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THEORY OF VARIABLE DYNAMICAL ELECTRICAL SYSTEMS

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BY

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THEORY OF VARIABLE DYNAMICAL-ELECTRICAL SYSTEMS.

BY H. W. NICHOLS.

COMPARED with the volume of literature on the subject of electrical and mechanical systems with invariable elements—resistance, inertia or inductance, stiffness or capacity—very little has been published concerning systems in which these elements are variable in a general way, and this notwithstanding the fact that such important applications as electric signaling depend upon the variability of some element of the system. The problem is usually to find the effect upon a steady or quasi-steady state of changes in some element, and the steady state is often not of interest, so that its existence is ignored. The ignoring of the undisturbed state but not of the energy which is transformed from it by the variable elements leads to many interesting and important problems, some of which are considered in this paper.

A more formal statement of the problem to be solved is: A dynamical-electrical system capable of description by means of differential equations obtained from a Lagrangian and a Dissipation function of the usual type is operating under the influence of given impressed forces. This state of the system is disturbed by changes in an inertia, resistance or stiffness element and the disturbed state is considered by itself, the undisturbed state being ignored if its motions are of types (defined later) different from those of the disturbed state.

It will be found that energy can in this way be added to the disturbed state in a manner similar to that in which it is "lost" from a dynamical system by transformation to a type with which the purely dynamical problem is not concerned, namely the energy of thermal agitation of systems whose coördinates are not required to be included in the Lagrangian function in order to obtain a satisfactory solution of the larger scale dynamical problem.

A very simple illustration of a system of this kind is an electric bell or "buzzer." From one very practical point of view the dynamical system of interest comprises only the button with its impressed force and the vibrating armature, so that the type of motion obtained bears no relation to that of the source (for example, a battery) nor is there any necessary relation between the energy of the motion and the work done by the impressed force at the button. From another point of view the system is a generator of oscillations, either electrical or mechanical, without energy supply at the frequency of the oscillations and deriving its energy from a source which may be ignored if only oscillatory states are of interest.

From the point of view here taken it is convenient to classify dynamical-electrical systems into the following types, the first two of which are the ones ordinarily considered and the last two are of particular interest because such systems are capable of bringing energy into play from sources of different kinds, as will be explained later.

Types of Systems.—1. All purely electrical systems (whose motions do not involve the changing of a mechanical coördinate) and those in which the variables are measured from equilibrium positions, the generalized displacements x being small, are characterized by Lagrangian and Dissipation functions which are homogeneous quadratic functions of the displacements and velocities and have constant coefficients. As a result the system of differential equations of motion is a set of linear equations of the form:

$$\begin{aligned} S_{11}x_1 - S_{12}x_2 - S_{13}x_3 &= e_1 \\ -S_{21}x_1 + S_{22}x_2 - S_{23}x_3 &= e_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \end{aligned}$$

Here any stiffness operator S has the form

$$S = lp^2 + rp + s; \quad p = d/dt$$

and e is a given function of the time, being an impressed force. This system is characterized by the fact that its operational determinant $|S_{11}S_{nn}|$ is symmetrical, since the Lagrangian and Dissipation functions are of the specified type, and by the fact that it satisfies the energy principle. Due to these facts, the reciprocal theorem holds, namely, if C_{jk} represents the operator which finds the displacement x_j from unit driving force at the place k, or the mutual compliance between j and k:

$$C_{jk} = \frac{D_{jk}}{D}$$

where D is the operational determinant of the system and D_{jk} the minor of row j and column k. But when a unit driving force is located at j it produces at k the symbolic displacement $C_{kj} = D_{kj}/D$; and because D is symmetrical, $D_{jk} = D_{kj}$. Hence the two mutual compliances are the same. Such a system has been called bilateral.

The set of linear equations above has the further property that, if the driving forces are periodic and are resolved into their Fourier components, all these components will in general appear in the particular solution for any x, and no others will ever appear. This is because the coefficients are constant, and this property will be described by saying that the system cannot change the *type* of driving force. This feature is important in many applications.

2. When the coördinates of interest are the small departures from zero values in a state of motion, the Lagrangian function is not a homogeneous quadratic one but leads to a set of differential equations of the form:[1]

$$\begin{aligned} S_{11}x_1 - (B_{12} + C_{12})x_2 - (B_{13} + C_{13})x_3 - \cdots &= e_1 \\ -(B_{21} - C_{21})x_1 + S_{22}x_2 - (B_{23} - C_{23})x_3 - \cdots &= e_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \end{aligned}$$

in which C_{jk} has the form $R_{jk}p = R_{kj}p$.

If the kth equation is multiplied by px_k and the results added, the terms in C cancel and hence do not enter into the equation of activity. Systems of this kind are different from those of the first class in that the reciprocal theorem does not hold. They resemble them, however, in reproducing the type of driving force. While the C's are of odd order in p, they do not contribute odd order terms to D, although they do to its minors. These centrifugal terms correspond formally to mutual resistances, but differ from ordinary dissipative terms in that they occur in pairs in such a way as to make the determinant D of even order in p so long as true resistances are not present. This is suggestive, as it indicates the possibility of compensating true resistances by similar terms, and the general conditions under which this may be accomplished will be discussed later. It is evident that in order to do this there must be a transformation of energy from that of an ignored type of driving force, otherwise an uncompensated flow of energy takes place out from the system through the resistances.

Energy dissipated as heat is of course not lost, but simply transformed into a type which is ignored in purely dynamical-electrical problems as outside the scope of the investigation. In the same way energy may be thought of as entering the system from an ignored source through suitable devices for changing its type into one with which the problem is concerned. The principal object of this paper is to investigate systems in which this occurs.

3. When the differential equations of the system are linear but with coefficients which are functions of the time, the system is characterized by the very important fact that it is able to execute motions whose types are different from those of the driving force, that is, the particular solu-

[1] See, for example, Whittaker, Analytical Dynamics, p. 84.

tions no longer correspond in component frequencies to the driving forces. If therefore some driving forces are of types which are ignored for the purposes of the investigation, perhaps because they have no direct influence upon certain parts of the system, it is possible to supply energy to the system in a way similar to that in which it is drawn off as heat. The definition chosen for the *type* of driving force or motion is now shown to be a suitable one, for when no energy is transformed to ignored types of any kind, the particular solutions depend upon the then pure imaginary roots of $D = 0$.

4. When the differential equations of the system are non-linear, the principle of superposition no longer holds, which fact is of considerable importance in some applications. Non-linearity often means that all the mechanism of the system has not been taken into account.

I. Systems with Invariable Elements.

The classification adopted is a convenient one for our purposes and will be followed in this treatment, beginning with the first two classes. These are the cases ordinarily considered, and will be taken up only very briefly to collect some useful results, all of which, however, are no doubt old.

The coefficients of p^n in the differential equations:

$$\begin{aligned} S_{11}x_1 - S_{12}x_2 - \cdots &= e_1 \\ -S_{21}x_1 + S_{22}x_2 - \cdots &= e_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \end{aligned}$$

being constants, p may be treated as an algebraic quantity, with the result that any x has the value

$$x_k = \frac{\Sigma D_{jk}e_j}{D}.$$

If e_j is the only driving force acting,

$$x_k = \frac{D_{jk}e_j}{D}$$

and the operator D_{jk}/D is called the mutual compliance, C_{jk}, between the parts k and j, being the operator which finds the displacement produced at k by a driving force at j. The use of such operators apparently originated with Heaviside. For systems of the first class, $C_{kj} = C_{jk}$. When $k = j$ the compliance is self instead of mutual.

In certain special cases in which the driving forces are special functions of the time these operators reduce to algebraic quantities. Some of the important results are considered below.

Steady State.—If in C_{jk}, the operator p is put equal to zero, the solution obtained is that appropriate to the final steady state under the influence of a constant driving force.

Harmonic State.—When p is put equal to $in + \delta$, the solution is that appropriate to an exponentially increasing or decreasing harmonic driving force of frequency $n/2\pi$ and damping δ, and in the extremely important case in which $\delta = 0$, the solution is that for an undamped harmonic driving force of frequency $n/2\pi$. The highly developed subject of alternating currents and of sustained simply periodic motions in general, depends largely for its practicaly value upon this method of reducing the differential equations to algebraic ones.[1]

When the driving force is harmonic, putting $p = in$ reduces any x to the quotient of two determinants whose elements are complex numbers. Further, since any S_{qs} enters linearly into the determinant D, any x_k must be of the form (omitting all subscripts for brevity):

$$x = \frac{aS + b}{cS + d},$$

or a bilinear transformation of S. This is the reason alternating current loci are circles.

When this transformation is thrown into its three constant form by division by c (which obviously cannot be zero for any physical system) it becomes:

$$x = \frac{a}{c} - \frac{1}{c^2}\frac{ad - bc}{S + d/c} = J - \frac{\Delta}{S + K},$$

the constants of which should and do have physical significance, namely:

When $\Delta = 0$ a change in S has no effect upon x; these two parts of the system are conjugate. Hence to have conjugate parts a system, if connected at all, must have at least three degrees of freedom.

J is the value of x when S is infinite, that is, when the branch having the operator S is removed.

When K is zero or Δ/J reduction of S to zero produces infinite and zero values, respectively, of x.

Many other relations may be found involving this transformation. It is useful in experimental work, being established by three points.

In practical work, for example in alternating currents, it is usual to ignore the operational nature of these quantities S, etc., and treat them in the same way as the forces, displacements and velocities themselves; that is, complex numbers $a + ib$, are used for both operators (O) and

[1] For a concise and valuable treatment of this case ($p = in + \delta$) see G. A. Campbell, Proc. A. I. E. E., April, 1911.

physical quantities (T), the latter being functions of the time, which variable is eliminated in single frequency problems by this method. Consequently no distinction is made between products such as OT and TT, which are physically very different things, the first being of frequency $n/2\pi$ and the second having a constant part and a part of double frequency, n/π, that is, not being capable of representation in the complex plane of T. Thus if

$$O = a + ib, \quad T = c + id,$$

the formal product OT gives a physically intelligible result, while the formal product $TT = c_1c_2 - d_1d_2 + i(c_1d_2 + c_2d_1)$ has no physical significance. In particular, it does not represent power, torque, etc. Steinmetz avoided this difficulty by giving up the rule $i \times 1 = 1 \times i$ for these products, but it seems better[1] to introduce a double frequency operator, say k, represented geometrically by a unit vector at right angles to 1 and i, such that

$$O = a + ib + ko,$$

$$T = c + id + ko,$$

as before, to retain the formal operations OT, but to define the complete product TT of two physical quantities as the sum of the scalar and vector products:

$$T \cdot T + T \times T = c_1c_2 + d_1d_2 + k\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} = P + kQ.$$

This gives the correct value (in contrast to formal multiplication) and moreover indicates, by the unit operators 1 and k, the nature of the result with respect to frequency. In problems in which both time differentiations and multiplications TT are required, there is some advantage in using this method and it will be used here when necessary.

Impulses and Initial Values.—When p is made infinite in the compliance operator, the initial values of the coördinates are found.

When a driving force e_j is impulsive, its impulse being

$$I_j = \operatorname*{Lim}_{\substack{t=0 \\ e_j=\infty}} \int^t e_j dt,$$

we have $e_j = pI_j$, which may be substituted in the differential equations to find the behavior under this kind of excitation. An advantage of this method of treatment is that the initial conditions may be found from the differential equations; thus the initial displacements are

$$x_k(o) = \operatorname*{Lim}_{p=\infty} (pC_{jk}I_j),$$

[1] Armour Engineer, January, 1912.

and the initial velocities

$$\dot{x}_k(o) = \lim_{p=\infty} (p^2 C_{jk} I_j).$$

In general, since $C_{jk} = D_{jk}/D$ and the order of D is two higher than that of D_{jk}, the initial displacements are zero, the initial velocities finite, and the initial accelerations infinite for impulsive excitation. This ceases to be the case when inertia terms are lacking from some of the elements, when some initial velocities may be infinite. $(pC_{jk})^{-1}$ and $(p^2C_{jk})^{-1}$, with p infinite, are respectively the initial resistance and initial inertia offered at the place k to a sudden disturbance acting at the place j.

Free Oscillation.—The condition of free oscillation of the system is that $D = 0$, which equation gives the values of $p = in + \delta$ corresponding to the frequencies and damping constants of the component oscillations. If the free oscillations are *sustained*, $\delta = 0$.

S nce the condition $D = 0$ is also the condition that the effect produced by a given driving force shall be the largest possible, it is clear that the two requirements of good signaling, namely that the effect x shall be both a large and a true copy of the cause e for all wave forms, are in general contradictory; for the condition that x shall be largest is also the one that the system shall oscillate without regard to the driving force. In this case the "quality" of reproduction is zero. (It is highly desirable to develop some dynamical specification of quality of reproduction which corresponds to and predicts data furnished by the senses.)

There are two obvious exceptions to this statement, one the case in which the compliance C is of one kind and also independent of p, the other the case of an infinite number of degrees of freedom, to which this argument does not necessarily apply.

II. Systems Having Variable Elements.

When the inertia, resistance, or stiffness factors are variable with the time only, the differential equations of the system are linear with variable coefficients and the importance of this class of systems depends, from our point of view, upon the fact that the particular solutions contain types of motion different from those of the driving force. The variation which is most important is that in which the magnitude never departs greatly from a mean value and in cases of physical interest it is then possible to find a solution in the form of a convergent series as has been done, for example, by Barkhausen,[1] Pupin and others, using a method of successive approximation. The object of this paper, however, is not primarily to find the coefficients in such an expansion, but to show

[1] "Problem der Schwingungserzeugung," 1907.

how the transformation of energy from one type to another leads to useful results.

Considering any generalized stiffness factor with constant elements,

$$S = lp^2 + rp + s,$$

it is clear that when l, r, and s are variable S must be written

$$p(lp) + rp + s = lp^2 + [r + (pl)]p + s,$$

so that if

$$l = l_0 + \lambda, \quad r = r_0 + \rho, \quad s = s_0 = \sigma,$$

the change in S is

$$\Delta S = \lambda p^2 + (\rho + \dot{\lambda})p + \sigma.$$

This interpretation of ΔS will be understood in what follows.

In the set of linear equations

$$\begin{aligned} S_{11}x_1 - S_{12}x_2 - \cdots &= e_1 \\ - S_{21}x_1 + S_{22}x_2 - \cdots &= e_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

let S_{11} be the variable element. This can always be done, if necessary, by a linear change of variable. Put $S_{11} = S_0 - \delta S$ and call D_0 the value of D when $\delta S = 0$. Also let

$$x_k = X_k + \xi_k$$

in which

$$X_k = \frac{\Sigma D_{jk} e_j}{D_0}.$$

If these values are substituted in the set of equations above we get

$$\begin{aligned} (S_0 - \partial S)\xi_1 - S_{12}\xi_2 - \cdots &= \delta S X_1 \\ - S_{21}\xi_1 + S_{22}\xi_2 - \cdots &= 0 \\ - S_{31}\xi_1 - S_{32}\xi_2 + \cdots &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{A}$$

and any ξ is therefore:

$$\xi_k = \frac{D_{k1}}{D}\delta S X_1 = \frac{D_{k1}}{D_0}\left(1 - \delta S \frac{D_{11}}{D_0}\right)^{-1} \delta S X_1$$

$$= \left\{1 + \delta S \frac{D_{11}}{D_0} - \left(\delta S \frac{D_{11}}{D_0}\right)^2 + \cdots\right\} \frac{D_{k1}}{D_0}\delta S X_1. \tag{1}$$

This equation shows that, so far as first order terms in δS are concerned, the disturbance superposed upon the unvaried system may be accounted for by supposing the sources of the unvaried motion removed and replaced by a driving force equal to $\delta S X_1$ whose seat is in the variable element, the constants of the system remaining the same. It shows also that, with a proper interpretation of the symbols, the driving force for large or small values of δS is

$$\Xi = \frac{\delta S x_1}{1 - \delta S \dfrac{D_{11}}{D_0}},$$

since when this is operated upon by D_{k1}/D_0 the result is ξ_k.

There is, however, one very important difference between this system and one without ignored sources and excited by the same driving force, for in the latter case all the power expended in the system comes from the driving force, while in the former case the power may come from impressed forces required to produce the variation δS (that is, from the equivalent driving force) or it may come from the sources maintaining the ignored state, in which case the variable element acts as a transformer of energy from one type to another. These two rôles are essentially different ones and will be discussed in detail shortly, after a few consequences of equation (1) are noted.

It follows from that equation that if ξ is intended to be a copy of the variation δS, the copy cannot be perfect unless $D_{11} = 0$, which means that the compliance of the system, measured from the variable branch, shall be zero. Consequently a perfect copy of the variation of a stiffness factor S cannot be obtained with finite displacements. The terms of degree higher than the first in δS indicate distortion, or departure from perfection of the copy otherwise than through resonant selectivity of the system. An illustration is a microphone telephone transmitter, in which the electrical copy of the motion of the diaphragm is desired to be perfect.

It is evident that forces f_j, of non-ignored type, may be added to the system in the usual way, and the right hand members of (A) will be supposed increased by these impressed forces of ξ-type.

To evaluate ξ in algebraic terms, $\delta S X_1$ must be reduced to a function of time, say $\phi(t)$, and $\delta S D_{11}/D_0$ to the form $F(t) \cdot P(p)$; then the solution is the sum of terms such as

$$\xi' = \frac{D_{11}}{D}\phi(t), \quad \xi'' = F(t) \cdot P(p)\xi', \quad \xi''' = - F(t)P(p)\xi'',$$

etc.

A few simple examples will be given to show the application.

(*a*) Consider an electrical circuit containing inductance l, resistance r, capacity $1/s$ and a constant source of E.M.F. E. The elements are in series. Let the stiffness of the condenser vary as $s_0(1 - a \cos nt)$. The steady state is $X_1 = E/s_0$ and

$$\xi = \left(1 + \frac{a \cos nt}{S_{0\,2}} - \cdots \right) \frac{aE \cos nt}{S_0},$$

where

$$S_0 = lp^2 + rp + s_0.$$

Since $1/S_0$ operating upon any periodic function can always be evaluated, the expansion can be carried out.

(*b*) If the battery is replaced by an alternator of voltage $E \cos qt$, we get

$$X_1 = \frac{Ee^{iqt}}{S_0(iq)}; \quad \delta S X_1 = A \cos [(q + n)t + \alpha] + B \cos [(q - n)t + \beta].$$

Hence ξ contains terms of frequencies proportional to $q \pm n$, and higher order terms of frequencies proportional to $q \pm kn$, $k = 1, 2, \cdots$.

(*c*) If, in this circuit, the resistance varied according to $r(1 - a \cos nt)$, we have $\delta S X_1 = ar \cos nt \cdot pX_1$ and no disturbance is produced unless X_1 is a function of the time, that is, unless the E.M.F. E is variable. If, however, the condenser is shunted by an infinite perfect inductance we have, for a constant E.M.F.

$$X_1 = E/rp; \quad \delta S X_1 = aE \cos nt.$$

(*d*) If, in the last circuit, the inductance is variable so that

$$\delta S = al \cos nt \cdot p - anl \sin nt \cdot p$$

with $X_1 = E/rp$ we get

$$\delta S X_1 = - aEnl/r \cdot \sin nt,$$

and the part $al \cos nt \cdot p$ of the stiffness δS has no influence because the current pX_1 is constant. Finally, if the circuit carries an alternating current

$$pX_1 = A \cos qt,$$

we find

$$\delta S X_1 = B[(q + n) \sin (q + n)t + (q - n) \sin (q - n)t].$$

So far no account has been taken of the manner in which the variation δS is produced, while if a complete description of the behavior of the system is to be had, the dynamics of the variable element must be included in the system of equations. If the energy represented by the ξ-system comes from forces required to produce the variation δS, the principle of energy will be satisfied by including these forces and no liberation of energy from the original state will take place. If, on the other hand, the energy of the ξ-system comes from the ignored state and is simply set free by the action of forces producing the variation, the principle of energy will not be satisfied for that system and there will be no particular relation between the energy of the latter forces and the energy set free by them. This is a very important distinction; for example, in the problem of the telephone amplifier and of generators of sustained oscillations the energy which is transformed must come from an auxiliary and ignored source.

To determine the source of the energy of the disturbed state, consider how variations in any element are produced. Any inertia, resistance or stiffness factor, or in the electrical case, any inductance, resistance or capacity, is fixed by geometrical coördinates and by quantities of the nature of permeability, dielectric constant, etc., depending upon the properties of materials. The geometrical coördinates and electrical charges are the variables chosen to represent the state of the system, together with these material constants whose dynamical natures are either not known or supposed not known. With these the Lagrangian and Dissipation functions are built up, the equations of motion being then found from

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial F}{\partial \dot{x}} = E.$$

Now L is a function of the coördinates x and their velocities $\dot{x}$ and is differentiated by each to find the reactions; hence any change in the geometrical shape of any system of electrical conductors or other bodies, and the forces thereby brought into play, are included in the dynamics of the system. It therefore follows that any energy derived from the change of geometrical form of any inductance, capacity, inertia or stiffness of the system comes from impressed forces tending to change this geometrical form and is taken account of in the equation of energy. In particular, in any complete cycle of operations, no energy comes from the undisturbed state of the system, hence *no energy of an ignored type is continuously transformed by the variation of the geometrical coördinates determining any inductance, capacity, inertia or stiffness factor of the system.*

On the other hand, forces due to the variation of material constants in the Lagrangian function L are not included in the dynamical equations obtained by Lagrange's method and hence the energy set free or transformed by their variations may come from that of the undisturbed state.

An example of this is found in the case of a deformable inductance coil. If the coil is energized by a battery and then deformed so as to vary periodically the inductance of the circuit and thus produce an alternating current in it, all the power represented by that alternating current will be derived from mechanical forces required to vary the shape of the circuit. Such a device could not be used to bring into play an auxiliary source of energy of different type—for example, it could not be made into a telephone amplifier. An ordinary telephone receiver is also a system of this kind.

The same thing is true in the case of a condenser whose geometrical dimensions only are varied, for here the forces resisting deformation are derived from the Lagrangian function and enter into the activity equation.

If in the coil the permeability of the medium is varied without mechanical motion of it as a whole, and hence the inductance varied without varying any geometrical coördinate entering into the Lagrangian function, the energy of the varied state must be derived from the battery. It may require energy to produce the variation in permeability, but the amount of this energy may be quite different from that transformed from the auxiliary source and will depend upon different things. The same remarks apply to the variation of the dielectric constant of a condenser without bodily motion of the medium itself.

The Dissipation function F is also a function of the coördinates and their velocities, together with constants of materials, but in the equations of motion only its partial derivatives with respect to the velocities appear. Hence if *either* the geometrical dimensions of resistances or the specific resistance constants of materials vary with the time, the impressed forces required are not part of the dynamical scheme described by the Lagrangian equations. The energy of the disturbed state must come from the sources of the undisturbed state, while the energy required to vary the resistances need have no necessary relation to that brought into the system from the ignored sources. Any resistance variation, however produced, is able to transform energy from the undisturbed state to the disturbed one.

From this discussion it follows that the only variations not already taken account of in the ordinary equations of motion are those in which a permeability, density, dielectric constant, elastic constant, or a resistance is changed. These are therefore the ways in which energy can be transformed from the sources of the undisturbed state, and in what immediately follows the variation δS of the stiffness will be supposed to contain explicitly only these parts, any other part being already included in the equations of motion as obtained by Lagrange's method. These variations, as well as the resulting motions, will first be supposed small in order that the equations may remain linear.

To take account of the variations produced in the way just discussed, we will now suppose that δS depends upon some mechanical or electrical coördinates $x_1 \cdots x_N, x_{N+1} \cdots x_{N+M}$ and consequently upon $\xi_1 \cdots \xi_{N+M}$ in a way described by the differential equation

$$\delta S = \sum_{j=1}^{N+M} \phi_{1j}(p)\xi_j,$$

which is apparently sufficiently general to include all cases in which the equations remain linear.

The variation δS may depend upon M coördinates not required for the specification of the original system which is not concerned with the mechanism producing this variation, as well as upon the N coördinates originally required. (We might also suppose that a number of elements were varied, with

$$\delta S_k = \sum_1^{N+M} \phi_{jk}(p)\xi_j; \quad k = 1 \cdots N,$$

but it follows from the previous discussion that no generality is added thereby.)

The system $(\xi_{N+1} \cdots . \xi_{N+M})$, which represents the additional mechanism by which δS is determined, will be supposed subject to laws capable of statement by linear differential equations, and the driving force $\delta S X_1$ will be written

$$\Sigma \phi_{1j}\xi_j X_1 = \Sigma Q_{1j}\xi_j.$$

The differential equations of the system, including impressed forces of ξ-type, will now be of the form:

$$\begin{array}{cccc|ccc}
(S_{11} - Q_{11})\xi_1 & -(S_{12} + Q_{12})\xi_2 & \cdots & -(S_{1N} + Q_{1N})\xi_N & -Q_{1,\,N+1}\xi_{N+1} & \cdots & = f_1 \\
-S_{21}\xi_1 & +S_{22}\xi_2 & \cdots & -S_{2,\,N}\xi_N & 0 & \cdots & = f_2 \\
\hline
0 & 0 & \cdots & 0 & S_{N+1\,N+1}\xi_{N+1} & \cdots & = f_{N+1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & 0 & \cdots & S_{N+M\,N+M}\xi_{N+M} & = f_{N+M}
\end{array}$$

Aside from the fact of the variability of S_{11} (which variation may be supposed to be small) this system differs from those discussed in part I in that *there is no necessary relation* between $S_{1k} + Q_{1k}$ and S_{k1}. Consequently *the equation of activity is not satisfied for the ξ-system, but energy is continuously drawn into the system from the ignored sources through the periodically variable element.* The energy so drawn is, per unit time:

$$P = \frac{1}{T}\sum \int_0^T Q_{1j}\xi_j\dot{\xi}_j dt.$$

We have now succeeded in making S_{kj} different from S_{jk} in a way not accounted for by centrifugal terms and such that $S_{jk} - S_{kj}$ may have practically any form, depending upon the dynamical nature of the mechanism by which S is varied. This system obviously need have no special relation to the original one; in fact, if no impressed forces act upon the additional coördinates $\xi_{N+1} \cdots \xi_{N+M}$ required to completely determine δS, that system has no effect upon the solution for any original coördinates except through the Q-terms, for if

$$D^{(N)} = |S_{11}S_{nn}|$$

any original coördinate is

$$\xi_k = \frac{\sum_1^N {}_1D_{jk}{}^{(N)}f_j}{D^{(N)}}, \quad k = 1 \cdots N,$$

and contains no explicit reference to the additional mechanism producing the variation δS.

Since we are treating the original system $\xi_1 \cdots \xi_N$ as the one whose dynamical nature is known, or as the only one upon which measurements are to be made, it is appropriate to eliminate the coördinates $\xi_{N+1} \cdots \xi_{N+M}$ from explicit appearance; moreover this is suggested by the form of the determinant above. For these reasons imagine the last M equations to be solved for the coördinates appearing in them. Let the result of this be substituted in the first N equations and put $Q_{N+j}\xi_{N+j} = g_{1j}$, the g's being expressed in this way in terms of the impressed forces $f_{N+1} \cdots f_{N+M}$. We then get for the equations of motion

$$\text{(B)} \quad \begin{cases} (S_{11} - Q_{11})\xi_1 - (S_{12} + Q_{12})\xi_2 \cdots - (S_{1N} + Q_{1N})\xi_N = f_1 + \Sigma g_{1j} = f_1 + g_1 \\ \quad - S_{21}\xi_1 \quad + S_{22}\xi_2 \qquad \cdots \qquad\qquad = f_2 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad - S_{N1}\xi_1 \qquad \cdots \qquad \cdots \quad + S_{NN}\xi_N \qquad = f_N. \end{cases}$$

which contain explicitly only the original variables, and from which the following fundamental theorem is obtained:

The effect of the ignored sources and of the mechanism by which the small variation δS is produced is to make the original system appear from dynamical-electrical measurements on the ξ-variables only as one in which S_{11} is changed to $S_{11} - Q_{11}$, any S_{1k} to $S_{1k} + Q_{1k}$, and the impressed force f_1 to $f_1 + g_1$.

These equations are the fundamental ones in the study of systems to which energy is added by transformation from ignored sources, and to make clear the meaning of the terms a few special cases will now be considered. First supposed that no impressed forces act upon the coördinates $\xi_{N+1} \cdots \xi_{N+M}$ so that $g_1 = 0$ ultimately. Also suppose that the Q's are zero except Q_{11} which will be written $ap^2 + bp + c$. The equations then show that the system behaves as one in which the stiffness S_{11} is $(l_{11} - a)p^2 + (r_{11} - b)p + (s_{11} - c)$ so that the inertia, resistance and stiffness, or the inductance, resistance and (capacity)$^{-1}$ have been decreased by the (positive or negative) amounts a, b, c, respectively. The quantity $-b$ is called a "negative resistance" for obvious reasons. The power added to the system by means of it, or the negative power dissipated by it, is $b\dot{\xi}_1^2$, hence proportional to the square of the velocity $\dot{\xi}_1$ as is the heat produced in the resistance r_{11}. Energy is thus brought into the system by the same general type of ignored mechanism as that by which it passes out of the purely dynamical or electrical system.

The effects of a and c are changes in the storage of power and hence in the resonant frequencies and phases of forced oscillations. An ordinary electric bell is a simple example of this kind of system, especially if the contact is shunted with a resistance of a few ohms so that the resistance changes are not too large.

Second, if every Q is zero, but g_1 is different from zero we have the simplest case, namely, energy added through the trigger or relay action of impressed forces which vary only the eliminated coördinates and do not act upon the original system. A variable resistance telephone transmitter and an electric switch are illustrations. The power added is $g_1\dot{\xi}_1$.

Third, if every Q is zero except one with unlike subscripts say Q_{12}, and if $g_1 = 0$, the most noteworthy effect is that the determinant of the coefficients is neither symmetrical nor has it only the special skew-symmetric elements appropriate to centrifugal forces. The reciprocal theorem does not hold and we get, for example, putting Δ for the value of D when $Q_{12} = 0$:

$$\frac{C_{21}}{C_{12}} = \frac{\Delta_{21} - Q_{12}M_2}{\Delta_{12}}$$

where M_2 is a second minor of Δ. The mutual compliances may therefore be made widely different in two opposite directions through the system. Similar expressions obtain for other mutual compliances. Examples of systems having these characteristics will be worked out later in this paper.

Free Oscillations.—One of the most important cases is that in which all the impressed forces are zero. In this case no forces of ξ-type act upon the system and if it is to move and do work all the energy required must be transformed from the ignored sources. Such a system is called an oscillation generator.

If the coördinates are not to be zero the determinant D must vanish and its vanishing will determine values $p = p_1$, p_2, etc., which give the frequencies and damping constants of the oscillations. Now for the invariable systems occurring in pure dynamics it can be proved that the real parts of the roots of $D = 0$ are negative if there is any resistance in the system, so that *sustained* free oscillations of those systems cannot take place. This proof, however, does not apply to systems having the more general determinant here found, and we may expect sustained oscillations under proper conditions.

The condition of sustained free oscillation is that $D = 0$ with $p = in$. Suppose p is given this value in the equation $D = 0$ which will then become an equation in n with both real and imaginary coefficients. This equation is equivalent to two equations, say

$$F(n) = 0, \quad G(n) = 0,$$

which when solved simultaneously will give certain values of n and certain corresponding relations between the n's and the Q's. The latter relations are those which must exist in order that p shall be pure imaginary, or that the oscillations of frequency $n/2\pi$ shall be sustained. Hence in order to make the system perform certain free oscillations n, the transforming mechanism must be adjusted to give the corresponding values of the Q's, and by changing these values different oscillations will in general be possible. In this respect this kind of system differs from one merely devoid of resistances, which oscillates simultaneously in all possible modes except in special cases of normal coördinates, when special starting conditions are required.

The power dissipated in a system executing harmonic oscillations is

$$(S_{11}\xi_1 - S_{12}\xi_2 - \cdots) \cdot p\xi_1 + (-S_{21}\xi_1 + S_{22}\xi_2 - \cdots) \cdot p\xi_2 + \cdots$$

and the power transformed is

$$(Q_{11}\xi_1 - Q_{12}\xi_2 - \cdots) \cdot p\xi_1.$$

In the *sustained* free oscillations of the system these are equal, hence in that case the variable element transforms just enough power to supply the dissipation.

Since the effect of the variable element is to supply the dissipated power, it might be thought that to calculate the frequencies of sustained oscillation it would be necessary only to ignore all resistances, or better the dissipation in each branch by making the resistances zero or infinite. But the frequencies so obtained will not in general be the correct ones, even when the variable element introduces only negative resistances and does not change any reactance. As an example consider the case of a transformer:

$$D = \begin{vmatrix} L_1p^2 + R_1p + S_1 - bp & -Mp^2 \\ -Mp^2 & L_2p^2 + R_2p + S_2 \end{vmatrix} = 0.$$

We get, with all losses suppressed:

$$(L_1L_2 - M^2)p^4 + (S_2L_1 + S_1L_2)p^2 + S_1S_2 = 0.$$

while equating terms in odd and even powers of p separately to zero gives the additional term in the frequency equation:

$$R_2(R_1 - b)p^2.$$

The odd order terms are

$$p(R_1 - b)(K_2p^2 + S_2) + pR_2(L_1p^2 + S_1)$$

and these equations are inclnsistent with $R_2(R_1 - b) = 0$ unless $M = 0$.

Moreover let $\pm p_1$ and $\pm p_2$ be the roots of the even part of $D = 0$ and substitute them in the equation for b. The result will be that in general $b(p_1)$ will be different from $b(p_2)$, so that the system, with a given value of b, will perform only a part of the possible oscillations.

Example.—A device which illustrates this theory and method is one for producing small alternating currents for laboratory use and known as the "microphone hummer." It consists of a bar B supported on knife edges above a magnet and carrying a carbon cell C through which flows current from a battery. Motion of the bar varies the resistance of the cell and consequently introduces an E.M.F. into its circuit which produces current in the magnet and sustains the oscillations under proper conditions. Sufficient energy is transformed from the battery to allow alternating current to be drawn off into a load resistance R. To solve this problem, take the case of the slightly more general arrangement shown in the next figure in which S_{12} represents any kind of coupling of the meshes 1 and 2 carrying the mesh currents ξ_1 and ξ_2. A' and A'' are infinite perfect inductances to restrict the direct currents to their proper paths. Impressed forces are included as shown.

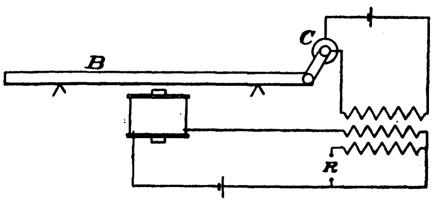

Fig. 1.

This system has a variable inductance in the magnetic circuit since the permanence of the magnetic circuit is a function of the displacement ξ_3. Since this variation is due to a change in the geometry of the figure it will be taken account of in the Lagrangian function as will be seen below. The inductance of the magnetic circuit is $L_0/(1 - a\xi_3)$ where L_0 is the average inductance, hence that part of the kinetic energy which depends upon the magnet is

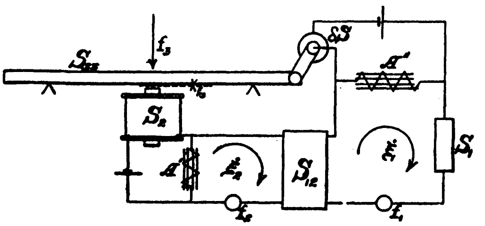
Fig. 2.

$$\tfrac{1}{2}L(I_2 + \xi_2)^2 = \tfrac{1}{2}L_0(1 + a\xi_3)(I_2 + \xi_2)^2,$$

where I_2 is the steady current through it. We therefore get, to first order terms:

$$\frac{\partial T}{\partial \xi_3} = aL_0 I_2 p\xi_2 + \text{constant} = \phi p\xi_2 + \text{constant}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \xi_2} = L_0 p^2 \xi_2 + \phi p \xi_3$$

and the equations of motion are therefore:

$$\begin{aligned} S_{11}\xi_1 - S_{12}\xi_2 \quad\quad\quad 0 &= \partial S X_1 + f_1 = Q_{13}\xi_3 + f_1 \\ - S_{\ 1}\xi_1 + S_{22}\xi_2 + \phi p\xi_3 &= f_2, \\ 0 \quad - \phi p \xi_2 + S_{33}\xi_3 &= f_3. \end{aligned}$$

Here $\delta S X_1$ is equal to the product of the direct current, I_1, through the microphone and its resistance change δr.

The determinant of the system is

$$D = \begin{vmatrix} S_{11} & -S_{12} & -Q_{13} \\ -S_{21} & S_{22} & \phi p \\ 0 & -\phi p & S_{33} \end{vmatrix}$$

It shows how the centrifugal terms ϕp appear due to the fact that the pull of the magnet is proportional to the square of the total current, and also how the symmetry fails when power is added through the term $Q_{13} = Q$.

Since the general theory is immediately applicable to the problem, only some very simple cases will be treated further. Suppose, for example, that the load is pure resistance, $S_1 = Rp$, and that the bar has effective mass m and stiffness s, its resistance being neglected. Also let the coupling $S_{12} = S_{21}$ be through a transformer of self inductances J, J, and mutual inductance M. The equations of the system now become:

$$\left|\begin{matrix} Jp^2 + Rp & -Mp^2 & -Q \\ -Mp^2 & Kp^2 & \phi p \\ 0 & -\phi p & mp^2 + s \end{matrix}\right| \begin{matrix} f_1 \\ f_2; \quad K = L + J. \\ f_3 \end{matrix}$$

Free Oscillations.—For this case, $D = 0$ with $p = in$, giving two equations with even and odd powers of p. Put $Q = A + Bp$ where A and B are even functions of p; then we get for $D = 0$:

$$\begin{aligned} G(mp^2 + s) + J\phi^2 &= M\phi B, \\ RK(mp^2 + s) + R\phi^2 &= M\phi A, \end{aligned} \qquad G = JK - M^2.$$

Here $A = a_0 + a_2 p^2 + \cdots$, $B = b \ + b_2 p^2 + \cdots$.

These equations give a great deal of information; for example we may inquire what are the conditions under which the bar will oscillate in its own natural frequency, for which $-p^2 = s/m = n_0^2$. For this case $A = R\phi/M$, $B = J\phi/M$, and we get for the simplest dynamical connection between the resistance change and the coördinate ξ_3:

$$\delta SX_1 = \delta r I_1 = Q\xi_3 = (R\phi/M + pJ\phi/M)\xi_3.$$

Hence the resistance change must depend upon the displacement and the velocity according to the law

$$\delta r = \frac{J\phi}{MI_1}\frac{d\xi_3}{dt} + \frac{R\phi}{MI_1}\xi_3$$

in order that the oscillations shall be sustained and have the required frequency. In the instrument the microphone is mounted upon an arm which can be set at various angles to the axis of the bar and this allows the correct adjustment to be approximated. The resistance change probably depends upon the acceleration also, which, for this motion, is proportional to the displacement. It is easy to see that the power drawn into the system from the battery is equal to that dissipated in the load R.

We might also wish to know the frequency at which the system could oscillate for a given dynamical connection; as an example suppose the microphone is so fastened to the bar that the resistance change is proportional to the displacement. Then $A = a_0$, $B = 0$ and

$$-p^2 = n_0^2\left(1 + \frac{J\phi^2}{Gs}\right),$$

$$a_0 = \frac{R\phi}{M}\left(1 - \frac{JK}{G}\right).$$

The frequency is therefore increased and the current I_1 must be adjusted to give a_0 the proper value as set by the second equation. Note that if $\phi = 0$, that is, if the bar does not react upon the magnet, the frequency will be $n_0/2\pi$ and no power will be required from the battery to maintain the oscillation. If the damping of the bar is not assumed to be zero more interesting problems arise which may easily be worked out but are too long to be included here. In that case the damping of the bar has considerable influence upon the frequency (not the same as in the damped free oscillation of the bar *alone*) and this effect can be noticed in the instrument by damping the bar without adding to its inertia.

Forced Vibrations.—Imagine an alternator of frequency $n/2\pi$ acting in the mesh 1. The current in the load R will be

$$\xi_1 = \frac{pD_{11}f}{D} = \frac{p\Delta_{11}f}{\Delta - Q\Delta_{13}},$$

where Δ is the value of D when $Q = 0$.

Now D/pD_{11} is the impedance offered by the system when measured from the terminals of the alternator, hence the effect of adding the transforming device is to change this impedance from

$$Z = \frac{\Delta}{p\Delta_{11}}$$

to

$$Z_Q = Z\left(1 - Q\frac{\Delta_{13}}{\Delta}\right).$$

The impedance may therefore be considerably decreased and also given very different reactance characteristics. This fact may be described by saying that the transforming device introduces into one of the terminals the *negative impedance* $- QZ(\Delta_{13}/\Delta)$. In fact, this is what would be indicated by measurements made with a Wheatstone bridge at those terminals.

If $Q(\Delta_{13}/\Delta) = 1$ we have $D = 0$ and the system offers no impedance at all. This is the case of free oscillations again.

Similar results obviously will be found if the alternator is connected into the other mesh or if a mechanical force acts upon the bar. The effect of the transforming mechanism is therefore to amplify the effect of an impressed force by supplying energy from a source of another type which has been ignored in the problem.

We may look at this problem in another way: thus suppose only electrical quantities can be measured so that the system is taken to be one of apparently but two degrees of freedom, both electrical, and information is to be gained only by operations upon them. Elimination of the mechanical coördinate by means of the last equation gives:

$$(Jp^2 + Rp)\xi_1 - \left(Mp^2 + \left(\frac{Q\phi p}{mp^2 + s}\right)\right)\xi_2 = f_1 + Qf_3/(mp^2 + s),$$

$$- Mp^2\xi_1 + \left(Kp^2 + \frac{\phi^2 p^2}{mp^2 + s}\right)\xi_2 = f_2 - \frac{\phi p f_3}{mp^2 + s}.$$

This system, especially in the neighborhood of $mp^2 + s = 0$, would act very differently from a purely electrical system. The two mutual compliances would be widely different (but still only if Q were not zero), the effect of the force f would be changed under the same circumstances, and for any finite coupling ϕ the effective stiffness factors would be changed.

Another Example.—A good illustration of free vibrations occurs in the case of the "howling telephone" which is formed by holding an electrically connected telephone transmitter and receiver together as shown in Fig. 3, the geometry of the system being there made as simple as possible. The fact that the dynamical connection between the receiver diaphragm and the variable element is a column of air (assumed to move in one dimension only) adds interest to the problem.

Let the variable current be ξ_1 and the displacement of the diaphragms ξ_2 and ξ_3. Then δr will be $h\xi_3$, say, and

$$I\delta r = Q\xi_2,$$

so that Q/hI expresses the dynamical connection between the two diaphragms.

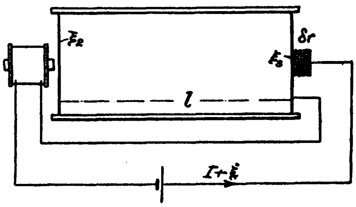

Fig. 3.

We may suppose h to be a number, although this is not necessary. The equations of motion are now similar to those of the last example, namely

$$(Lp^2 + Rp)\xi_1 - (Q - \phi p)\xi_2 = 0,$$

$$- \phi p\xi_1 + S_{22}\xi_2 = 0,$$

in which S_{22} means the stiffness offered by the diaphragm to an impressed force and consequently includes the loading due to the air column.

The condition of free vibration is

$$(Lp^2 + Rp)S_{22} = \phi p(Q - \phi p)$$

or for sustained oscillations

$$(inL + R)S_{22} = \phi(Q - in\phi),$$

and to solve the problem we must know S_{22} and Q for harmonic motions. These depend only upon the air column with the two diaphragms and may be found in two ways. To do this directly from the differential equations of the fluid motion, consider the tube of air and let S_0 and S_l be the stiffnesses of receiver and transmitter diaphragms alone, and also put:

ρ = mean density of fluid,

δP = increase of pressure over mean,

V = velocity potential in fluid,

$qV = \partial V/\partial x$,

f = impressed force on unit area of the receiver diaphragm.

The differential equations of the fluid motion are, for $p = in$

$$(n^2 + a^2q^2)V = 0; \quad \delta P = - in\rho V,$$

from which we get

$$V = e^{int}\left(A \cos\frac{nx}{a} + B \sin\frac{nx}{a}\right),$$

$$\delta P = - in\rho e^{int}\left(A \cos\frac{nx}{a} + B \sin\frac{nx}{a}\right)$$

The boundary conditions are:

At $x = 0$
$$S_0\xi_2 = f - \delta P,$$
$$p\xi_2 = qV.$$
At $x = l$
$$S_q\xi_3 = \delta P,$$
$$p\xi_3 = qV.$$

These four equations allow us to eliminate A and B and solve for ξ_2 and ξ_3 by means of:

$$\left(S_0 \cos\frac{nl}{a} - an\rho \sin\frac{nl}{a}\right)\xi_2 + \quad S_q\xi_3 = f\cos\frac{nl}{a},$$

$$\left(S_0 \sin\frac{nl}{a} + an\rho \cos\frac{nl}{a}\right)\xi_2 - an\rho\ \xi_2 = f\sin\frac{nl}{a}.$$

From these we get:

$$S_{22} = \frac{(S_0S_q - a^2n^2\rho^2)\sin\frac{nl}{a} + an\rho(S_0 + S_q)\cos\frac{nl}{a}}{an\rho\cos\frac{nl}{a} + S_q\sin\frac{nl}{a}},$$

$$Q = \frac{hI \cdot an\rho}{an\rho\cos\frac{nl}{a} + S_q\sin\frac{nl}{a}},$$

and the problem is solved provided the resulting transcendental equations can be solved for the values of n and hI.

These values of S_{22} and Q are interesting in themselves: thus $S_{22} = 0$ gives the free vibrations of a pipe with arbitrary terminal conditions, and by putting, in the equivalent expression,

$$\tan\frac{nl}{a} = \frac{an\rho(S_0 + S_q)}{(an\rho)^2 - S_0S_q},$$

zero and infinite values of S_0, S_q, we get the frequencies of open and closed pipes. Other values give the effects of yielding ends.

There is one value of diaphragm stiffness which is unique and important. Suppose $S_q = ian\rho$ so that the transmitter diaphragm offers only the resistance $a\rho = r_0$. Then

$$S_{22} = S_0 + inr_0,$$

$$\tan\frac{nl}{a} = i,$$

and we get the result that no finite free oscillations exist, so that no reflextion takes place at S_q, all the energy sent out is absorbed, and the tube behaves as one of infinite length. We therefore get the effect of an infinite column of fluid upon a vibrating diaphragm and the energy radiated to infinity. Hence r_0 may be called the radiation resistance of the fluid. For air it is about 40 C.G.S.

Although the equations of free vibration cannot be solved in general, the character of the solution can be seen. There will be a number of values of n which satisfy the frequency equation, and for each value of n a corresponding value of hI which is required to sustain the oscillations of that frequency.

These examples are sufficient to show the applications which may be made of this theory and method. Others, together with a treatment of non-linear connections, will be given in another paper.