1)\,\delta_3 = (v_1u_2 - v_2u_1)\,\delta_1 - (w_1u_2 - w_2u_1)\,\delta_0,$$

and from the last two

$$(u_2x_3 - u_3x_2)\,\delta_2 = (v_2x_3 - v_3x_2)\,\delta_1 - (w_2x_3 - w_3x_2)\,\delta_0\,,$$

(35)

$$(u_2x_3 - u_3x_2)\,\delta_3 = (v_2u_3 - v_3u_2)\,\delta_1 - (w_2u_3 - w_3u_2)\,\delta_0\,.$$

It will always be possible to determine δ_2 and δ_3 from at least one of the equations (34), (35). The only apparent exception is in the case where

$$u_1x_2 - u_2x_1 = u_2x_3 - u_3x_2 = 0\,; \qquad \text{i. e.} \qquad u_1 : u_2 : u_3 = x_1 : x_2 : x_3\,.$$

But this requires $\delta_0 = 0$, which has been excluded. Hence we may find δ_1 by differentiating δ_0, and then δ_2 and δ_3 from the above equations.

8. *The invariants of the differential equation of an associated curve.* The calculation of the invariants of (24) may be greatly simplified by means of the auxiliary equation

(36) $$\sigma''' + 3y_1\,\sigma'' + 3y_2\,\sigma' + y_3\,\sigma = 0\,,$$

where

(37) $$-3q_1 = \frac{\delta_1}{\delta_0}\,, \quad 3q_2 = \frac{\delta_2}{\delta_0}\,, \quad -q_3 = \frac{\delta_3}{\delta_0}\,.$$

Substituting (37) in (32), we find

(38) $$\begin{aligned} r_1 &= p_1 + q_1\,, \\ r_2 &= p_2 + 2p_1\,q_1 + q_2\,, \\ r_3 &= p_3 + 3p_2\,q_1 + 3p_1\,q_2 + q_3. \end{aligned}$$

Let

$$R_2 = r_2 - r_1^2 - r'_1 \quad , \qquad R_3 = r_3 - 3r_1\,r_2 + 2r_1^3 - r''_1\,,$$

$$Q_2 = q_2 - q_1^2 - q'_1 \quad , \qquad Q_3 = q_3 - 3q_1\,q_2 + 2q_1^3 - q''_1.$$

Then from (38) we find

$$R_2 = P_2 + Q_2 \quad , \qquad R_3 = P_3 + Q_3\,;$$

and finally

$$\Phi_3 = \theta_3 + \psi_3 , \tag{39}$$

where

$$\Phi_3 = R_3 - \frac{3}{2} R'_2 \quad , \quad \psi_3 = Q_3 - \frac{3}{2} Q'_2 .$$

9. *The tangent to an associated curve.* Let us denote by P_u, P_v, the points defined by the expressions $u_1 y + u_2 z + u_3 \rho$, $v_1 y + v_2 z + v_3 \rho$. Then, if C_τ denotes the curve which is described by the point P_τ, it us clear from (25) and (27) that P_u is a point on the tangent to C_τ at P_τ, and P_v is some other point in the plane. Hence the equation of the tangent to C_τ at P_τ, referred to the moving triangle of reference $P_y P_z P_\rho$, is

$$\begin{vmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ u_1 & u_2 & u_3 \\ x_4 & x_2 & x_3 \end{vmatrix} = 0, \tag{40}$$

where $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ are the current coordinates.

10. *A new transformation, and discussion of the simplest case in which it is periodic.* We have seen that the two points P_z, P_ρ, which are associated with P_y to form the standard local triangle of reference, are defined by the semi-covariants (2). Now the curve described by P_z is an integral curve of (24) in the case $x_1 = 0$, $x_2 = 1$, $x_3 = 0$. Hence, associated with P_z, are two points P_{z_1}, P_{ρ_1}, defined by

$$z_1 = \tau' + r_1 \tau ,$$

$$\rho_1 = \tau'' + 2 r_1 \tau' + r_2 \tau ,$$

which, together with P_z, might form a triangle of reference for studiyng the properties of the curve described by P_z. Let us investigate the conditions under which these two triangles are identical.

For the point P_z we find

$$\begin{array}{llll} x_1 = 0 \quad , & u_1 = - P_z \quad , & v_1 = - P_3 \quad , & w_1 = - P'_3 , \\ x_2 = 1 \quad , & u_2 = 0 \quad , & v_2 = - 2P_2 \quad , & w_2 = - 3P'_2 , \\ x_3 = 0 \quad , & u_3 = 1 \quad , & v_3 = 0 \quad , & w_3 = 0. \end{array}$$

Hence the differential equation of the curve C_z is

$$(41) \qquad P_3\tau''' + (3p_1P_3 - P'_3)\tau'' + (3p_2P_3 - 2p_1P'_2)\tau' +$$
$$+ (p_3P_3 - p_2P'_3 - 2P_2P'_3 + 3P'_2P_3)\tau = 0.$$

Substuting the values of τ', τ'', r_1, r_2 from (25), (27), (41), we find

$$z_1 = - P_2y - \frac{P'_3}{3P_3}z + \rho,$$

$$\rho_1 = - P_3y - 2P_2z - \frac{2P'_3}{3P_3}\rho.$$

Hence, a necessary and sufficient condition that P_ρ and P_y be defined by z_1 and ρ_1, respectively, is that

$$(42) \qquad P_2 = 0 \quad , \quad P_3 = \text{constant}.$$

It is clear that (42) implies $\theta_8 = 0$.

If we consider the point P_ρ it can be shown in like manner that (42) is a necessary and sufficient condition that the points P_y and P_z be defined by z_1 and ρ_1, respectively.

If we think of (2) *as a transformation which carries* P_y *into* P_z *and upon repetition into* P_ρ, *then* (42) *is the condition that this transformation be of period three.* That is, three successive appllcations of this transformation will carry the point P_y into P_z, P_ρ, P_y.

If these conditions are satisfied, equation (41) and (1) are identical.

Hence C_z is projective with C_y. The same thing is true sf the curve C_ρ.

Hence a modified form of Halphen's theorem: [1] *If* $\theta_8 = 0$, *and only in this case, the independent variable may be chosen in such a way that each vertex of the triangle* $P_yP_zP_\rho$ *shall describe a curve of coincidence points as well as* P_y.

11. *The condition that an associated point shall remain fixed.* As the point P_y moves along the curve C_y, the triangle of reference, $P_yP_zP_\rho$, continually changes its position and shape. Thus, if the point P_τ is defined by the expression

$$\tau = x_1y + x_2z + x_3\rho,$$

---

[1] W., p. 70.

its coordinates, x_1, x_2, x_3, referred to the moving triangle of reference, will be variable if it is a fixed point of the plane. We propose to find the conditions which the functions x_1, x_2, x_3 must satisfy in order that P_τ be a fixed point.

Since P_u is a point on the tangent to C_τ at P_τ, *a necessary and sufficient condition that P_τ remain fixed is that its coordinates satisfy the relation*

$$u_1 : u_2 : u_3 = x_1 : x_2 : x_3, \tag{43}$$

which may be written in the form of a system of linear homogeneous differential equations of the first order

$$\begin{aligned} x'_1 &= \omega x_1 + P_2 x_2 + \left(\theta_3 + \frac{1}{2}P'_2\right)x_3, \\ x'_2 &= -x_1 + \omega x_2 \qquad + 2P_2 x_3, \\ x'_3 &= \qquad\quad -x_2 \qquad + \omega x_3, \end{aligned} \tag{44}$$

where ω is an arbitrary function of t. In this case the determinant δ_0 must vanish for all values of v_1, v_2, v_3.

It is interesting to note that if we put $u_i = \omega x_i$, it follows from (31) that $\delta_0 = \delta_1 = 0$, $\delta_3 = \omega\,\delta_2$. Equation (25) and (30) then both reduce to

$$\tau' + (p_1 - \omega)\tau = 0.$$

12. *The condition that an associated point shall move along a straight line.* It is clear that, if the path of P_τ is a straight line, the point P_v must lie on the tangent to C_τ at P_τ, for all values of the independent variable. Hence, *if an associated point does not remain fixed, a necessary and sufficient condition that its path be a straight line is that its coordinates satisfy the differential equation*

$$\delta_0 = \begin{vmatrix} x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0. \tag{45}$$

13. *Applications.* We shall now make some applications of the preceding theory to a few of the points which are covariantly connected with the curve C_y. We shall assume in each case that $\theta_3 \neq 0$; that is, that the curve C_y is not a conic.

The point B, which is the second vertex of the canonical triangle, is defined by the covariant

$$\theta'_3 y + 3\theta_3 z ,$$

We find

$$x_1 = \theta'_3 , \qquad u_1 = \theta''_3 - 3P_2\theta_3 ,$$

$$x_2 = 3\theta_3 , \qquad u_2 = 4\theta'_3 ,$$

$$x_3 = 0 , \qquad u_3 = 3\theta_3 ,$$

$$v_1 = \theta'''_3 - 3\theta^2_3 - 6P_2\theta'_3 - \frac{9}{2}P'_2\theta_3 ,$$

$$v_2 = 5\theta''_3 - 6P_2\theta_3 ,$$

$$v_3 = 7\theta'_3 ,$$

$$w_1 = \theta^{(4)}_3 - 12\theta_3\theta'_3 - 9P_2\theta''_3 - \frac{27}{2}P'_2\theta'_3 - \frac{9}{2}P''_2\theta_3 ,$$

$$w_2 = 6\theta'''_3 - 18P_2\theta'_3 - 9P'_2\theta_3 ,$$

$$w_3 = 12\theta''_3 .$$

The condition (43) requires $\theta_3 = 0$. Hence B cannot remain fixed. The equation of the tangent to the locus of B is

$$(46) \qquad 18\theta^2_3 x_1 - 6\theta_3\theta'_3 x_2 + [(\theta'_3)^2 - 9P_2\theta^2_3 - \theta_8]\, x_3 = 0.$$

It is easily shown that this line meets CA in a point which is defined by the covariant

$$[(\theta'_3)^2 + 9P_2\theta^2_3 + \theta_8]\, y + 6\theta_3\theta'_3 z + 18\theta^2_3 \rho .$$

Hence, *the lines* BA , BB_b *and* BC *form a harmonic pencil with the tangent to the locus of* B.

It will be found that

$$2\delta_0 = \theta_{12} - 54\theta^4_3 \quad , \qquad 2\theta_3\delta_1 = \theta_{16} + 8\theta'_3\delta_0 ,$$

$$6\theta^2_3\delta_2 = 7\theta'_3\theta_{16} - 8[3\theta_3\theta''_3 - 7(\theta'_3)^2]\,\delta_0 ,$$

(47)

$$18\theta^3_3\delta_3 = -[15\theta_3\theta''_3 - 18P_2\theta^2_3 - 28(\theta'_3)^2]\,\theta_{16} +$$

$$+ 2[18\theta^2_3\theta'''_3 - 108\theta_3\theta'_3\theta''_3 + 112(\theta'_3)^3 + 18P_2\theta^2_3\theta'_3 - 27P'_2\theta^3_3]\,\delta_0.$$

Hence the second vertex of the canonical triangle will describe a straight line if, and only if, $\mu - 54 \equiv 0$. It may happen that $\mu - 54 = 0$ for certain values of the independent voriable t. These values of t will give the points of inflection of the locus of B. The differential equation of the locus of B may now be found by substituting the values of δ_0, δ_1, δ_2, δ_3 in (30).

From (37), (38), (39), we find

$$(48)\quad 54\Phi_3 = 54\theta_3 - \frac{2\theta_{12}}{\theta_3} - \frac{\theta_{24} - 4\theta_8\theta_{16}}{\theta^3_3(\theta_{12} - 54\theta^4_3)} + \frac{15\theta_{16}\theta_{20}}{\theta^3_3(\theta_{12} - 54\theta^4_3)^2} - \frac{40\theta^3_{16}}{\theta^3_3(\theta_{12} - 54\theta^4_3)^3},$$

A necessary and sufficient condition that the locus of B be a conic is obtained by equation Φ_3 to zero.

The point C, which is the third vertex of the canonical triangle, is defined by the covariant

$$[(\theta'_3)^2 + 9P_2\theta^2_3]\,y + 6\theta_3\theta'_3 z + 18\theta^2_3\rho.$$

We find

$$x_1 = (\theta'_3)^2 + 9P_2\theta^2_3 , \qquad u_1 = 2\theta'_3\theta''_3 + 12P_2\theta_3\theta'_3 - 18\theta^3_3 ,$$

$$x_2 = 6\theta_3\theta'_3 , \qquad u_2 = \theta_8 + 14(\theta'_3)^2 ,$$

$$x_3 \equiv 18\theta^2_3 , \qquad u_3 = 42\,\theta_3\theta'_3 ,$$

$$v_1 = 2\theta'_3\theta'''_3 + 2(\theta''_3)^2 - 96\theta^2_3\theta'_3 + 6P_2\theta_3\theta''_3 + 6P_2(\theta'_3)^2 + 36P^2_2\theta^2_3 - 9P'_2\theta_3\theta'_3 ,$$

$$v_2 = 6\theta_3\theta'''_3 + 22\theta'_3\theta''_3 - 18\theta^3_3 - 120P_2\theta_3\theta'_3 - 27P'_2\theta^2_3 ,$$

$$v_3 = 48\theta_3\theta''_3 + 49(\theta'_3)^2 - 9P_2\theta^2_3 ,$$

$$w_1 = 2\theta'_3\theta_3^{(4)} + 6\theta''_3\theta'''_3 - 144\theta^2_3\theta''_3 - 240\theta_3(\theta'_3)^2 + 216P_2\theta_3\theta'_3 +$$

$$+ 108P_2P'_2\theta^2_3 - 27P'_2\theta_3\theta''_3 - 27P'_2(\theta'_3)^2 - 9P''_2\theta_3\theta'_3 ,$$

$$w_2 = 6\theta_3\theta_3^{(4)} + 30\theta'_3\theta'''_3 + 24(\theta''_3)^2 - 144\theta^2_3\theta'_3 - 198P_3\theta_3\theta''_3 -$$

$$- 198P_2(\theta'_3)^2 - 180P'_2\theta_3\theta'_3 - 27P''_3\theta^2_3 ,$$

$$w_3 = \theta'_8 + 48\theta_3\theta'''_3 + 176\theta'_3\theta''_3.$$

The condition (43) requires $\theta_3 = 0$. Hence C cannot remain fixed. The equation of the tangent to the locus of C is

$$(49) \qquad 18\theta_3^2\theta_8 x_1 - (6\theta_3\theta'_3\theta_8 - 324\theta_3^5)x_2 + [(\theta'_3)^2\theta_8 - 9P_2\theta_3^3\theta_8 - 108\theta_3^4\theta'_3]x_3 = 0.$$

It will be found that

$$\delta_0 = -\theta_8^3 - 108\theta_3^4(\theta_{12} - 54\theta_3^4),$$

$$\theta_3\delta_1 = -\theta_8^2\theta_{12} - 108\theta_3^4\theta_{16} + 8\theta'_3\delta_0,$$

$$(50) \qquad 3\theta_3^2\delta_2 = \theta_8^2\theta_{16} - 1620\theta_3^4\theta'_3\theta_{16} - \theta_8\theta_{12}^2 - 15\theta'_3\theta_8^2\theta_{12} + 54\theta_3^4\theta_8\theta_{12} - $$
$$- 8[3\theta_3\theta''_3 - 11(\theta'_3)^2]\,\delta_0,$$

$$27\theta_3^3\delta_3 = -\theta_{12}(21\theta'_3\theta_8 + 54\theta_3^4)(\theta_{12} - 54\theta_3^4) + \theta_8\theta_{16}(21\theta'_3\theta_8 + 54\theta_3^4) +$$
$$+ [69\theta_8\theta''_3 - 238(\theta'_3)^2](\theta_8^2\theta_{12} + 108\theta_3^4\theta_{16}) - 4[18\theta_3^2\theta'''_3 - 198\theta_3\theta'_3\theta''_3 + 308(\theta'_3)^3]\delta_0.$$

Hence the locus of C will be a straight line if, and only if,

$$\lambda + 108(\mu - 54) \equiv 0.$$

Then differential equation of the locus of C may now be found from (30).

After computing the relative invariant of weight three as in the preceding case it will be found that the locus of C is a conic if, and only if,

$$54\theta_3^4\delta_0^3 - [108\theta_3^4\theta_{24} + 30\theta_8\theta_{12}\theta_{16} - 594\theta_3^4\theta_8\theta_{16} - 54\theta_3^4\theta_{12}^2 - 54\cdot 108\theta_3^8\theta_{12} + 5\theta_{12}^3]\delta_0^2 +$$
$$(51) \qquad + 3(\theta_8^2\theta_{12} + 108\theta_3^4\theta_{16})[10\theta_8^2\theta_{16} + 15\theta_8\theta_{12}^2 + 5400\theta_3^4\theta_{20} - 270\theta_3^4\theta_8\theta_{12}]\delta_0 -$$
$$- 40(\theta_8^2\theta_{12} + 108\theta_3^4\theta_{16})^3 \equiv 0.$$

The point D is defined by the covariant

$$\left[(\theta'_3)^2 + 9P_2\theta_3^2 + \frac{1}{3}\,\theta_8\right]y + 6\theta_3\theta'_3 z + 18\theta_3^2\rho.$$

We find

$$3x_1 = 6\theta_3\theta''_3 - 4(\theta'_3)^2, \qquad 3u_1 = \theta'_8 + 6\theta'_3\theta''_3 + 36P_2\theta_3\theta'_3 - 54\theta_3^3,$$

$$3x_2 = 18\theta_3\theta'_3, \qquad 3u_2 = 4\theta_8 + 42(\theta'_3)^2,$$

$$3x_3 = 54\theta_3^2, \qquad 3u_3 = 126\theta_3\theta'_3,$$

$$3v_1 = \theta''_8 + 6\theta'_3\theta'''_3 + 6(\theta''_3)^2 - 288\theta_3^2\theta'_3 - 4P_2\theta_8 + 42P_2\theta_3\theta''_3 - 10P_2(\theta'_3)^2 - 27P'_2\theta_3\theta'_3,$$

$$3v_2 = 5\theta'_8 + 90\theta'_3\theta''_3 - 54\theta_3^3 - 198P_2\theta_3\theta'_3,$$

$$3v_3 = 2\theta_8 + 138\theta_3\theta''_3 + 154(\theta'_3)^2.$$

The condition (43) requires $\theta_3 = 0$. Hence D cannot remain fixed. The equation of the tangent to the locus of D is

$$(52)\quad 108\theta^2_3\theta_8 x_1 - 9\theta_3[\theta_{12} + 4\theta'_3\theta_8 - 162\theta^4_3]x_2 +$$
$$+]3\theta'_3\theta_{12} + 10(\theta'_3)^2\theta_8 - 12\theta_3\theta''_3\theta_8 - 486\theta^4_3\theta'_3]x_3 = 0.$$

It will be found that

$$-27\delta_6 = 72\theta_8\theta_{16} - 30\theta^2_{12} + 32\theta^3_8 + 2\cdot 54^2\theta^4_3\theta_{12} - 54^3\theta^8_3.$$

Hence, a necessary and sufficient condition that D move along a straight line is

$$(53)\qquad 36\nu - 15\mu^2 + 16\lambda + 54^2\mu - 27\cdot 54^2 = 0.$$

The line CA touches its envelope at the point E_b which is defined by the covariant

$$[(\theta'_3)^2 + 9P_2\theta^2_3 - \theta_8]y + 6\theta_3\theta'_3 z + 18\theta^2_3\rho.$$

We find

$x_1 = (\theta'_3)^2 + 9P_2\theta^2_3 - \theta_8$, $\qquad u_1 = -\theta'_8 + 2\theta'_3\theta''_3 - 18\theta^3_3 + 12P_2\theta_3\theta'_3$,

$x_2 = 6\theta_3\theta'_3$, $\qquad u_2 = 14(\theta'_3)^2$,

$x_3 = 18\theta^2_3$, $\qquad u_3 = 42\theta_3\theta'_3$,

$v_1 = -\theta''_8 + 2\theta'_3\theta'''_3 + 2(\theta''_3)^2 - 90\theta^2_3\theta'_3 + P_2\theta_8 + 13P_2(\theta'_3)^2 + 63P^2_2\theta^2_3 - 9P'_2\theta_3\theta'_3$,

$v_2 = -\theta'_8 + 30\theta'_3\theta''_3 - 18\theta^3_3 - 66P_2\theta_3\theta'_3$,

$v_3 = 7\theta_8 + 105(\theta'_3)^2 + 207P_2\theta^2_3$.

The condition (43) requires $\theta_{12} + 54\theta^4_3 = 0$. Hence, a necessary and sufficient condition that E_b remain fixed is $\mu + 54 = 0$. In this case the line CA rotates about this point. The equation of the tangent to the locus of E_b is

$$3\theta_3 x_2 - \theta'_3 x_3 = 0,$$

which is, of course, the equation of the line CA. It will be found that

$$\delta_0 = 2(\theta_{12} + 54\theta^4_{,3})^2.$$

Hence E_b cannot describe a straight line.

The line BC touches its envelope at the point E_a defined by the covariant

$$[(\theta'_{,3})^2\theta_{,8} + 9P_2\theta^2_{,3}\theta_{,8} + 108\theta^4_{,3}\theta'_{,3}]y + (6\theta_{,3}\theta'_{,3}\theta_{,8} + 324\theta^5_{,3})z + 18\theta^2_{,3}\theta_{,8}\rho.$$

In this case we obtain the following results:

(*a*) A necessary and sufficient condition that the line BC rotate about a fixed point is

$$\lambda - 108(\mu + 54) \equiv 0.$$

(*b*) The point cannot describe a straight line.

All the cubic which have seventh order contact with C_y at P_y meet in the ***Halphen point*** (1), which is defined by the covariant

$$7(5\theta'_{,3}\theta_{,8} - 756\theta^4_{,3})^2 + 25\theta^3_{,8} + 1575P_2\theta^2_{,3}\theta^2_{,8}]y + 210\theta_{,3}\theta_{,8}(5\theta'_{,3}\theta_{,8} - 756\theta^4_{,3})z + 3150\theta^2_{,3}\theta^2_{,8}\cdot d.$$

We find:

(*a*) This point cannot remain fixed.

(*b*) A necessary and sufficient condition that it describe a straight line is

$$
\begin{aligned}
&2^6\cdot5^6\lambda^3 + 2^6\cdot3^8\cdot5\cdot7^4\mu^3 + 2^2\cdot3^2\cdot5^5\cdot7^2\cdot83\,\lambda^3\mu + 2^9\cdot3^6\cdot5^4\cdot7^4\lambda\mu^2 - \\
&- 2^2\cdot3^5\cdot7^3\cdot5^4\cdot323\,\lambda^2 + 2^7\cdot3^{11}\cdot5^2\cdot7^6\cdot23\,\lambda - 2^9\cdot3^{12}\cdot6^2\cdot7^7\cdot\mu^2 + \\
(54)\qquad &+ 2^{10}\cdot3^{15}\cdot5\cdot7^8\cdot\mu - 2^6\cdot3^{10}\cdot5^4\cdot7^5\lambda\mu - 3\cdot5^6\cdot7\,\lambda^2\mu^2 + 2^3\cdot5^6\cdot7\,\lambda^2\nu + \\
&+ 2^4\cdot3^4\cdot5^5\cdot7\,\lambda\mu^3 - 2^{11}\cdot3^{17}\cdot7^8 \equiv 0.
\end{aligned}
$$

The tangent to C_y at P_y meets the osculating cubic again in the point

(1) W., p. 68.

which is defined by the covariant [1]

$$\left(567\ \theta_3^3 - \frac{5}{2}\theta'_8\right)y - 20\ \theta_8 z.$$

We find:

(*a*) This point cannot remain fixed.

(*b*) A necessary and sufficient condition that it describe a straight line is

$$\begin{aligned} &2^6\cdot 5^3\sigma - 2^3\cdot 3^3\cdot 5^3\mu\nu + 2^4\cdot 3^5\cdot 5^2\cdot 7\cdot 11\nu - 2\cdot 3^4\cdot 5^2\cdot 7\cdot 179\mu^2 + 2^2\cdot 3^{12}\cdot 5\cdot 7^2\mu + 2^6\cdot 5^3\lambda\mu + \\ (55) \qquad &+ 3\cdot 5^3\cdot 17\mu^3 + 2^7\cdot 3^2\cdot 5^3\cdot 43\lambda - 2^3\cdot 3^{14}\cdot 7^3 \equiv 0' \end{aligned}$$

14. *The invariant triangle of an anharmonic curve.* An anharmonic curve may be defined as a curve along which the absolute invariant $\lambda\left(=\frac{\theta_8^3}{\theta_3^8}\right)$ is constant. It is convenient to divide these curves into two classes as follows;

$$\text{First class,}\qquad \text{where } \lambda \neq \frac{3^9}{4},$$

$$\text{Second class,}\qquad \text{»}\quad \lambda = \frac{3^9}{4}.$$

Associated with each anharmonic curve of the first class is a triangle' called the *invariant triangle,* which has the following property. Suppose any tangent is drawn to the curve. The anharmonic ratic of the point of contact and the three points of intersection of this tangent with the three sides of the triangle is constant for all points of the curve. We now propose to find the coordinates of the vertices of this triangle.

Since the vertices remain fixed as P_y moves along C_y, their coordinates must be solutions of the system of differential equations (44). This system of equations has ∞^3 solutions corresponding to the ∞^2 points in the plane, any one of which may be thought of as a fixed point. However, since we are only interested in covariant points, the only solutions to be considered are those where $x_1 y + x_2 z + x_3 \rho$ is a covariant.

Since λ is constant, it follows by differentiation that θ_{12} and all the successive Jacobians of θ_3 and θ_8 vanish identically. Hence the only inde-

(1) W., p. 69.

pendent non-vanishing relative invariants of an anharmonic curve are θ_3 and θ_8; so that λ is the only independent non-vanishing absolute invariant of an anharmonic curve. Let Φ_w be any relative invariant of weight w. Then

$$\frac{\Phi_w}{\theta^p{}_3\,\theta^q{}_8},$$

where

$$3p + 8q = w,$$

will be an absolute invariant, which must be a fuction of λ. For the absolute invariants of any curve are quotients of properly chosen powers of relative invariants. Consequently, *any relative invariants of an anharmonic curve may be written in the form*

$$\Phi_w = a\theta^p{}_3\,\theta^q{}_8\,,$$

where a, p, q, are constants and $3p + 8q = w$. Then

$$\Phi'_w = a\theta_3^{p-1}\,\theta_8^{q-1}(p\theta'_3\,\theta_8 + q\theta_3\,\theta'_8).$$

$$J(\theta_3\,,\,\Phi_w) \equiv 3\theta_3\,\Phi'_w - w\theta'_3\,\Phi_w$$

$$= a\theta^p{}_3\,\theta_8^{q-1}[3q\,\theta_3\,\theta'_8 - (w - 3p)\,\theta'_3\,\theta_8] =$$

$$= aq\,\theta^p{}_3\,\theta_8^{q-1}\,\theta_{12}.$$

Hence

$$J(\theta_3\,,\,\Phi_w) = 0 \tag{56}$$

for all values of w.

Any covariant point which is not on the tangent at P_y may be represented by

$$c_8 + \Phi_4 c_4 + \Phi_8 c_0\,, \tag{57}$$

where Φ_4 and Φ_8 are either equal to zero or to relative invariants of the total weights indicated, and C_0, C_4, C_8 are the fundamental covariants defined by (18). Our problem is to determine Φ_4 and Φ_8 so that the coordinates of the point represented by (57) shall be a solution of (44). We find, from (26) and (57),

$$
\begin{aligned}
x_1 &= (\theta'_3)^2 + 9P_2\theta^2_3 + \theta'_3\Phi_4 + \Phi_8, \\
x_2 &= 6\theta_3\theta'_3 + 3\theta_3\Phi_4, \\
x_3 &= 18\theta^2_3, \\
u_1 &= \Phi'_8 + \theta''_3\Phi_4 + \theta'_3\Phi'_4 + 2\theta'_3\theta''_3 - 3P_2\theta_3\Phi_4 + 12P_3\theta_3\theta'_3 - 18\theta^3_3, \\
u_2 &= 4\theta'_3\Phi_4 + 3\theta_3\Phi'_4 + \Phi_8 + \theta_8 + 14(\theta'_3)^2, \\
u_3 &= 42\theta_3\theta'_3 + 3\theta_3\Phi_4.
\end{aligned}
\tag{58}
$$

Substituting in (43) and reducing by means of (56), we find that the point will remain fixed if, and only if, Φ_4 and Φ_8 satisfy the equation

$$\Phi^2_4 - 2\Phi_8 - 2\theta_8 = 0,$$

$$\Phi_4\Phi_8 - \Phi_4\theta_8 + 108\theta^4_3 = 0.$$

After eliminating Φ_6 we have

$$\Phi^3_4 - 4\theta_8\Phi_4 + 216\theta^4_3 = 0.$$

Dividing the roots of this equation by $6\theta_2$, we obtain

$$r^3 - \frac{\theta_8}{9\theta^2_3} r + \theta_3 = 0. \tag{59}$$

Hence a necessary and sufficient condition that the point represented by (57) remain fixed is that

$$\Phi_4 = 6\theta_3 r_i \quad , \quad \Phi_8 = 18\theta^2_3 r^2_i - \theta_8, \qquad (i = 1, 2, 3),$$

where r_1, r_2, r_3 are the three roots of the cubic (59).

Any point which is on the tangent to C_y at P_y may be represented by

$$C_4 + \psi_4 C_0. \tag{60}$$

We find

$$
(61) \qquad
\begin{aligned}
x_1 &= \theta'_3 + \psi_4 , & u_1 &= \psi'_4 + \theta''_3 - 3P_2\theta_3 , \\
x_2 &= 3\theta_3 , & u_2 &= 4\theta'_3 + \psi_4 , \\
x_3 &= 0 , & u_3 &= 3\theta_3 .
\end{aligned}
$$

The condition (43) requires $\theta_3 = 0$. Hence no point represented by a covariant of the type (60) can remain fixed as t varies. Thus we have the theorem : *There are, at most, three covariant points which remain fixed as* P_y *moves along an anharmonic curve. Their coordinates are given by*

$$
(62) \qquad
\begin{aligned}
x_1^{(i)} &= 18\theta^2_3 r^2_i + 6\theta_3\theta'_3 r_i + (\theta'_3)^2 + 9P_2\theta^2_3 - \theta_8 , \\
x_2^{(i)} &= 18\theta^2_3 r_i + 6\theta_3\theta'_3 , \qquad (i = 1 , 2 , 3) \\
x_3^{(i)} &= 18\theta^2_3 ,
\end{aligned}
$$

where r_1 , r_2 , r_3, *are the three roots of the cubic* (59).

It is known from the theory of differential equation that the three soutions (62) will form a fundamental set of solutions of (44) if, and only if, the determinant

$$
D = \begin{vmatrix}
x_1^{(1)} & x_2^{(1)} & x_3^{(1)} \\
x_1^{(2)} & x_2^{(2)} & x_3^{(2)} \\
x_1^{(3)} & x_2^{(3)} & x_3^{(3)}
\end{vmatrix}
$$

does not vanish identically. It will be found that

$$
D = 324\theta^4_3(r_1 - r_2)(r_1 - r_3)(r_2 - r_3).
$$

Hence the three points defined by (62) will form a non-degenerate triangle if, and only if, the roots of (59) are distinct.

If $\lambda = \dfrac{3^9}{4}$ the cubic (59) has two equal roots. Hence in the case of an anharmonic curve of the second class the three points are replaced by two. This justifies our apparently arbitrary division of anharmonic curves into two classes depending on the value of λ.

Let us suppose that the three covariant fixed point (62) are distinct and denote then by A_1 , A_2 , A_3. They form a non-degenerate triangle and the equation of the side opposite A_i is found to be

$$(63) \quad u_i \equiv 18\theta^2_3 x_1 + (18\theta^3_3 r_i - 6\theta_3\theta'_3)x_2 + \qquad (i = 1, 2, 3).$$

$$+ [18\theta^2_3 r^2_i - 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - 9P_2\theta^2_3 - \theta_8]x_3 = 0.$$

The tangent to C_y at P_y outs the sides of the triangle $A_1A_2A_3$ in the points

$$(64) \qquad Q_i \equiv (-3\theta_3 r_i + \theta'_3, 3\theta_3, 0). \qquad (i = 1, 2, 3).$$

The anharmonic ratio of the four points P_y, Q_1, Q_2, Q_3 is found from (23) to be

$$(65) \qquad (P_y, Q_2, Q_1, Q_3) = \frac{r_3 - r_1}{r_2 - r_1} \equiv k.$$

It is evident from (59) that

$$r_1 + r_2 + r_3 = 0, \quad r_1r_2 + r_2r_3 + r_3r_1 = -\frac{\theta_8}{9\theta^2_3}, \quad r_1r_2r_3 = -\theta_8.$$

From these equations we find

$$(66) \qquad \lambda = \frac{\theta^3_8}{\theta^8_3} = 3^9 \frac{(k^2 - k + 1)^3}{(h - 2)^2(1 - 2k)^2(k + 1)^2}.$$

Hence, the anharmonic ratio (65) is constant for all points on the curve. In other words, the triangle defined by (62) is the *invariant triangle of the anharmonic curve.*

15. *The curves which are associated with an anharmonic curve.* We have seen that the vertices of the invariant triangle are the only points that remain fixed as P_y moves along an anharmonic curve. Let us now investigate the character of the curves described by the other covariant points in the plane. Reaves [1] has studied certain special types of anharmonic curves and has shown that some of the points defined in the early part of this paper describe straight lines, while others describe curves which are projective with the given curve. His results will be found to agree with the general theorem which we are about to derive.

Since the vertices of the invariant triangle are fixed, its sides will also be fixed. Consequently the points on the three lines which form the sides of

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this triangle move along these lines. It will be shown later that these are the only covariant points in the plane which describe straight lines as P_y moves along C_y.

It will be convenient to study next the points which are not on the tan- to C_y at P_y, that is, points defined by the covariant (57). From (58) and (63) we find that the point P_τ is on a side of the invariant triangle if, and only if,

$$\alpha_8^{(i)} \equiv 18\theta^2_3 r^2_i + 3\theta_3\Phi_4 r_i + \Phi_8 - \theta_8 = 0. \qquad (i = 1, 2, 3). \tag{67}$$

Let us suppose that $\alpha_8^{(i)} \neq 0$. From (40), (56), and (58) we find that the tangent to C_τ has the equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0, \tag{68}$$

where

$$a_1 = -18\theta^2_3[\Phi^2_4 - 2\Phi_8 - 2\theta_8],$$

$$a_2 = 6\theta_3[\Phi_4\Phi_8 - \Phi_4\theta_8 + 108\theta^4_3] + 6\theta_3\theta'_3[\Phi^2_4 - 2\Phi_8 - 2\theta_8],$$

$$a_3 = (\Phi_8 - (\theta'_3)^2 + 9P_2\theta^2_3)[\Phi^2_4 - 2\Phi_9 - 2\theta_8] -$$

$$- (\Phi_4 + 2\theta'_3)[\Phi_4\Phi_8 - \Phi_4\theta_8 + 108\theta^4_3].$$

Eliminating x_1 between (63), (68), and reducing by means of (59), we obtain

$$3\theta_3\alpha_8^{(i)}[\Phi_4 - 6\theta_3 r_i]\,x_2 - \alpha_8^{(i)}[\Phi_8 + \theta_8 + \theta'_3\Phi_4 - 6\theta_3\theta'_3 r_i - 3\tau_2\Phi_4 r_i]\,x_3 = 0.$$

Dividing by $\alpha_8^{(i)}$, we have

$$3\theta_3[\Phi_4 - 6\theta_3 r_i]\,x_2 - [\Phi_8 + \theta_8 + \theta'_3\Phi_4 - 6\theta_3\theta'_3 r_i - 3\theta_3\Phi_4 r_i]\,x_3 = 0. \tag{69}$$

Eliminating x_2 between (63) and (69), we have

$$18\theta^2_3[\Phi_4 - 6\theta_3 r_i]\,x_1 + 6\theta_3 r_i[(\theta'_3)^2 + 9P_2\theta^2_3 + \theta'_3\Phi_4 + \Phi_8]\,x_3 - \Phi_4[(\theta'_3)^2 + 9P_2\theta^2_3 + \theta_8]\,x_3 -$$

$$- [2\theta'_3(\Phi_8 + \theta_8) - 108\theta^4_3]\,x_3 = 0.$$

Hence the coordinates of the points of intersection of (68) with the three sides of the invariant triangle are

$$
(70)\quad
\begin{aligned}
x_1^{(i)} &= -6\theta_3 r_i[(\theta'_3)^2 + 9P_2\theta^2_3 + \theta'_3\Phi_4 + \Phi_8] + \Phi_4[(\theta'_3)^2 + 9P_2\theta^2_3 + \theta_8] + \\
&\qquad + 2\theta'_3(\Phi_8 + \theta_8) - 108\theta^4_3 , \\
x_2^{(i)} &= -6\theta_3 r_i(6\theta_3\theta'_3 + 3\theta_3\Phi_4) + 6\theta_3\theta'_3\Phi_4 + 6\theta_3(\Phi_8 + \theta_8) , \\
x_3^{(i)} &= -108\theta^3_3 r_i + 18\theta^2_3\Phi_4. \qquad (i = 1 , 2 , 3).
\end{aligned}
$$

These points are defined by the covariants

$$
(71)\quad (\Phi_4 - 6\theta_3 r_i)C_8 + 2(\Phi_8 + \theta_8 - 3\theta_3\Phi_4 r_i)C_4 - (6\theta_3\Phi_8 r_i - \Phi_4\theta_8 + 108\theta^4_3)C_0.
$$

$$
(i = 1 , 2 , 3).
$$

From (23) we find that the anharmonic ratio of P_τ and the three points (70) is $\dfrac{r_3 - r_1}{r_2 - r_1}$.

If P_τ is on the tangent to C_y at P_y it is defined by the covariant (60). If it is also on a side of the invariant triangle, we have

$$
(72)\qquad \beta_4^{(i)} \equiv 3\theta_3 r_i + \varphi_4 = 0. \qquad (i = 1 , 2 , 3).
$$

Let us assume $\beta_4^{(i)} \neq 0$. Proceeding as above, we find that the tangent to C_τ cuts the three sides of the invariant triangle in the points defined by the covariants

$$
(73)\qquad C_8 + 2(\varphi_4 - 3\theta_3 r_i)C_4 + (\theta_8 - 6\theta_3 r_i)C_0 \qquad (i = 1 , 2 , 3).
$$

We find, as before, that the anharmonic ratio of the four poits is $\dfrac{r_3 - r_1}{r_2 - r_1}$.

Thus we are led to the theorem: *As* P_y *moves along an anharmonic curve*

(a) *The vertices of the invariant triangle remain fixed,*

(b) *The covariant points on the sides of this triangle describe straight lines, and*

(c) *All other covariant points describe anharmonic curves which are projective with the given anharmonic curve.*

16. *The polar triangle.* There is another triangle associated with the invariant triangle which has some interesting geometric properties. Let us denote by B_1 , B_2 , B_3 , the poles of the lines A_2A_3 , A_2A_1 , A_1A_2 , with respect to the

osculating conic. We find the coordinates of B_i to be

$$x_1^{(i)} = 18\theta_3^2 r_i^2 - 6\theta_3\theta'_3 r_i + (\theta'_3)^2 + 9P_2\theta_3^2 - \theta_8,$$

(74) $$x_2^{(i)} = -18\theta_3^2 r_i + 6\theta_3\theta'_3, \qquad (i = 1, 2, 3),$$

$$x_3^{(i)} = 18\theta_3^2.$$

The equations of the three sides of this polar triangle are

$$v_i \equiv 18\theta_3^2 x_1 - (18\theta_3^2 r_i + 6\theta_3\theta'_3)x_2 + [18\theta_3^2 r_i^2 + 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - 9P_2\theta_3^2 - \theta_8]x_3 = 0$$

$$(i = 1, 2, 3).$$

Thus we see that in order to obtain the coordinates of the vertices and the equations of the sides of the polar triangle it is only necessary to change the sign of θ_3 in (59), (62) and (63). Hence this polar triangle goes over into the invariant triangle for a curve $\overline{C}_y$ which corresponds to C_y by a dualistic transformation (1).

The equation of the line A_iB_i is

$$18\theta_3^2 x_1 - 6\theta_3\theta'_3 x_2 - [18\theta_3^2 - (\theta'_3)^2 + 9P_2\theta_3^2 - \theta_8]x_3 = 0. \qquad (i = 1, 2, 3).$$

These three lines are concurrent in the point $B \equiv (\theta'_3, 3\theta_3, 0)$. That is, the two triangles are perspective from B. Hence the points of intersection of corresponding sides are collinear. If S_i denote the points of intersection of these lines, it will be found that

(75) $$S_i \equiv (-18\theta_3^2 r_i^2 + (\theta'_3)^2 + 9P_2\theta_3^2 + \theta_8, 6\theta_3\theta'_3, 18\theta_3^2). \qquad (i = 1, 2, 3).$$

These points lie on the line CA, the polar of B with respect to the osculating conic.

If the point of intersection of the tangent at P_y with the sides of the polar triangle are denoted by R_i, we find

(76) $$R_i \equiv (3\theta_3 r_i + \theta'_3, 3\theta_3, 0), \qquad (i = 1, 2, 3),$$

and the anharmonic ratio of the four points P_y, R_1, R_2, R_3, is given by

$$(P_y, R_2, R_1, R_3) = \frac{r_3 - r_1}{r_2 - r_1} \equiv k.$$

(1) W., p. 61.

Although the invariant triangle and its polar give the same anharmonic ratio at every point of an anharmonic curve, there is a fundamental difference in the way they are related to the given curve. The invariant triangle remain fixed as P_y moves along an anharmonic curve, while the polar triangle is continually changing its position and shape. Since $r_i = 0$ only when $\theta_3 = 0$, the two triangles are always distinct.

It is easily seen that the anharmonic ratio

$$(P_y\,,\,Q_i\,,\,B\,,\,R_i) = -1. \qquad (i = 1\,,\,2\,,\,3).$$

That is, P_y and B are harmonic conjugates with respect to any homologous pair of points Q_i, R_i. It follows from (21), (62), (76) that the lines A_1B_1, A_2B_2, A_3B_3 are concurrent at E_b. The lines H_1Q_1, H_2Q_2, H_3Q_3 are concurrent in the same point.

17. *Anharmonic curves of the second class.* The cubic (59) may be written in the form

$$(77) \qquad \left(r - \frac{27}{2}\frac{\theta^3_3}{\theta_8}\right)^2 \left(r + 27\,\frac{\theta^3_3}{\theta_8}\right) + \left(\frac{\theta_8}{9\theta^2_3}\,r - \theta_3\right)\left(\frac{3^9}{4}\frac{\theta^8_3}{\theta^3_8} - 1\right) = 0.$$

Hence, if $\lambda = \dfrac{3^9}{4}$, the roots are

$$(78) \qquad r_1 = r_2 = \frac{27}{2}\frac{\theta^3_3}{\theta_8}\,, \qquad r_3 = -27\,\frac{\theta^3_3}{\theta_8}\,.$$

Substituting in (62) and (63), we find that the two fixed points are

$$(79) \qquad \begin{aligned} A'_{12} &\equiv \left(81\,\frac{\theta^4_3\theta'_3}{\theta_8} + (\theta'_3)^2 + 9P_2\theta^2_3 - \frac{1}{3}\,\theta_8\,,\; 243\,\frac{\theta^5_3}{\theta_8} + 6\theta_3\theta'_3\,,\; 18\theta^2_3\right), \\ A'_3 &= \left(-162\,\frac{\theta^4_3\theta'_3}{\theta_8} + (\theta'_3)^2 + 9P_2\theta^2_3 + \frac{5}{3}\,\theta_8\,,\; -243\,\frac{\theta^5_3}{\theta_8} + 6\theta_3\theta'_3\,,\; 18\theta^2_3\right), \end{aligned}$$

and the equations of the two fixed lines are

$$(80) \qquad \begin{aligned} u'_{12} &= 18\theta^2_3x_1 + 3\left(81\,\frac{\theta^5_3}{\theta_8} - 2\theta_3\theta'_3\right)x_2 + \left[-81\,\frac{\theta^4_3\theta'_3}{\theta_8} + (\theta'_3)^2 - 9P_2\theta^2_3 - \frac{1}{3}\,\theta_8\right]x_3 = 0, \\ u'_3 &= 18\theta^2_3x_1 - 6\left(81\,\frac{\theta^5_3}{\theta_8} + 2\theta_3\theta'_3\right)x_2 + \left[162\,\frac{\theta^4_3\theta'_3}{\theta_8} + (\theta'_3)^2 - 9P_2\theta^2_3 + \frac{5}{3}\,\theta_8\right]x_3 = 0. \end{aligned}$$

As P_y moves along an anharmonic curve of the second class, the two points (79) remain fixed, while every covariant point which lies on either of the lines (80) moves along that line. All other covariant points describe anharmonic curves of the second class.

18. *The osculating anharmonic curve.* If C_y is not anharmonic curve, the points defined by (62) will not remain fixed as the independent variable changes. However, even in this case, they are still of special interest. Let P_y be a fixed point on the curve C_y, corresponding to $t = t_0$. If we make use of the special triangle of reference which corresponds to the Laguerre-Forsyth canonical form of the differential equation, we find from (62) and (63),

$$(81)\quad A_i \equiv (18\theta^2_3 r^2_i + 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \theta_8 \,,\, 18\theta^2_3 r_i + 6\theta_3\theta'_3 \,,\, 18\theta^2_3), \quad (i = 1 , 2 , 3),$$

and

$$(82)\quad u_i \equiv 18\theta^2_3 x_1 + (18\theta^2_3 r_i - 6\theta_3\theta'_3)x_2 + [18\theta^2_3 r^2_i - 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \theta_8]x_3 \,,$$

$$(i = 1 , 2 , 3),$$

where r_1 , r_2 , r_3 are the roots of (59) and θ_3 , θ_8 are defined by

$$\theta_3 = P_3 \quad , \quad \theta_8 = 6P_3P''_3 - 7(P'_3)^2.$$

The triangle $A_1A_2A_2$ will be the invariant triangle of an anharmonic curve whose equation, referred to the fixed triangle $P_yP_zP_\rho$, is [1]

$$(83)\qquad u_1^{r_2-r_3} u_2^{r_3-r_1} u_3^{r_1-r_2} - 1 = 0.$$

If $t = t_0$ is an ordinary point for the function P_3 the three solutions y_1, y_2 , y_3, of the differential equations may be expressed as a power series in $t - t_0$, which converges for values of $|t - t_0|$ sufficiently small. We may, without loss of generality, take $t_0 = 0$ sinec, by the transformation
the series may be reduced to this form. Wilczynski [2] has shown that

$$t - t_0 = \bar{t} \,,$$

(1) Encyklop. d. math. Wissensch. III. D. 4, s. 206.
(2) W., p. 62.

the three power series are

$$y_1 = 1 - \frac{P_3}{3!}t^3 - \frac{P'_3}{4!}t^4 - \frac{P''_3}{5!}t^5 - \frac{P'''_3 - P^2_3}{6!}t^6 - \ldots,$$

(84) $$y_2 = t - \frac{P_3}{4!}t^4 - \frac{2P'_3}{5!}t^6 - \frac{3P''_3}{6!}t^6 - \ldots,$$

$$y_3 = \frac{1}{2}t^2 - \frac{P_3}{5!}t^5 - \frac{3P'_3}{6!}t^6 - \ldots.$$

Let

(85) $$\overline{u}_i = 18\theta^2_3 y_1 + (18\theta^2_3 r_i - 6\theta_3\theta'_3)y_2 + [18\theta^2_3 r^2_i - 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \theta_8]y_3$$

$$(i = 1, 2, 3),$$

where y_1, y_2, y_3 are defined by (84). If now we express the combination

(86) $$\overline{u}_1^{\,r_2 - r_3}\,\overline{u}_2^{\,r_3 - r_1}\,\overline{u}_3^{\,r_1 - r_2} - 1$$

as a power series in t, it will be found that the power series contains no terms of lower degree than the eighth. Hence the anharmonic curve (83) has at P_y a contact of the seventh order with C_y; that is, it has eight consecutive points in common with it. In other words, it is the ***osculating anharmonic curve.***

As P_y moves along C_y the points A_1, A_2, A_3 will describe certain loci, but for every position of P_y they will determine the invariant triangle of an anharmonic curve which osculating C_y at P_y. Of course when P_y passes through a point where $\lambda = \frac{3^9}{4}$ the three points will be replaced by two, and the osculating anharmonic will be of the second class.

19. *Penosculating anharmonic curves.* Let us consider the three points

(87) $$B_i \equiv (18\theta^2_3 r^2_i + 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \omega,\ 18\theta^2_3 r_i + 6\theta_3\theta'_3,\ 18\theta^2_3),$$

$$(i = 1, 2, 3),$$

where r_1, r_2, r_3 are the roots of the cubic

$$r^3 - \frac{\omega}{9\theta_3^2} r + \theta_3 = 0, \tag{88}$$

and ω is an arbitrary function of t. The equation of the three sides of this triangle are $v_i = 0$, where

$$v_i \equiv 18\theta_3^2 x_1 + (18\theta_3^2 r_i - 6\theta_3\theta'_3)x_2 + [18\theta_3^2 r_i^2 - 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \omega]x_3. \tag{89}$$

$$(i = 1, 2, 3).$$

The equation of an anharmonic curve, of which B_1, B_2, B_3 is the invariant triangle, is

$$v_1^{r_2-r_3} v_2^{r_3-r_1} v_3^{r_1-r_2} - 1 = 0. \tag{90}$$

Let

$$\overline{v_i} = 18\theta_3^2 y_1 + (18\theta_3^2 r_i - 6\theta_3\theta'_3)y_2 + [18\theta_3^2 r_i^2 - 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \omega]y_3, \tag{91}$$

$$(i = 1, 2, 3),$$

where y_1, y_2, y_3 are defined by (84). If now we express the combination,

$$\overline{v_1}^{r_2-r_3}\, \overline{v_2}^{r_3-r_1}\, \overline{v_3}^{r_1-r_2} - 1, \tag{92}$$

as a power series in t, it will be found that the series will begin with the terms

$$\frac{(r_1 - r_3)(r_3 - r_2)}{10 \times 7\,!}\theta_3(\theta_8 - \omega)t^7.$$

Hence the anharmonic curve (90) has at P_y a contact of the sixth order with C_y. In other words, equation (90) defines a one parameter family of *penosculating* anharmonic curves. For each value of ω we obtain one of the curves and, in particular, for $\omega = \theta_8$ we have the osculating anharmonic.

The values of θ_3 and θ'_3, for $t = t_0$, will be the same for all curves of the family as for C_y, while θ''_3 will vary from one curve to another, being equal to its value for C_y only when $\omega = \theta_8$. As ω varies, the points B_i will describe certain loci whose equations, referred to the fixed local triangle of reference, may be obtained by eliminating ω and r_i between the equations

$$x_1^{(i)} = 18\theta_3^2 r_i^2 + 6\theta_3\theta'_3 r_i + (\theta'_3)^2 - \omega,$$

$$\text{(93)} \qquad x_2^{(i)} = 18\theta_3^2 r_i + 6\theta_3\theta'_3, \qquad (i = 1, 2, 3),$$

$$x_3^{(i)} = 18\theta_3^2.$$

It should be noted that these loci differ from those considered in the earlier part of this paper in that they are connected with a fixed point P_y.

After performing the elimination, we obtain

$$(x_2^2 - 2x_1x_3)(\theta'_3x_3 - 3\theta_3x_2) + 3\theta_3^2x_3^3 = 0.$$

The three vertices of the invariant triangle will lie on this cubic for all values of the parameter ω. In deriving this equation we have used the special triangle of reference corresponding to the Laguerre-Forsyth canonical form of the differential equation. However, in order to transform to the general triangle of reference, it is only necessary to replace x_1 by $x_1 - \frac{1}{2}P_2x_3$. The cubic then becomes

$$\text{(94)} \qquad (x_2^2 - 2x_1x_3 + P_2x_3^2)(\theta'_3x_3 - 3\theta_3x_2) + 3\theta_3^2x_3^3 = 0.$$

This cubic has a node at P_y with tangents $x_3 = 0$ and $\theta'_3x_3 - 3\theta_3x_2 = 0$. That is, it has the same node and the same nodal tangents as the eight-pointic nodal cubic. It will be found that the inflectional line of the cubic (94) is

$$\text{(95)} \qquad 18\theta_3^2x_1 - 6\theta_3\theta'_3x_2 + [(\theta'_3)^2 - 9P_2\theta_3^2]x_3 = 0.$$

A glance at equation (17) shows that the two cubics have the same inflectional line.

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